IFT-Perimeter-SAIFR
Journeys into Theoretical Physics 2025
Saturday Exam

• Write your name on each page

• Number each page used to solve a given problem as $1/n, 2/n, \ldots, n/n$ where n is the number of pages used to solve that problem

• Do not solve more than one problem per page – these exams will be split apart and graded by different people.

• Problem 1 (Some Frustration in the Ising Model): 25%
• Problem 2 (Quantum Amplification): 25%
• Problem 3 (Gravitation, springs, and integrability): 25%
• Problem 4 (Trajectories in Cosmic Shower): 25%

Suggestion: Try to first do the easiest parts of each exercise, and then try to do the harder parts on as many exercises as possible. This is a difficult exam, so do not be discouraged if you get stuck on an exercise.

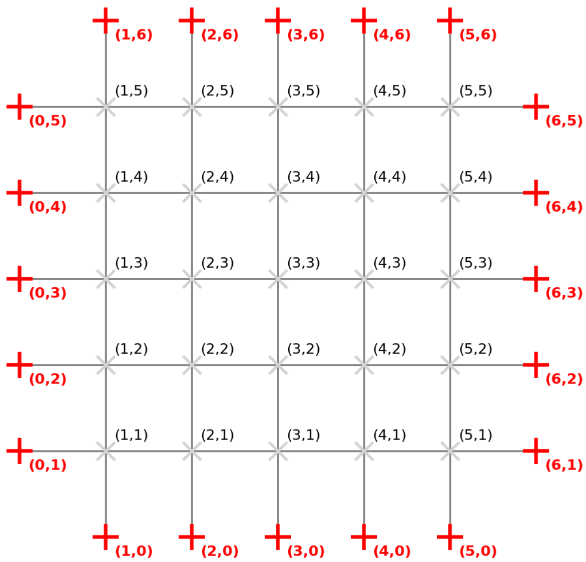

Figure 1: Square lattice example for $L = 5$. We have L^2 dynamical spins (the crosses) and $4L$ boundary spins fixed to be pointing up (the pluses). We sum over all vertical and horizontal links when computing the energy, $H = -J \sum_{i=0}^{L} \sum_{j=1}^{L} \sigma_{i,j}\sigma_{i+1,j} - J \sum_{i=1}^{L} \sum_{j=0}^{L} \sigma_{i,j}\sigma_{i,j+1}$

1 Some Frustration in the Ising Model

Consider the two dimensional Ising model on a square two dimensional grid with $L \times L$ dynamical spins plus $4L$ frozen spins at the boundary. At each vertex with coordinates $x = (i, j)$ we have a spin σ_x which can take values $+1$ (up spin) or -1 (down spin). We fix the spins in the lattice boundary, that is in the first row, last row, first column and last column to be pointing up,

$$\sigma_x = +1\,, \qquad \text{for} \qquad x = (0, i)\,, (L+1, i), (i, 0) \text{ or } (i, L+1)\,,$$

see figure. Then the partition function is the sum over all possible spin configurations,

$$Z = \sum_{\{\sigma_x = \pm 1\}} \exp(-\beta H)\,, \qquad H = -J \sum_{\langle x,y \rangle} \sigma_x \sigma_y$$

In the first sum we have a sum over all possible bulk dynamical spin configurations. In the energy functional H, the sum is over all neighbooring x and y connected by an edge, see figure and caption. Note that there are edges between bulk points and between bulk and boundary points but there are no links between boundary points[1]

Ferromagnetic at Low and High Temperature

Consider first the ferromagnetic case $J > 0$.

1. **[2pt]** In the low temperature ($\beta \to \infty$) limit the ground state configuration dominates. Show that in this limit

$$Z \simeq a\, e^{\beta A} \tag{1}$$

[1]Since those spins are frozen, the contribution from those edges would lead to an immaterial constant anyway.

with $a = 1$ and $A = 2JL(L+1)$.
Hint: How many edges are there?

2. [**2pt**] Expanding further in this limit we get the first correction

$$Z \simeq a e^{\beta A} + a' e^{\beta A'} + \ldots$$

What is a' and A' in the first subleading term in the low temperature expansion?

3. [**2pt**] What is the partition function Z equal to in the high temperature limit $\beta \to 0$.

4. [**2pt**] Explain why the first small beta correction in this limit is of order β^2.
Hint: You don't need to compute it; just explain why the linear term of $O(\beta)$ vanishes.

5. [**4pt**] What is the leading behavior of the magnetization

$$m = \frac{1}{L^2} \sum_{x \in \text{bulk}} \langle \sigma_x \rangle$$

in the two limits

$$\beta \to \infty \quad \text{and} \quad \beta \to 0\,.$$

You can state the result without computation if you explain it physically.

Anti-ferromagnetic at Low Temperature

Consider now the anti-ferromagnetic case $J < 0$.

6. [**3pt**] What is the leading term in the low temperature expansion (1) for $L = 1$? What about $L = 2$?
Hint: Compute the full partition function for $L = 1$ and read the next sentence before attacking $L = 2$.

Through brute force enumeration, we find $a = 1, 7, 1, 8, 1, \ldots$ for $L = 1, 2, 3, 4, 5, \ldots$ respectively. We see that the low temperature ground state is degenerate for some L's but not for others.

7. [**3pt**] Come up with a simple guess for a and A for L odd in (1) by identifying the single ground state configuration for L odd.
Hint: That configuration has $2L - 2$ edges connecting spins with the same sign.

8. [**3pt**] Explain first why this is a very hard problem for general even L compared with the same problem with periodic boundary conditions and even L. What would a and A be for periodic boundary conditions (i.e. $\sigma_{i,j} \equiv \sigma_{i+L,j} \equiv \sigma_{i,j+L}$)? Explain why the anti-ferromagnetic periodic boundary condition problem would be more intricate for L odd as far as degeneracy counting goes.

9. [**4pt**] Argue that the thermodynamical value of the free energy per site

$$f \equiv - \lim_{\beta \to \infty} \lim_{L \to \infty} \frac{1}{\beta L^2} \log Z$$

equals

$$f = 2J\,.$$

with our fixed $\sigma_{\text{boundary}} = +1$ boundary conditions. What would it be for periodic boundary conditions?

2 Quantum Amplification

Suppose we have a state

$$|\psi\rangle = \sum_{i=1}^{N} \psi(i)|i\rangle \,, \tag{2}$$

where all amplitudes $\psi(i)$ are real numbers of similar magnitude; N is large[2]. We want to act on this state with a bunch of unitary transformations to amplify one of the states. That is, after acting with a bunch of unitary transformations, the state takes again the same form but the amplitude $\psi(i)$ multiplying one of the states $|i\rangle$ should now stand out compared to all other states. This problem is about this amplification process. Two key processes will play a role as illustrated in figure 3. One flips the sign of the ket we want to amplify so that $\psi(i) \to -\psi(i)$; the other reflects all amplitudes $\psi(i)$ around their mean value. As illustrated in this figure, combining these two operations can nicely amplify a state. We will see that for large N we need to act sequentially $O(\sqrt{N})$ times with these two transformations on a state to efficiently amplify it so that the probability of measuring that state is above 50%.

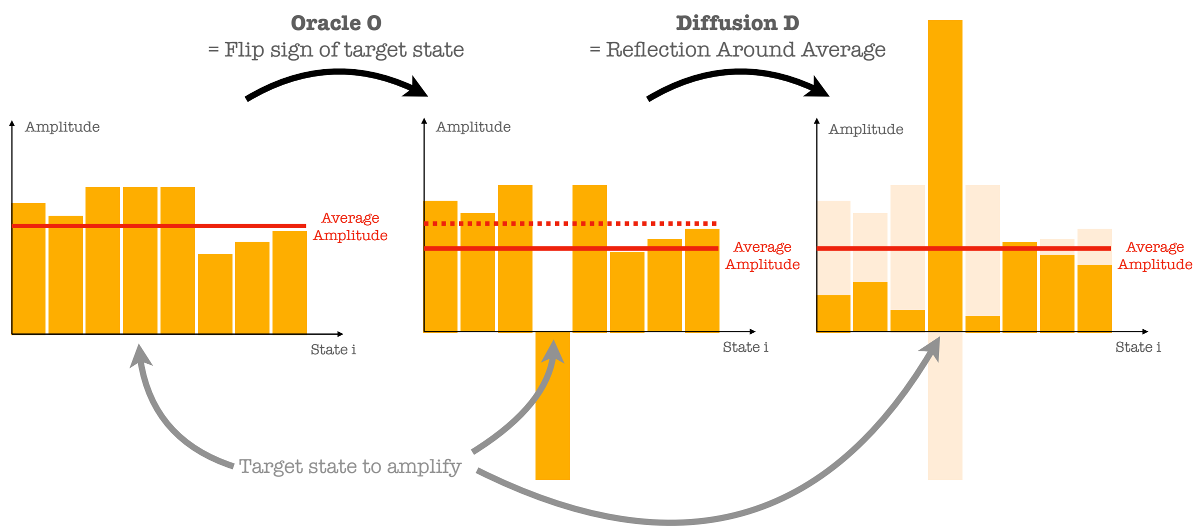

Figure 2: The Oracle Operator flips the sign of a target state (in the figure we took the state to be amplified to be the 4th in a list of twelve states). The Diffusion Operator reflects all amplitudes around the average of the amplitudes. We will see in this problem that both transformations are nice unitary transformation. What is also true (and very important for practical applications) but we will *not* see in this problem is that they can be realized by combining local unitary transformations.

Suppose that we have a normalized state, which is a linear combination of some basis states $|i\rangle$ where $i = 1...N$ with the same amplitude. In other words, we start with the initial state

$$|s_{\text{initial}}\rangle = \sum_{i=1}^{N} \frac{1}{\sqrt{N}}|i\rangle. \tag{3}$$

We want to amplify the amplitude multiplying $|1\rangle$ say. The oracle operator $\mathcal{O}$ is defined as

$$\mathcal{O}|1\rangle = -|1\rangle \,, \qquad \mathcal{O}|i\rangle = |i\rangle \text{ if } i > 1 \,,$$

[2] $|i\rangle$ could be a complete set of states of a spin-1/2 chain of length L, e.g., in which case $N = 2^L$.

and the diffusion operator $\mathcal{D}$ is defined as

$$\mathcal{D}|i\rangle = -|i\rangle + \frac{2}{N}\sum_{j=1}^{N}|j\rangle \tag{4}$$

1. **[3pt]** Show that the oracle and the diffusion operator are unitary operators.

2. **[2pt]** What happens to the average amplitude $\bar{\psi} \equiv \frac{1}{N}\sum_i \psi(i)$ when we flip the target state in $|s_{\text{initial}}\rangle$ by acting on the state with the oracle operator $\mathcal{O}$? You only need to answer it qualitatively. Does it get bigger or smaller or remain the same?

3. **[3pt]** Show that the diffusion operator does act like a reflection around average operator; namely show that upon acting with $\mathcal{D}$ on a state the amplitudes of the resulting states are related to the original amplitudes by a reflection around the mean amplitude value $\bar{\psi}$. How does $\mathcal{D}$ affect the mean amplitude of a state?

After repeated action of oracles and diffusion we have an intermediate state

$$|s_{\text{intermediate}}\rangle = \sqrt{1-C^2}|1\rangle + \frac{C}{\sqrt{N-1}}\sum_{i=2}^{N}|i\rangle \tag{5}$$

where C is real and positive.

4. **[3pt]** Show that the state is properly normalized. If the probability of finding the state $|1\rangle$ in the intermediate state is an order one value still smaller than 50%, what can you say about C?

5. **[5pt]** What is the state after the operator $\mathcal{O}$ acts on it? At large N what is the approximate mean amplitude of the state $|s_{\text{intermediate}}\rangle$? What about of the state $\mathcal{O}|s_{\text{intermediate}}\rangle$?

6. **[5pt]** Now we apply the diffusion to this state to get $\mathcal{D}\mathcal{O}|s_{\text{intermediate}}\rangle$. Again at large N, what is the approximate change in the amplitude of the target state $|1\rangle$ from $|s_{\text{intermediate}}\rangle$ to $\mathcal{D}\mathcal{O}|s_{\text{intermediate}}\rangle$? (You should find that under the conditions of problem 4 it increases by at least $\sqrt{2}/\sqrt{N}$.) What is the approximate change in the amplitude of the other states?

7. **[4pt]** Show that there exists a number M less than $\sqrt{N}/2$ such that after M repetitions of the oracle plus diffusion loop the amplitude of the target state $|1\rangle$ will exceed $1/\sqrt{2}$ and thus the probability of measuring this state will be bigger than 50%.

These ideas are used in quantum computation to create search algorithms of $\sqrt{N}$ complexity beating their classical counterparts which scale like N.

3 Gravitation, springs, and integrability

Central force motion is ubiquitous in physics, governed by potentials of the form $V(\vec{x}) = -k\|\vec{x}\|^n$ for $\vec{x} \in \mathbb{R}^3$. Motion of a particle in such a potential is described by the Hamiltonian

$$H(\vec{x},\vec{p}) = \frac{1}{2}\|\vec{p}\|^2 + V(\vec{x}) = \frac{1}{2}\frac{\|\vec{p}\|^2}{m} - k\|\vec{x}\|^n \tag{6}$$

This motion can be bound, as in orbits, or unbound, as in the deflection of a passing particle. Bertrand's theorem states that of the possible central force laws, only those associated with gravitation ($n = -1$) and harmonic springs ($n = 2$) produce bound motion with *closed* orbits. In this problem we will see how this property arises from the *integrability* of the gravitational and Hookian systems. Integrability is a concept that arises often in the study of many-body quantum systems with extensively many conserved quantities.

Useful identities for this problem:

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}) \tag{7}$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} \tag{8}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \tag{9}$$

1. [**3pt**] Show explicitly that the three components of angular momentum $\vec{L} = \vec{x} \times \vec{p}$ and the energy are conserved. What are the symmetries associated with these conserved quantities?

   Hint: a quantity $f(\vec{x},\vec{p})$ is conserved in Hamiltonian dynamics if its Poisson bracket $\{f, H\}$ with the Hamiltonian is zero, where the Poisson bracket is defined by

   $$\{f,g\} = \frac{\partial f}{\partial \vec{x}} \cdot \frac{\partial g}{\partial \vec{p}} - \frac{\partial f}{\partial \vec{p}} \cdot \frac{\partial g}{\partial \vec{x}} \tag{10}$$

2. [**3pt**] What is the dimension of the phase space for this system? What is the dimension of the phase space consistent with setting the angular momentum and the energy to specific values? Describe the geometric relationship between the angular momentum $\vec{L}$ and the available phase space.

3. [**3pt**] Define the *Runge–Lenz* vector $\vec{A} = \vec{p} \times \vec{L} - km\hat{x}$, where $\hat{x} = \vec{x}/\|x\|$ is the unit vector in the $\vec{x}$ direction. Show that the Runge–Lenz vector is conserved in gravitational systems with $n = -1$.

4. Additional conserved quantities are only useful if they are independent from each other (consider the conserved quantities H^2, H^3, ...).

   (a) [**1pt**] If the Runge–Lenz vector were independent from the angular momentum and energy, what is the dimension of the phase space consistent with fixing all three?

   (b) [**1pt**] What is $\vec{A} \cdot \vec{L}$ equal to?

   (c) [**2pt**] Show that $\vec{A} \cdot \vec{A}$ produces constraints between $\vec{A}$, $\vec{L}$, and the energy.

   (d) [**1pt**] Given the above, what is the actual dimension of the phase space region consistent with fixing all three conserved quantities? Relate this to the fact that bound orbits are closed for gravitational systems.

5. **[3pt]** Derive the path of motion in terms of the magnitudes of $\vec{A}$ and $\vec{L}$, the radius $r = \|\vec{x}\|$, and θ the angle between $\vec{A}$ and $\vec{x}$.

   Hint: in this 'integrable' system, equations of motion can be solved without integrating anything. Start with $\vec{A} \cdot \vec{x}$.

6. **[2pt]** Give a geometric interpretation of the Runge–Lenz vector.

7. **[2pt]** In a system with a potential different from $n = -1$, the Runge–Lenz vector is not conserved. How does it evolve with time? What are the implications for orbits in a system with n only slightly different from -1?

8. **[4pt]** Now define the 3×3 tensor $A_{ij} = \frac{1}{2} p_i p_j - kmx_i x_j$. Show that for a Hookian spring system ($n = 2$) A is conserved. Show that $\operatorname{Tr} A$, $\operatorname{Tr} A^2$, and $A\vec{L}$ imply constraints on the components of A. What is the dimension of the phase space consistent with fixing the tensor A, $\vec{L}$, and the energy?

4 Trajectories in Cosmic Shower

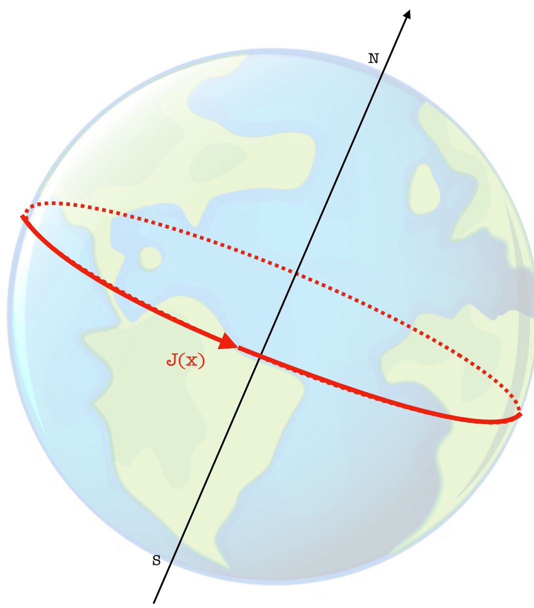

Figure 3: The Earth has a magnetic field which can be approximated by a magnetic dipole created by a current loop with current $J(x)$ in the plane of the equator.

In a cosmic shower, a negatively charged pion decays into a muon and a neutrino after entering the Earth?s atmosphere.

1. **[2pt]** The negatively charged pion has rest mass M_π and charge e, and is moving with relativistic velocity v_π in the direction of the Earth when it decays. If its half-life at rest is T_π, what is its half-life when moving with velocity v_π?

2. **[3pt]** If the muon has rest mass M_μ and the neutrino has rest mass zero, what is the velocity of the pion v_π if the muon has zero velocity at the moment the pion decays?

The Earth has a magnetic field which can be approximated by a magnetic dipole created by a current loop with current $J(x)$.

3. **[4pt]** Use $B = \nabla \times A$ and $\nabla \times B = \frac{4\pi}{c} J$ to derive the vector potential $A(x)$ in terms of $J(x)$ in the gauge $\nabla A = 0$.
   Hint: Useful formula: $\frac{d}{dx^j}\frac{d}{dx^j}|x|^{-1} = -4\pi\delta^3(x)$. (4 points)

4. **[2pt]** If $J(x)$ is approximated by a circular current loop in the plane of the equator as in the figure, draw a cartoon of the magnetic field lines around the earth.

5. **[4pt]** If the current loop has radius R, find the value of the B field on the axis perpendicular to the loop that goes through the North and South Poles as a function of the distance from the center of the loop.
   Hint: At the center of the loop you should find $|B| = 2\pi|J|/cR$.

In a gravitational field $g_{mn}(x)$ and electromagnetic field described by the vector potential $A_m(x)$, a relativistic spin-zero particle of charge e and rest mass M has the Lagrangian

$$L = Mc\sqrt{g_{mn}(x)\frac{dx^m}{d\tau}\frac{dx^n}{d\tau}} + eA_m(x)\frac{dx^m}{d\tau}$$

where $m, n = 0$ to 3.

6. **[5pt]** Compute the relativistic equations of motion coming from extremizing the Lagrangian (Euler-Lagrange equations) and show that they imply the expected non-relativistic equations in the limit where g_{mn} is constant and $v \ll c$.
7. **[2pt]** Describe in words the trajectories of the muon and neutrino depending on the latitude in which they enter the Earth's atmosphere.
8. **[3pt]** Using the fact that the Lagrangian is the kinetic energy minus the potential energy, add an appropriate term to the Lagrangian depending on the spin of the particle $\vec{S}$ and the electromagnetic field. Describe in words how this term affects the trajectories of the muon and neutrino which are entering at zero degrees latitude (i.e. above the North Pole).