IFT-Perimeter-SAIFR
Journeys into Theoretical Physics 2025
Sunday Exam

- Write your name on each page

- Number each page used to solve a given problem as $1/n, 2/n, \ldots, n/n$ where n is the number of pages used to solve that problem

- Do not solve more than one problem per page – these exams will be split apart and graded by different people.

- Problem 1 (Some more uses for entanglement): 25%
- Problem 2 (Toric code on a disk): 25%
- Problem 3 (Modeling and managing the population dynamics of a Caribbean reptile): 25%
- Problem 4 (Horizon Problem and the CMB): 25%

Suggestion: Try to first do the easiest parts of each exercise, and then try to do the harder parts on as many exercises as possible. This is a difficult exam, so do not be discouraged if you get stuck on an exercise.

1 Some more uses for entanglement

Superdense coding

This problem has you work through the super-dense coding protocol, which is a method to send 2 classical bits of information using a single qubit of communication.

1. **[5pt]** Let's first of all make a naive attempt at sending two bits of classical data using a single qubit of message. Suppose Alice has message bits $a, b \in \{0, 1\}$, which she wants to communicate to Bob. We have Alice prepare one of the following four states based on her message bits,

$$
\begin{aligned}
00 &\to |0\rangle \\
01 &\to |1\rangle \\
10 &\to |+\rangle \\
11 &\to |-\rangle
\end{aligned} \tag{1}
$$

   Alice then sends the resulting state to Bob. Bob measures his state to try to learn Alice's message. Explain why this idea doesn't work, and in particular why Bob cannot determine the message with perfect correctness (an argument suffices, you don't need a formal proof).

2. **[5pt]** Next let's try a different protocol, where we add the ingredient of shared entanglement between Alice and Bob. Suppose Alice and Bob share an EPR pair $|\Psi_{00}\rangle_{AB}$. Alice chooses one of the four operations $\{I_A, X_A, Z_A, X_A Z_A\}$ and applies it to $\mathcal{H}_A$. Write down the four states on AB that result from Alice's four possible operations.

3. **[5pt]**After Alice applies her choice of Pauli operator, can Bob tell which operation she has applied? Why or why not?

4. **[5pt]** Alice now sends the A system to Bob. Explain what Bob should do to learn Alice's message.

Remote state preparation.

The teleportation protocol allows Alice to send an unknown quantum state $|\psi\rangle$ to Bob. We can also explore the setting where Alice, the sender, knows the quantum state $|\psi\rangle$ to be sent but Bob, the receiver, does not. To send the known state to Bob, the naive method would be for Alice to send a classical description of the state. This could require many classical bits to describe accurately, since there are two complex numbers α, β needed to describe an arbitrary qubit. In fact though, for a large class of quantum states we can get away with only a small amount of classical communication. Focus on a simplified setting where

$$
|\psi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + e^{i\phi} |1\rangle \right). \tag{2}
$$

5. **[5pt]** Show that if Alice and Bob share entanglement, Alice can have Bob prepare $|\psi\rangle$ in his lab by sending only a single classical bit of message.

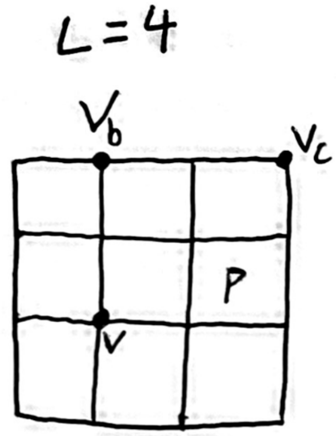

Figure 1: A lattice with open boundary condition, $L = 4$.

2 Toric code on a disk

We have seen toric code on torus, which has 4 degenerate ground states. Now, let us work out what things look like with open boundary conditions.

Consider an $L \times L$ square lattice with open boundary condition, namely, different edges are not glued together and do not talk to each other when they are separated far away. Qubits live on the edges (links) of the lattice. We shall consider Hamiltonians of the form

$$H = -\sum_v A_v - \sum_p B_p - H_{bd}, \tag{3}$$

where the first two terms are familiar: v runs over vertices with 4 connecting edges and p runs over plaquettes containing 4 edges (see Fig. 1), and

$$A_v = \prod_{i \in v} \sigma_i^x, \quad B_p = \prod_{i \in p} \sigma_i^z. \tag{4}$$

The boundary Hamiltonian H_{bd} contains only local terms near the physical edges.

1. **[5pt]** Consider the following two different possibilities: (a) $H_{bd}^{(a)} = -\sum_i \sigma_i^z$ for all i on the boundary.

   (b) $H_{bd}^{(b)} = -\sum_{v_b} \prod_{i \in v_b} \sigma_i^x - \sum_{v_c} \prod_{i \in v_c} \sigma_i^x$, where $\{v_b\}$ includes all boundary vertices with 3 (not 4!) connecting edges, and the last corner term runs over the four corner vertices v_c, each having 2 connecting edges (see Fig. 1).

   Show that in both cases, we get a **unique gapped** ground state.

2. **[5pt]** The situation becomes more interesting when parts of the edges have (a) and the other parts have (b). Specifically, let the boundary Hamiltonian be $H_{bd}^{(a)}$ on the upper and lower boundaries, and $H_{bd}^{(b)}$ on the left and right boundaries (without the corner terms).

   Show that we get **two degenerate gapped** ground states.

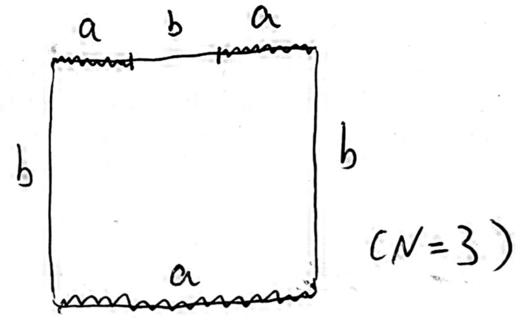

Figure 2: Boundary interactions with $N = 3$ segments of $H_{bd}^{(a)}$ and $H_{bd}^{(b)}$.

3. **[5pt]** Recall that ground-state degeneracy is not always stable against local perturbation (it may be "accidental" or "fine-tuned"). Also recall that on torus, we can argue that the ground state degeneracy is stable due to a set of operators, living on long strings, that are commuting with the Hamiltonian but non-commuting among themselves. It turns out that a similar argument also works for the 2-fold degeneracy in (2). Find the relevant string operators in (2).

4. **[5pt]** In (2), there are two segments on the boundary with interaction (a) and two with interaction (b). We can consider even more alternations of the two boundary interactions. Consider a situation with N segments in (a) and N segments in (b); see Fig. 2 for illustration. What is the ground-state degeneracy? What are the relevant string operators?

5. **[5pt]** We may want to interpret the ground state degeneracy as some kind of effective degrees of freedom (like spins) living on the domain walls between the two boundary interactions (a) and (b). Based on the ground state degeneracy obtained from (4), argue that this "domain wall degree of freedom" cannot really be a local spin with a well-defined local Hilbert space.

3 Modeling and managing the population dynamics of a Caribbean reptile

The Aruba island rattlesnake is a critically endangered species, with several attributes that make it particularly susceptible to extinction. First, the species range is limited to a $\sim$30km^2 region within the small island of Aruba. Second, the population has been declining due to human persecution and habitat degradation. After some work, you were able to measure a per-capita birth rate $b = 0.70$ year^{-1} and a per-capita death rate $d = 0.69$ year^{-1}. You could also estimate a current population size of 250 individuals and an island carrying capacity of $K = 500$ individuals.

Assuming that the Aruba island rattlesnake reproduces asexually[1], the Master equation for the probability of having a population size n at time t is

$$\frac{\partial P(n,t)}{\partial t} = \sum_{\ell=\pm 1} (E^{\ell} - 1)[\Omega_{n\to n-\ell} P(n,t)], \tag{5}$$

with $\Omega_{n\to n+1} = bn$ and $\Omega_{n\to n-1} = dn + \gamma n(n-1)/L^2$ where L^2 is the species range size. The equation for the dynamics of the mean population size is

$$\frac{\mathrm{d}\langle n\rangle}{\mathrm{d}t} = -\sum_{\ell} \Big\langle \ell\, \Omega_{n\to n-\ell} \Big\rangle. \tag{6}$$

1. **[4pt]** Obtain the value of the parameter γ.
2. An Aruban NGO working on the conservation of the island rattlesnake has read your model and is interested in using it to quantify extinction risks. The *minimum viable population*, MVP, is the abundance threshold below which the survival probability of the population in the next 50 years is lower than 95%.
   - A.- **[6pt]** How would you obtain the MVP for the Aruba island rattlesnake if you were able to solve analytically any of the equations you derived in the previous step? You do not need to compute anything here; just explain how you would obtain this information if you had the solutions of your models *(2.5 pts)*.
   - B.- **[6pt]** Explain, step by step, a numerical simulation approach that would give you this information.
3. **[5pt]** Some months after you delivered your results to the Aruban NGO, they contact you again because they have realized that some of your parameter measurements were wrong and they provide you with the updated values. The new demographic rates are $b = 1.12$ year^{-1} and a per-capita death rate $d = 1.11$ year^{-1}. Describe the qualitative impact of this parameter change on the island's carrying capacity and the MVP. Do not forget to explain the reasoning that supports your conclusions.
4. **[4pt]** How would you refine your model to account for the fact that this species has sexual reproduction? That is, the birth of a new individual, which can be a male or a female with the same probability, depends on the encounter between a male and a female.

[1] Asexual reproduction means that new organisms appear from the division of a parent in two $A \to 2A$, as we saw in all the examples in class

4 Horizon Problem and the CMB

In this exercise you will learn about the horizon problem in the cosmic microwave background (CMB) in the standard hot big bang model and possibly how to solve it.

The CMB originates from the time when the universe cools enough so that electrons and protons can bind forming neutral hydrogen atoms. This process is called recombination and happens quickly at a redshift of approximately $z_{rec} = 1100$. The photons from the CMB are measured today in the microwave range.

1. **[2pt]** Recall that the conformal time is defined as:

$$d\eta = \frac{dt}{a(t)} \tag{7}$$

   Show that the conformal time interval in a time interval $(t_i.t_f)$ can be written as:

$$\eta(t_f) - \eta(t_i) = \int_{a_i}^{a_f} \frac{da}{Ha^2} \tag{8}$$

2. **[3pt]** Show that if the universe is is composed of a single fluid with constant equation of state w then:

$$\eta(t_f) - \eta(t_i) = \frac{2}{H_0(1+3w)} \left[(a_f)^{(1+3w)/2} - (a_i)^{(1+3w)/2}\right] \tag{9}$$

3. **[2pt]** What are the limits of η_i as one approaches the Big Bang singularity $a_i \to 0$ which defines $t_i = 0$, if $w > -1/3$ and if $w < -1/3$?

4. **[6pt]** Consider a universe filled with only matter and radiation, and that their contributions to the energy density of the universe are equal at a_{eq}. In this case show that the Hubble parameter is given by:

$$H = H_0 \Omega_m^{1/2} a^{-2} (a + a_{eq})^{1/2} \tag{10}$$

5. **[3pt]** Show that in the above case the conformal time is given by:

$$\eta(a) = \int_0^a \frac{da'}{Ha'^2} = \frac{2}{H_0 \Omega_m^{1/2}} \left[\sqrt{a + a_{eq}} - \sqrt{a_{eq}}\right] \tag{11}$$

6. **[3pt]** The angle subtended by the causal patch at recombination today is given by the ratio of the comoving size of the horizon at recombination and the comoving distance from today to the recombination time:

$$\theta_h = \frac{2\eta_{rec}}{\eta_0 - \eta_{rec}} \tag{12}$$

   Estimate this angle in degrees.

7. **[3pt]** How many causally disconnected patches are there in the sky today?

8. **[3pt]** This is called the horizon problem. Looking at the 3rd part of this question, how can you solve this problem?