IV School on Light and Cold Atoms
Course on 2D quantum gases
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Hélène Perrin
helene.perrin[at]univ-paris13.fr

Lecture 1

Bose-Einstein condensation and superfluidity

This course on two-dimensional quantum gases, given at the ICTP São Paulo fourth school on light and cold atoms, is organized into three lectures. The first introductory lecture addresses Bose-Einstein condensation in an ideal gas, the effect of interactions in the weakly interacting limit, Gross-Pitaevskii equation and its hydrodynamic formulation, as well as superfluidity. The second lecture is devoted to the main features of two-dimensional quantum gases: importance of the interactions, enhanced phase fluctuations and the emergence of a quasi long-range order, superfluidity through the Berezinskii-Kosterlitz-Thouless mechanism, scaling symmetry. The third lecture has taken the form of a seminar on fast rotating quasi two-dimensional Bose gases and thermal melting of the vortex lattice.

I would like to thank the organizers of the school, Raul Celistrino, Patricia Castilho, Mathilde Hugbart and Romain Bachelard for this nice opportunity to prepare this lecture series and encourage me to prepare lecture notes. I also thank ICTP SAIFR for their efficient help during the school, and the attendees for their active participation, questions and feedback. The slides accompanying these notes can be found on the school website: `https://www.ictp-saifr.org/slca2025/`

The present lecture is devoted to Bose-Einstein condensation and superfluidity. To prepare the lecture, I have mostly used my lecture notes prepared for Master program ICFP at ENS and the lectures given by Jean Dalibard at Collège de France during the academic year 2015-2016 [1], which are available online, in French and in English. In addition, the following general references may be useful:

1. F. Dalfovo, S. Giorgini, L. Pitaevskii and S. Stringari, *Theory of Bose-Einstein condensation in trapped gases*, Rev. Mod. Phys. **71**, 463 (1999) [2];
2. Y. Castin, *Bose–Einstein condensates in atomic gases: simple theoretical results*, Proceedings of Les Houches LXXII Summer School [3];
3. C. J. Pethick and H. Smith, *Bose–Einstein Condensation in Dilute Gases*, Second edition, Cambridge (2008) [4];
4. Lev Pitaevskii and Sandro Stringari, *Bose-Einstein condensation*, Oxford (2003) [5];
5. Lev Pitaevskii and Sandro Stringari, *Bose-Einstein Condensation and Superfluidity*, Oxford (2016) [6].

1 Reminder: BEC in an ideal gas

Let us first examine the possibility for Bose-Einstein condensation (BEC) for a gas of non interacting bosons. BEC will occur if the number of particles in excited states saturates, i.e. if the total number of particles N exceeds $N_{\mathrm{exc}}^{\max}(T)$, the maximum number of particles in the excited states at a given temperature T.

1.1 BEC: a saturation of the excited states

We describe the bosonic gas in the grand canonical ensemble and assume that the ground state of the single-particle Hamiltonian is non degenerate. The average energy is fixed by the temperature T and the average atom number N is fixed by the chemical potential μ. The average occupation of each state j of energy E_j, with E_0 the energy of the ground state, is given by

$$\boxed{n_j = \frac{1}{e^{(E_j-\mu)/kT} - 1},} \tag{1}$$

where k is Boltzmann's constant. $n_j \geq 0$, which imposes $\mu < E_j$ for all j, which is fulfilled as soon as $\mu < E_0$. The occupation in the ground state is

$$n_0 = \frac{1}{e^{\beta(E_0-\mu)} - 1} = \frac{z}{1-z} \tag{2}$$

where $\beta = 1/(kT)$ and we have introduced the *fugacity*

$$\boxed{z = e^{\beta(\mu-E_0)} < 1.} \tag{3}$$

Using z, we can also write n_j as

$$n_j = \frac{1}{z^{-1}e^{\beta(E_j-E_0)} - 1}, \tag{4}$$

such that the number of particles in excited states writes

$$N_{\mathrm{exc}}(z,T) = \sum_{j>0} \frac{1}{z^{-1}e^{\beta(E_j-E_0)} - 1}, \tag{5}$$

which is an increasing function of z. BEC will occur if $N_{\mathrm{exc}}(z,T)$ stays finite when z approaches 1, while n_0 is not bounded and diverges as $z \to 1$. This depends on how E_j depends on j.

Using the formula

$$\sum_{n=1}^{\infty} x^n = \frac{x}{1-x} \qquad \text{if } |x| < 1, \tag{6}$$

we can write Eq. (5) as

$$N_{\mathrm{exc}}(z,T) = \sum_{j>0}\sum_{n=1}^{\infty} z^n e^{n\beta(E_0-E_j)} = \sum_{n=1}^{\infty} z^n \sum_{j>0} e^{n\beta(E_0-E_j)}. \tag{7}$$

1.2 Semi-classical approximation: using the density of states

If kT is much larger than the spacing $E_{j+1} - E_j$ between consecutive energies in the spectrum, we can make a semi-classical approximation and replace the discrete sum by an integral

$$\sum_{j>0} \to \int_{E_0}^{+\infty} \rho(\varepsilon) d\varepsilon, \tag{8}$$

where $\rho(\varepsilon)$ is the density of states at energy ε, defined such that $\rho(\varepsilon)d\varepsilon$ is the number of states of the Hamiltonian $\mathcal{H}(\mathbf{r}, \mathbf{p})$ with an energy between ε and $\varepsilon + d\varepsilon$, or more formally

$$\boxed{\rho(\varepsilon) = \frac{1}{h^D} \int d\mathbf{p}\, d\mathbf{r}\, \delta\left(\mathcal{H}(\mathbf{r}, \mathbf{p}) - \varepsilon\right),} \tag{9}$$

with h the Planck constant and D the dimension of the system. We thus have

$$N_{\rm exc}(z, T) = \int_{E_0}^{+\infty} d\varepsilon\, \rho(\varepsilon) n_\varepsilon(\varepsilon) = \int_{E_0}^{+\infty} d\varepsilon\, \frac{\rho(\varepsilon)}{z^{-1} e^{\beta(\varepsilon - E_0)} - 1}. \tag{10}$$

The convergence of the integral at $+\infty$ is ensured as soon as $\rho(\varepsilon)e^{-\beta\varepsilon}$ has a converging integral, which is easy with the exponential: $\rho(\varepsilon)$ should not increase exponentially with the energy. When $z \to 1$, the limit at $\varepsilon \to E_0$ is more tricky: the integrand in Eq. (10) is approximately $kT\rho(\varepsilon)/(\varepsilon - E_0)$ such that $\rho(\varepsilon)$ should converge to 0 when $\varepsilon \to E_0$ to ensure the convergence of the integral. For example, $N_{\rm exc}$ diverges in ρ does not depend on ε.

Introducing an infinite sum over n, we can recast Eq. (10) into:

$$N_{\rm exc}(z, T) = \sum_{n=1}^{\infty} z^n \int_{E_0}^{+\infty} d\varepsilon\, \rho(\varepsilon) e^{-n\beta(\varepsilon - E_0)} = \sum_{n=1}^{\infty} z^n \int_0^{+\infty} d\varepsilon\, \rho(\varepsilon - E_0) e^{-n\beta\varepsilon}. \tag{11}$$

Using the change in integration variable $u = n\beta\varepsilon$ or $\varepsilon = ukT/n$, we can also recast this integral as

$$\boxed{N_{\rm exc}(z, T) = kT \sum_{n=1}^{+\infty} \frac{z^n}{n} \int_0^{+\infty} du\, \rho\left(\frac{kT}{n} u - E_0\right) e^{-u}.} \tag{12}$$

1.3 Simple expressions for a power-law density of states

If the density of state ρ is a power law, i.e.

$$\rho(\varepsilon) = \frac{1}{\epsilon_0} \left(\frac{E_0 + \varepsilon}{\varepsilon_0}\right)^q, \tag{13}$$

then we get for the total number of particles in the excited states

$$N_{\rm exc}(z, T) = \frac{kT}{\varepsilon_0} \sum_{n=1}^{+\infty} \frac{z^n}{n} \int_0^{+\infty} du \left(\frac{kT}{n\varepsilon_0}\right)^q u^q e^{-u} = \left(\frac{kT}{\varepsilon_0}\right)^{q+1} \sum_{n=1}^{+\infty} \frac{z^n}{n^{q+1}} \int_0^{+\infty} du\, u^q e^{-u}, \tag{14}$$

and finally

$$\boxed{N_{\mathrm{exc}}(z,T) = \Gamma(q+1)\left(\frac{kT}{\varepsilon_0}\right)^{q+1} \mathrm{Li}_{q+1}(z).} \tag{15}$$

Γ is the Euler Gamma function, and

$$\mathrm{Li}_s(z) = \sum_{n=1}^{+\infty} \frac{z^n}{n^s} \tag{16}$$

is the polylogarithmic function of order s. When it converges in the limit $z \to 1$, i.e. for $s > 1$, its value for $z = 1$ is the Riemann zeta function:

$$\mathrm{Li}_s(1) = \zeta(s). \tag{17}$$

We thus have a simple criterion for BEC to occur: if $q > 0$, $N_{\mathrm{exc}}^{\max}(T) = N_{\mathrm{exc}}(1,T) = \Gamma(q+1)\zeta(q+1)\left(\frac{kT}{\varepsilon_0}\right)^{q+1}$ is finite and the number of atoms in the excited states saturates if $N > N_c(T) = N_{\mathrm{exc}}^{\max}(T)$,

$$\boxed{N_c(T) = \Gamma(q+1)\zeta(q+1)\left(\frac{kT}{\varepsilon_0}\right)^{q+1},} \tag{18}$$

or equivalently if $T < T_c(N)$ with

$$\boxed{kT_c = \frac{\varepsilon_0 N^{\frac{1}{q+1}}}{[\Gamma(q+1)\zeta(q+1)]^{\frac{1}{q+1}}}.} \tag{19}$$

Below T_c, the number of particles in the excited states saturates to $N_{\mathrm{exc}} = N_c(T)$, and fraction of atoms in the ground state is given by

$$\frac{N_0}{N} = 1 - \frac{N_{\mathrm{exc}}}{N} = 1 - \frac{N_c(T)}{N}. \tag{20}$$

Using $N_c(T) \propto T^{q+1}$ and $N = N_c(T_c)$ by definition of T_c, we get

$$\boxed{\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{q+1}.} \tag{21}$$

1.4 From box traps to harmonic traps

Let us consider the case where the Hamiltonian describes a particle of mass M in dimension D in a generic, isotropic[1] power-law trap

$$V(\mathbf{r}) = V_0\left(\frac{r}{R}\right)^{2/\alpha} \tag{22}$$

[1]If the trap is anisotropic, it is easy to come back to the isotropic case by a mere dilation of the axes. The conclusion for BEC still holds.

with $\alpha \geq 0$ some exponent. This expression describes the case of an harmonic trap, with $\alpha = 1$, but also a box trap for $\alpha = 0$, where $V = 0$ for $r < R$ and $V = +\infty$ for $r > R$. The Hamiltonian is then

$$\mathcal{H}(\mathbf{r}, \mathbf{p}) = \frac{p^2}{2M} + V_0 \left(\frac{r}{R}\right)^{2/\alpha}. \tag{23}$$

Let us compute the density of states for this case, taking advantage of the symmetry of the trap and the kinetic energy:

$$\rho(\varepsilon) = \frac{1}{h^D} \int_0^{+\infty} C(D) r^{D-1} dr \int_0^{+\infty} C(D) p^{D-1} dp\, \delta\left(\frac{p^2}{2M} + V_0 \left(\frac{r}{R}\right)^{2/\alpha} - \varepsilon\right). \tag{24}$$

Here, $C(D)$ is a constant that depends on the dimension and gives the result of angular integration: $C(1) = 1$, $C(2) = 2\pi$, $C(3) = 4\pi$.

We make the change in integration variable $K = p^2/2M$, such that $dK = p\,dp/M$ and $p = \sqrt{2MK}$. We get

$$\rho(\varepsilon) = \frac{C(D)^2}{2} \left(\frac{2M}{h^2}\right)^{D/2} \int_0^{+\infty} r^{D-1} dr \int_0^{+\infty} K^{D/2-1} dK\, \delta\left(K + V_0 \left(\frac{r}{R}\right)^{2/\alpha} - \varepsilon\right). \tag{25}$$

The delta function integrates directly with $K = \varepsilon - V_0 \left(\frac{r}{R}\right)^{2/\alpha}$ with the condition $K \geq 0$, i.e. $r \leq r_{\max} = R\,(\varepsilon/V_0)^{\alpha/2}$:

$$\rho(\varepsilon) = \frac{C(D)^2}{2} \left(\frac{2M}{h^2}\right)^{D/2} \int_0^{r_{\max}} dr\, r^{D-1} \left(\varepsilon - V_0 \left(\frac{r}{R}\right)^{2/\alpha}\right)^{D/2-1}. \tag{26}$$

We now use the variable $u = r/r_{\max}$:

$$\rho(\varepsilon) = \frac{C(D)^2}{2} \left(\frac{2MR^2}{h^2}\right)^{D/2} \left(\frac{\varepsilon}{V_0}\right)^{D\alpha/2} \varepsilon^{D/2-1} \int_0^1 du\, u^{D-1} \left(1 - u^{2/\alpha}\right)^{D/2-1}. \tag{27}$$

The integral is now just a number $\eta(\alpha, D) = D^{-1}\Gamma(D/2)\Gamma(1+\alpha D/2)/\Gamma[(1+\alpha)D/2]$. We get:

$$\rho(\varepsilon) = \frac{\eta(\alpha, D) C(D)^2}{2} \left(\frac{2MR^2}{h^2}\right)^{D/2} V_0^{-D\alpha/2} \varepsilon^{D(\alpha+1)/2-1} = \frac{1}{\epsilon_0} \left(\frac{\varepsilon}{\varepsilon_0}\right)^q \tag{28}$$

with

$$\varepsilon_0 = \left[\frac{2}{\eta(\alpha, D) C(D)^2} \left(\frac{h^2}{2MR^2}\right)^{D/2} V_0^{D\alpha/2}\right]^{\frac{2}{D(\alpha+1)}} \tag{29}$$

and

$$\boxed{q = \frac{D(\alpha+1)}{2} - 1.} \tag{30}$$

We can now conclude: BEC can occur only if $q > 0$, i.e. only if

$$\boxed{D(\alpha+1) > 2.} \tag{31}$$

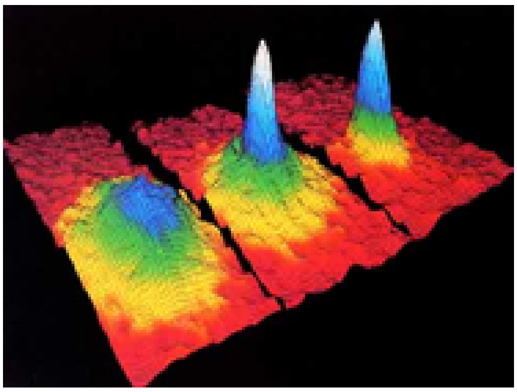
- In dimension $D = 1$, we need $\alpha > 1$, i.e. a trap steeper than a harmonic trap, a linear trap for example. BEC doesn't occur in a box, nor in a harmonic trap.

- In dimension $D = 2$, the condition is $\alpha > 0$. Any trapping potential with a power-law with positive exponent is sufficient to have a finite critical temperature for BEC. However, BEC doesn't occur in a box trap, which corresponds to the critical case $\alpha = 0$.
- Finally, in dimension $D = 3$, the condition writes $\alpha > -1/3$, which is always fulfilled as $\alpha \geq 0$. BEC occurs in any trap, even in a box (uniform density).

Exercise: give the expressions for $N_c(T)$ and $T_c(N)$ for the cases discussed above: harmonic trap of frequency ω_0 in 2D and 3D, and box trap with an atomic density n in 3D.

1.5 Experimental results

The first experimental demonstration of atomic Bose-Einstein condensation has been performed in the group of Eric Cornell and Carl Wieman [7]. They observed BEC of rubidium 87 atoms confined in a three-dimensional harmonic magnetic trap ($q = 2$), by releasing the atoms to access the momentum distribution[2], see Fig. 1. As temperature is lowered, a central peak appears in the distribution, corresponding to the condensate. In this case, condensation happens both in real and momentum space. They verified approximately the expected law for the condensate fraction [8].



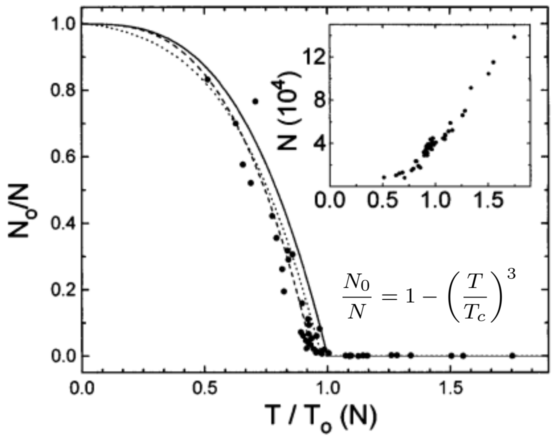


Figure 1: **Left**: First observation of Bose-Einstein condensation in atomic gases. Rubidium atoms initially confined in a harmonic trap are released in a time-of-flight experiment. The gas expands, and the density distribution (integrated along the imaging axis) after expansion is linked to the initial momentum distribution. Figure from Cornell's group. **Right**: Condensed fraction as a function of temperature in units of the predicted critical temperature T_0 for a non interacting gas, compared with the prediction for a non interacting gas. BEC occurs at a temperature slightly lower than predicted by the non interacting model. Inset: remaining atoms as a function of T/T_0. Figure adapted from [8].

[2]It was understood soon after that the shape of the central peak is linked to the initial interaction energy rather than to kinetic energy.

It took almost twenty more years to be able to realize a three-dimensional box potential for cold atoms ($q = 1/2$). In this case, condensation occurs in momentum state only. Interactions are kept extremely small, the density being independent of temperature in a box and much smaller than in the previous case. This experiment was first performed in the group of Zoran Hadzibabic in Cambridge [9], see Fig. 2. The absence of interactions makes the agreement with theory easier, and they were able to observe the saturation of the excited states [10].

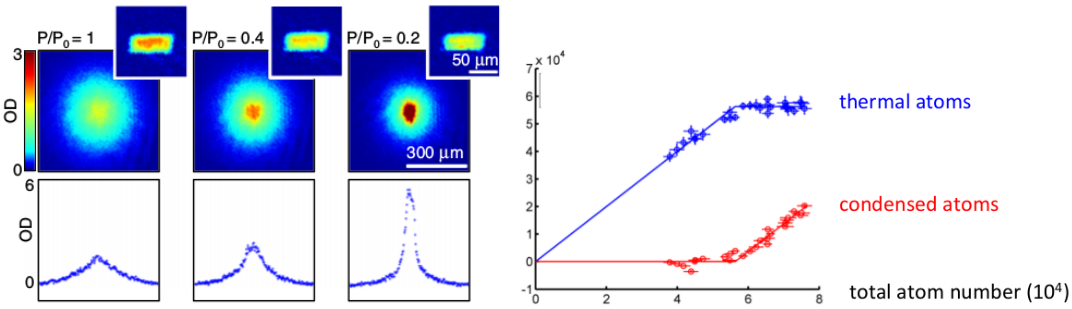


Figure 2: **Left**: Observation of BEC for a 3D Bose gas confined in a box trap. Figure from Ref. [9]. Upper panel: momentum distribution as observed after a 50 ms time-of-flight expansion. Lower panel: cut in the momentum distribution. Inset: in situ distribution before expansion. **Right**: Saturation of the number of atoms in the excited states in this situation. Figure from Ref. [1], adapted from a figure provided by Zoran Hadzibabić.

2 Weakly interacting degenerate Bose gas

From the early results presented in Fig. 1, it appears already that there is a shift in the critical temperature with respect to the prediction for the non interacting case. The expansion of the condensate is also not only due to its very small kinetic energy, but mainly to the effect of the repulsive interactions [11]. Even if the interactions remain weak, they affect qualitatively the shape of the condensate and its dynamics. In this section, we will derive the equation that describes the condensate in its ground state from a mean-field approach, using a variational procedure.

2.1 S-wave scattering in a nutshell

I will not recall the scattering theory here, nor derive explicitly the expression of the effective interaction potential and its link to the scattering length. I refer the interested reader to Ref. [12]. the main idea is that at low energies (below $\sim 1\,\mathrm{K}$), collisions occur only in s-wave and are described by a single parameter, the scattering length a.

In brief, we consider a spherical interaction potential $V_{\mathrm{int}}(|\mathbf{r}_1 - \mathbf{r}_2|)$ depending only on the relative distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$ between the two identical colliding particles of mass M. We assume that $V_{\mathrm{int}}(r)$ is short-range, with an attractive part faster than $1/r^3$ and a sharp potential barrier near $r = 0$ due to the electronic clouds of the two atoms. We introduce the center-of-mass frame:

$$\hat{\mathbf{P}} = \hat{\mathbf{p}}_1 + \hat{\mathbf{p}}_2 \qquad \text{and} \qquad \hat{\mathbf{p}} = \frac{1}{2}(\hat{\mathbf{p}}_1 - \hat{\mathbf{p}}_2), \tag{32}$$

$$\hat{\mathbf{R}} = \frac{1}{2}(\hat{\mathbf{r}}_1 + \hat{\mathbf{r}}_2) \qquad \text{and} \qquad \hat{\mathbf{r}} = \hat{\mathbf{r}}_1 - \hat{\mathbf{r}}_2. \tag{33}$$

The Hamiltonian $\hat{H} = \frac{\hat{p}_1^2}{2M} + \frac{\hat{p}_2^2}{2M} + V_{\rm int}(|\hat{\mathbf{r}}_1 - \hat{\mathbf{r}}_2|)$ writes with these new coordinates:

$$H = \frac{1}{4M}\left((\hat{\mathbf{p}}_1 + \hat{\mathbf{p}}_2)^2 + (\hat{\mathbf{p}}_1 - \hat{\mathbf{p}}_2)^2\right) + V(\hat{\mathbf{r}}) = \frac{\hat{\mathbf{P}}^2}{4M} + \frac{\hat{\mathbf{p}}^2}{M} + V_{\rm int}(\hat{\mathbf{r}}). \tag{34}$$

$\hat{\mathbf{P}}$ and $\hat{\mathbf{R}}$ describe the motion of the center of mass (mass $2M$) and $\hat{\mathbf{p}}$ and $\hat{\mathbf{r}}$ describe the relative motion, equivalent to the motion of a particle of mass $M/2$. The two motions are independent and we concentrate on the relative motion to describe the collision. The problem corresponds to the scattering of a fictitious particle of mass $M/2$ on the potential $V_{\rm int}(r)$.

It is solved by looking at the long-distance steady state of the wave function, with $r \to \infty$, under the form:

$$\psi_{\mathbf{k}}(\mathbf{r}) \sim e^{i\mathbf{k}.\mathbf{r}} + f(k, \mathbf{n}, \mathbf{n}')\frac{e^{ikr}}{r}, \tag{35}$$

with $\mathbf{k} = k\mathbf{n}$ corresponding to the incoming wave of energy $\hbar^2k^2/M$, and $\mathbf{r} = r\mathbf{n}'$ to the outgoing wave. $V_{\rm int}(r)$ being isotropic, the scattering amplitude $f(k, \theta)$ only depends on k and on the angle $\theta = (\mathbf{n}, \mathbf{n}')$ between the incoming and the outgoing waves. The scattering amplitude is linked to the interaction potential through

$$f(k, \mathbf{n}, \mathbf{n}') = -\frac{M}{4\pi\hbar^2}\int d\mathbf{r}'\, e^{-ik\mathbf{n}'\cdot\mathbf{r}'} V_{\rm int}(r')\psi_{\mathbf{k}}(\mathbf{r}'). \tag{36}$$

The wave function can be written as a product of a function of r and a function of θ, and the angular part can be decomposed on the spherical harmonics. For partial waves with a non zero angular momentum $\ell > 0$, the radial potential is modified by a centrifugal term $\hbar^2\ell(\ell+1)/Mr^2$, which constitutes a potential barrier and prevent atoms with a very small energy to reach the region where $V_{\rm int}(r)$ is non zero. As a result, at low energy only partial waves with $\ell = 0$, also called s-waves by analogy with the atomic structure, can contribute to the scattering amplitude. As s-waves are isotropic, the scattering amplitude is independent of θ. The *scattering length* is given by the limit at low energy of $f(k)$:

$$a = -\lim_{k\to 0} f(k). \tag{37}$$

The scattering properties at low energy are entirely contained in this single parameter. We can then replace the true interaction potential $V_{\rm int}(\mathbf{r})$ by an effective potential that will give the same scattering length a. The simplest choice[3] is

$$\boxed{V_{\rm eff}(\mathbf{r}) = g\delta(\mathbf{r}),} \tag{38}$$

where we need to choose

$$\boxed{g = \frac{4\pi\hbar^2 a}{M}} \tag{39}$$

to get the correct scattering amplitude (to lowest order, $\psi_{\mathbf{k}}(\mathbf{r}') \sim e^{i\mathbf{k}.\mathbf{r}'}$ in Eq. (36)).

[3]In reality, the potential should be regularized in zero and take a more sophisticated form, see Ref. [13].

2.2 Gross-Pitaevskii equation

To describe the effect of weak interactions on the condensate, we will take a mean-field approach. We will assume that the interactions are weak enough to neglect the correlations between particles in the many-body wave function $|\Psi\rangle$, and write it as a product state of N identical single particle states $|\phi\rangle$, where N is the number of atoms in the condensate:

$$|\Psi\rangle = |\phi\rangle_1 \otimes |\phi\rangle_2 \otimes |\phi\rangle_3 \otimes \cdots \otimes |\phi\rangle_N. \tag{40}$$

However, we now allow that this single-particle state $|\phi\rangle$ differs from the ground state of the single-particle Hamiltonian $h^{(1)} = p^2/2M + V(\mathbf{r})$, that we take with the kinetic term and an optional potential term. We then apply a variational method and look for the single particle wave function $|\phi\rangle$ which would minimize the energy $E[\phi]$ for the state $|\Psi\rangle$, under the constraint $\langle\phi|\phi\rangle = 1$.

Using the effective potential introduced in the previous section, the many-body Hamiltonian for N interacting atoms writes

$$H = \sum_{i=1}^{N} h_i^{(1)} + \frac{g}{2}\sum_{i\neq j} \delta(\mathbf{r}_i - \mathbf{r}_j) \tag{41}$$

where $h_i^{(1)} = \mathbb{1}\otimes\mathbb{1}\otimes\cdots\otimes h^{(1)}\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1}$ with $h^{(1)} = p^2/2M + V(\mathbf{r})$ at the i^{th} position is the single-particle Hamiltonian acting on atom i. The second term, with $N(N-1)$ terms in the sum, describes the contact interactions.

The energy $E[\phi]$ writes, given the state $|\psi\rangle$ of wave function $\phi(\mathbf{r}) = \langle\mathbf{r}|\phi\rangle$:

$$\begin{aligned} E[\phi] &= N\int d\mathbf{r}\left\{\phi^*(\mathbf{r})\left[-\frac{\hbar^2}{2M}\nabla^2\phi\right] + \phi^*(\mathbf{r})V(\mathbf{r})\phi(\mathbf{r})\right\} \\ &\quad +\frac{g}{2}N(N-1)\int d\mathbf{r}\,d\mathbf{r}'\,\phi^*(\mathbf{r}')\phi^*(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')\phi(\mathbf{r})\phi(\mathbf{r}') \\ &\simeq N\int d\mathbf{r}\left\{\phi^*(\mathbf{r})\left[-\frac{\hbar^2}{2M}\nabla^2\phi\right] + \phi^*(\mathbf{r})V(\mathbf{r})\phi(\mathbf{r})\right\} + g\frac{N^2}{2}\int d\mathbf{r}\,\phi^*(\mathbf{r})^2\phi(\mathbf{r})^2. \end{aligned} \tag{42}$$

In the last line, we have replaced $N(N-1)$ by N^2, which is valid if the condensate contains many atoms. Instead of minimizing $E[\phi]$ with the constraint

$$\int d\mathbf{r}\phi^*(\mathbf{r})\phi(\mathbf{r}) = 1, \tag{43}$$

we use the Lagrange multiplier approach and minimize

$$F(\phi,\phi^*) = E[\phi] - \lambda N\int d\mathbf{r}\phi^*(\mathbf{r})\phi(\mathbf{r}), \tag{44}$$

where ϕ and ϕ^* act as independent variables. The change in F when ϕ^* is modified by an infinitesimal amount $\delta\phi^*$ writes

$$\frac{\delta F}{N} = \int d\mathbf{r}\delta\phi^*(\mathbf{r})\left[-\frac{\hbar^2}{2M}\nabla^2\phi + V(\mathbf{r})\phi(\mathbf{r}) - \lambda\phi(\mathbf{r}) + gN\phi^*(\mathbf{r})\phi(\mathbf{r})^2\right]. \tag{45}$$

F is minimum if ϕ obeys the following equation:

$$-\frac{\hbar^2}{2M}\nabla^2\phi + V(\mathbf{r})\phi(\mathbf{r}) + gN|\phi(\mathbf{r})|^2\phi(\mathbf{r}) = \lambda\phi(\mathbf{r}). \tag{46}$$

There is a simple interpretation of the Lagrange multiplier λ. The energy of the ground state with N particles writes

$$E(N) = NE_1 + \frac{N(N-1)}{2}E_2 \tag{47}$$

with E_1 the single-particle energy corresponding to $|\phi\rangle$ and E_2 the two-particle energy for two atoms in state $|\phi\rangle$. The energy needed to add a particle in the condensate is given by

$$E(N+1) - E(N) = E_1 + N\frac{N+1-(N-1)}{2}E_2 = E_1 + NE_2 = \lambda. \tag{48}$$

The last equality comes directly from the integration of Eq. (46) after multiplication by $\psi^*(\mathbf{r})$. In other words, λ is the *chemical potential* of the gas.

Introducing the wave function normalized to N atoms $\psi = \sqrt{N}\phi$ and using μ as a notation for the *chemical potential*, we arrive at the usual form of the time-independent *Gross-Pitaevskii equation*:

$$\boxed{-\frac{\hbar^2}{2M}\nabla^2\psi + V(\mathbf{r})\psi(\mathbf{r}) + g|\psi(\mathbf{r})|^2\psi(\mathbf{r}) = \mu\psi(\mathbf{r}).} \tag{49}$$

Under this form, the square modulus of the wave function is simply the atomic density of the condensate:

$$\boxed{n(\mathbf{r}) = |\psi(\mathbf{r})|^2.} \tag{50}$$

2.3 Thomas-Fermi limit

It is easy to check using a Gaussian ansatz for ψ that the condensate would be unstable with attractive interactions, i.e. $g < 0$, unless the atom number is below some maximum atom number [3]. This maximum atom number begin quite small (typically of the order of 100 atoms), we will only consider the case of repulsive interactions in this lecture, i.e. $g > 0$.

Let us estimate the three contributions to the energy per particle in the Gross-Pitaevskii equation (GPE), assuming that the condensate with N atoms has a size of order R. We get (in 3D):

$$E_k \simeq \frac{\hbar^2}{2MR^2} \tag{51}$$

$$E_{\text{int}} \simeq \frac{gN}{R^3} = 8\pi\frac{\hbar^2}{2M}\frac{Na}{R^3} \tag{52}$$

$$E_{\text{pot}} \simeq V(R) = \frac{1}{2}M\omega_0^2R^2 = \frac{\hbar^2}{2M}\frac{R^2}{a_0^4} \tag{53}$$

if we assume a harmonic potential of frequency ω_0 and $a_0 = \sqrt{\hbar/M\omega_0}$ is the size of its ground state. These energies can be recast in units of $\hbar\omega_0 = \hbar^2/Ma_0^2$:

$$E_k \simeq \frac{\hbar\omega_0}{2}\frac{a_0^2}{R^2} \tag{54}$$

$$E_{\mathrm{int}} \simeq \frac{\hbar\omega_0}{2} \times 8\pi\frac{Na}{a_0}\frac{a_0^3}{R^3} \tag{55}$$

$$E_{\mathrm{pot}} \simeq \frac{\hbar\omega_0}{2}\frac{R^2}{a_0^2}. \tag{56}$$

If N is sufficiently large, the interaction term will be large, and the size of the condensate R will be significantly larger than the harmonic ground state a_0 due to the repulsive interactions. In this limit, both the interaction term and the potential term (which scales as R^2/a_0^2) will be much larger than the kinetic term, which instead is reduced by a factor a_0^2/R^2 with respect to the non interacting situation. For a harmonic trap, the potential energy will be much larger than $\hbar\omega_0$, indicating that interactions induce a population of the excited states of the harmonic oscillator.

The Thomas-Fermi approximation consists in neglecting the kinetic term in the GPE (49), which becomes a simple equation for ψ (no derivative). This justified if $8\pi Na \gg R$, see Eqs. (51) and (52). We can simplify by ψ, which leads to the solution for the density:

$$\boxed{n_{\mathrm{TF}}(\mathbf{r}) = \frac{1}{g}\left[\mu - V(\mathbf{r})\right], \qquad \text{for } V(\mathbf{r}) \leq \mu.} \tag{57}$$

The condensate thus takes a shape that is the opposite of the potential —a parabola with a maximum in the trap center if the trap is harmonic, see Fig. 3. The condition $V(\mathbf{r}) \leq \mu$ sets the limits of the condensate, where the density vanish. For an isotropic trap for instance, the radius is given by $V(R) = \mu$. Around this region, of course, the term $gn(\mathbf{r}) \simeq \mu - V(\mathbf{r})$ in the GPE becomes very small, and the kinetic term must play a role.

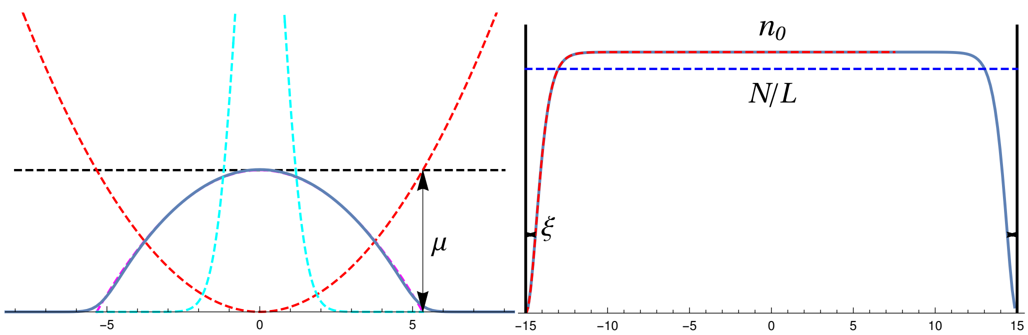


Figure 3: **Left**: Solution of Gross-Pitaevskii equation for a condensante in a harmonic trap (full blue line), compared to the Thomas-Fermi profile (dashed magenta, very well superposed except on the edges) and the non interacting solution (dashed cyan). Also shown are the trapping potential (dashed red) and the chemical potential (dashed black). **Right**: Same thing in a box potential. The density is uniform, except in a small region of size ξ near the edge.

The size of this region can be estimated easily in the case of a cubic box trap. The size ξ over which the density drops is found by equating the kinetic energy due to the wave function bending, $\hbar^2/2M\xi^2$, and the interaction energy $gn_0 = \mu$. We find

$$\boxed{\xi = \frac{\hbar}{\sqrt{2M\mu}},} \tag{58}$$

which is called the *healing length*. Near one of the edges, for example in $(x, L/2, L/2)$ with $x \ll L$ if the box is defined by $x, y, z \in (0, L)$, the wave function is given by

$$\psi(x, 0, 0) \simeq \sqrt{n_0} \tanh\left(\frac{x}{\xi\sqrt{2}}\right). \tag{59}$$

Its square (the density) is represented in Fig. 3, right. The central density is a little higher due to this depletion at the edges, on the order of $n_0 \simeq N/(L - 2\xi)^3$.

Exercise: using the Thomas-Fermi approximation, give the expression for the chemical potential μ in the case of atoms confined in a harmonic trap in dimension 2 and 3, with trapping frequencies ω_x and ω_y (and ω_z). Deduce the Thomas-Fermi radius R_i in each direction, i.e. the radius at which the density vanishes.

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3 Time-dependent GPE and hydrodynamic equation

3.1 Time-dependent GPE

The time-dependent version of GPE is very similar to its time-independent counterpart Eq. (49) [3,5]:

$$\boxed{i\hbar\partial_t\psi = -\frac{\hbar^2}{2M}\nabla^2\psi + V(\mathbf{r})\psi(\mathbf{r}) + g|\psi(\mathbf{r})|^2\psi(\mathbf{r}).} \tag{60}$$

If a wave function ψ_0 satisfies the time-independent GPE, then $\psi(\mathbf{r}, t) = \psi_0(\mathbf{r})\exp(-i\mu t/\hbar)$ is a solution of the time-dependent GPE, Eq. (60). It corresponds to the ground state of the condensate at rest. The time-dependent GPE will also us instead to explore the excitations and the dynamics of the condensate.

3.2 Hydrodynamic formulation

We can give an equivalent of Eq. (60), which implies a single complex classical field, in terms of coupled hydrodynamic equations on two real classical fields, the density and the velocity field.

Let use write the wave function in terms of density $n = |\psi|^2$ and phase:

$$\psi(\mathbf{r}, t) = \sqrt{n(\mathbf{r}, t)}e^{i\theta(\mathbf{r},t)}. \tag{61}$$

We will inject this expression into (60), multiply by ψ^* and get two equations for the imaginary part and the real part.

We start from the easy terms that do not involve derivatives. We get a real term:

$$\psi^*(\mathbf{r})\left[V(\mathbf{r})\psi(\mathbf{r}) + g|\psi(\mathbf{r})|^2\psi(\mathbf{r})\right] = n(\mathbf{r})\left[V(\mathbf{r}) + gn(\mathbf{r})\right]. \tag{62}$$

Then we examine the Laplacian term:

$$\begin{aligned}
\psi^*\nabla^2\psi &= \sqrt{n}e^{-i\theta}\nabla\cdot\left[\nabla\left(\sqrt{n}e^{i\theta}\right)\right] \\
&= \sqrt{n}e^{-i\theta}\nabla\cdot\left[\left(\nabla\sqrt{n}\right)e^{i\theta}+i\sqrt{n}e^{i\theta}\nabla\theta\right] \\
&= \sqrt{n}e^{-i\theta}\left[\left(\nabla^2\sqrt{n}\right)e^{i\theta}+2ie^{i\theta}\nabla\left(\sqrt{n}\right)\cdot\nabla\theta+i\sqrt{n}e^{i\theta}\nabla\cdot\nabla\theta-\sqrt{n}e^{i\theta}\nabla\theta^2\right] \\
&= n\left[\frac{\nabla^2\sqrt{n}}{\sqrt{n}}-\nabla\theta^2\right]+i\left(\nabla n\cdot\nabla\theta+n\nabla\cdot\nabla\theta\right). \qquad (63)
\end{aligned}$$

By analogy with the probability current in quantum mechanics $\mathbf{J}=i\hbar(\psi\nabla\psi^*-\psi^*\nabla\psi)/2M$, we introduce the fluid velocity

$$\boxed{\mathbf{v}=\frac{\hbar}{M}\nabla\theta.} \qquad (64)$$

Using this notation, the Laplacian writes finally

$$-\frac{\hbar^2}{2M}\psi^*\nabla^2\psi=n\left[-\frac{\hbar^2}{2M}\frac{\nabla^2\sqrt{n}}{\sqrt{n}}+\frac{1}{2}Mv^2\right]-i\frac{\hbar}{2}\left(\nabla n\cdot\mathbf{v}+n\nabla\cdot\mathbf{v}\right). \qquad (65)$$

The time derivative of the left-hand-side writes

$$\begin{aligned}
i\hbar\psi^*\partial_t\psi &= i\hbar\sqrt{n}e^{-i\theta}\left[\frac{1}{2\sqrt{n}}\partial_t ne^{i\theta}+i\partial_t\theta\sqrt{n}e^{i\theta}\right] \\
&= i\frac{\hbar}{2}\partial_t n-n\hbar\partial_t\theta. \qquad (66)
\end{aligned}$$

Identifying the imaginary term, we get the *continuity equation*:

$$\boxed{\partial_t n+\nabla(n\mathbf{v})=0.} \qquad (67)$$

This equation describes the conservation of the flow.

Simplifying the real parts by n and taking the gradient, we arrive at a *Euler-type equation*:

$$\boxed{M\partial_t\mathbf{v}=-\nabla\left[-\frac{\hbar^2}{2M}\frac{\nabla^2\sqrt{n}}{\sqrt{n}}+\frac{1}{2}Mv^2+V+gn\right].} \qquad (68)$$

Eqs. (67) and (68) together are equivalent to Eq. (60). Eq. (68) can also be recast into

$$M\frac{D\mathbf{v}}{dt}=-\nabla\left[-\frac{\hbar^2}{2M}\frac{\nabla^2\sqrt{n}}{\sqrt{n}}+V+gn\right] \qquad (69)$$

where

$$\frac{D\mathbf{v}}{dt}=\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v} \qquad (70)$$

is the particle derivative in the flow. The right-hand side has a potential energy, the interaction term gn that plays the role of the pressure and a quantum pressure term involving $\hbar$ which has no classical counterpart. This terms plays a role on smaller scales (the healing length ξ typically) and is responsible for the edges of the wave function in a trap.

Neglecting this term, which is less restrictive than the Thomas-Fermi approximation that would also neglect the v^2 term, leads to a description of the condensate dynamics at larger scales. This is the *hydrodynamic approximation*, useful for example to derive the collective modes of a trapped Bose gas.

3.3 Wave-like excitations

The time-dependent GPE, or the hydrodynamics equations, gives us access to the elementary excitations on the condensate. Let us first consider the easy case of a homogeneous gas. At equilibrium, the density is $n_0 = \mu/g$, independent of position, and the velocity field is zero. We look for small amplitude excitations. We write the density as $n(\mathbf{r}, t) = n_0 = \delta n(\mathbf{r}, t)$ and will keep only first order terms in δn and $\mathbf{v}$.

Under these assumptions, the continuity equation at first order writes

$$\partial_t \delta n + n_0 \nabla \cdot \mathbf{v} = 0. \tag{71}$$

Using the expansion $\sqrt{n} = \sqrt{n_0} + \delta n / 2\sqrt{n_0}$, we get for the Euler equation at first order:

$$M \partial_t \mathbf{v} = -\nabla \left[-\frac{\hbar^2}{2M} \frac{\nabla^2 \delta n}{2n_0} + g\delta n \right]. \tag{72}$$

Taking the time derivative of Eq. (71), we get

$$\partial_t^2 n = \frac{n_0}{M} \nabla^2 \left[-\frac{\hbar^2}{2M} \frac{\nabla^2 \delta n}{2n_0} + g\delta n \right] = \nabla^2 \left[-\frac{\hbar^2}{4M^2} \nabla^2 \delta n + \frac{g n_0}{M} \delta n \right]. \tag{73}$$

We look for solutions of the form $A\cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \varphi)$ for both the density δn and the velocity, which leads to the following dispersion relation for the small amplitude excitations:

$$\boxed{\omega(k) = \sqrt{\frac{\hbar^2 k^4}{4M^2} + \frac{g n_0 k^2}{M}} = \sqrt{\frac{\hbar^2 k^4}{4M^2} + c^2 k^2},} \tag{74}$$

where we have introduced the quantity

$$\boxed{c = \sqrt{\frac{g n_0}{M}} = \sqrt{\frac{\mu}{M}}} \tag{75}$$

which has the dimension of a velocity.

The relation $\omega(k)$ in Eq. (74) is known as the *Bogolubov spectrum*. It is represented in Fig. 4. It behaves differently at low or high momenta:

- In the low momentum limit, the term in k^2 dominates and the frequency writes approximately $\omega(k) = ck$. The dispersion relation is thus linear. This corresponds to *sound waves*, with a speed of sound given by c. This is very different from the ideal gas, which has a dispersion relation in k^2.

- In the high momentum limit, the term in k^4 is larger. We can make an expansion to first order in $c^2 k^2$, and get

  $$\omega(k) = \frac{\hbar^2 k^2}{2M} \sqrt{1 + \frac{4M}{\hbar^2 k^2} \mu} \simeq \frac{\hbar^2 k^2}{2M} + \mu. \tag{76}$$

  The spectrum corresponds to free particles with an energy shofted by μ due to the interactions with the majority atoms in the condensate.

The boundary between these two regimes occurs for

$$\frac{\hbar^2 k^2}{2M} = \mu \qquad \text{or} \qquad k = \frac{\sqrt{2M\mu}}{\hbar} = \xi^{-1}. \tag{77}$$

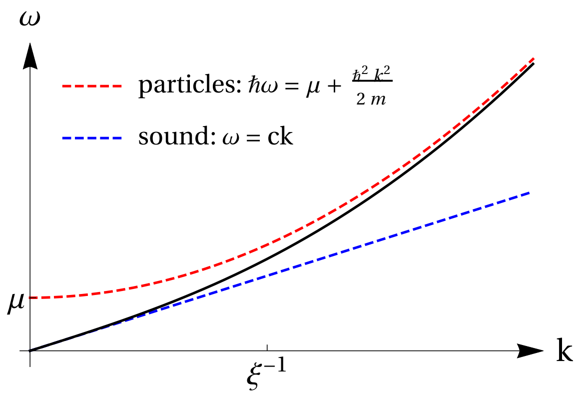

Figure 4: Bogolubov spectrum in a homogeneous gas.

3.4 A first criterion for superfluidity: existence of a critical velocity

The linear behavior of the dispersion relation has an important consequence. Indeed, let us consider a condensate at rest, in which we launch an very small object of mass m at speed $\mathbf{v}$. At which condition will the motion of this object be damped through the creation of some excitation in the fluid? Note that the situation is equivalent to the one of the fluid moving at $-\mathbf{v}$, if we look in the frame moving at $-\mathbf{v}$, such that it also describes the possible damping of the flow by excitation of the fluid due to small imperfections in a pipeline.

The problem is in variant in a translation (at least along $\mathbf{v}$, and energy and total momentum are conserved. Let use write energy and momentum conservation:

	Before excitation	After excitation
Energy	$0 + \frac{1}{2}mv^2$	$c\hbar k + \frac{1}{2}mv'^2$
Momentum	$\mathbf{0} + m\mathbf{v}$	$\hbar\mathbf{k} + m\mathbf{v}'$

Equating the total momentum before and after excitation gives the new velocity $\mathbf{v}' = \mathbf{v} - \hbar\mathbf{k}/m$. Reporting this expression into the condition for energy conservation, we get

$$\frac{1}{2}mv^2 = c\hbar k + \frac{1}{2}mv^2 - \hbar\mathbf{k}\cdot\mathbf{v} + \frac{\hbar^2k^2}{2m} \Leftrightarrow \mathbf{v}\cdot\mathbf{k} = ck + \frac{\hbar^2k^2}{2m}. \tag{78}$$

As $\hbar^2k^2/2m$ is always positive, the last equality implies $\mathbf{v}\cdot\mathbf{k} \geq ck$ and thus $v \geq c$. The speed of sound c appears to be a *critical velocity* for the creation of excitations. If the object has a velocity smaller than c, its motion is not damped —or if the fluid flows at a speed smaller than c, it is not damped. This corresponds to the *Landau criterion for superfluidity*, a first hint of superfluidity for quantum gases. As the speed of sound is proportional to $\sqrt{g}$, it vanishes in the absence of interactions. Interactions are required for the quantum gas to be a superfluid.

The existence of a critical velocity in quantum gases has been demonstrated by the group of Wolfgang Ketterle in 1999. A nice illustration is given in Fig. 5, from an experiment performed in the group of Jean Dalibard in Paris. The measured critical velocity v_c

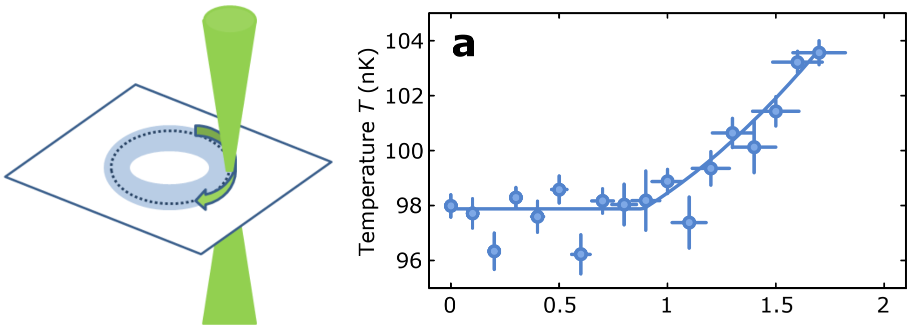

Figure 5: Demonstration of a critical velocity in a 2D gas. **Left**: A focused laser is stirred around the gas, at some fixed distance from the center, and at a fixed linear speed v. **Right**: The energy transferred to the gas is measured in a time-of-flight experiment. At low v, no energy is transferred. Above a critical value v_c, the energy increases quadratically with $v - v_c$, due to the creation of excited particle out of the BEC. Figure from Ref. [14].

is below c as the focused laser beam used to create the excitation also depletes the density locally, resulting in a smaller local speed of sound.

3.5 Collective modes in a trap

We consider now the case where the gas is confined in a trap $V(\mathbf{r})$, and we look again for the elementary excitations away from the density at equilibrium given by $n_0(\mathbf{r}) \simeq [\mu - V(\mathbf{r})]/g$, and write $n(\mathbf{r},t) = n_0(\mathbf{r}) + \delta n(\mathbf{r},t)$. In the trap, with a large enough atom number, we can neglect the quantum pressure and use the hydrodynamic approximation, such that the hydrodynamic equations read

$$\partial_t n + \nabla(n\mathbf{v}) = 0, \tag{79}$$

$$M\partial_t \mathbf{v} = -\nabla\left[\frac{1}{2}Mv^2 + V(\mathbf{r}) + gn\right]. \tag{80}$$

We keep only the first order terms in δn and $\mathbf{v}$:

$$\partial_t \delta n + \nabla(n_0\mathbf{v}) = 0, \tag{81}$$

$$M\partial_t \mathbf{v} = -\nabla\left[V(\mathbf{r}) + gn_0 + g\delta n\right]. \tag{82}$$

We use $V(\mathbf{r}) + gn_0 \simeq \mu$ to neglect its gradient in Eq. (82). We take again the time derivative of the continuity equation inject it in Euler's equation. We get [15]

$$\partial_t^2 \delta n = \nabla \cdot \left[\frac{gn_0(\mathbf{r})}{M}\nabla\delta n\right] = \nabla\cdot\left[c^2(\mathbf{r})\nabla\delta n\right]. \tag{83}$$

We have introduced the local speed of sound $c(\mathbf{r}) = \sqrt{gn_0(\mathbf{r})/M}$.

In the case of an isotropic 3D harmonic trap of frequency ω_0, the rotational invariance allows us to give the frequencies of the collective modes as a function of three quantum numbers (n_r, ℓ, m) for the number radial nodes, total angular momentum ℓ and its projection $m = -\ell \ldots \ell$ along z. The frequency depends only on n_r and ℓ and reads [15]

$$\omega_{n_r,\ell} = \omega_0\sqrt{2n_r^2 + 2n_r\ell + 3n_r + \ell}. \tag{84}$$

These frequencies get a small correction if we take into account the terms beyond the Thomas-Fermi approximation [16].

In two dimensions, the relevant quantum numbers are n_r and m, and we get almost the same formula (notice however the factor 2 instead of 3) [17]

$$\omega_{n_r,m} = \omega_0 \sqrt{2n_r^2 + 2n_r|m| + 2n_r + |m|}. \tag{85}$$

Let us discuss a few important modes:

- The lowest frequency modes are the center-of-mass or dipole modes, at the trap frequencies.
- The quadrupole modes, corresponding to a quadrupole deformation of the trap with $n_r = 0$ and $m = \pm 2$ ($\ell = 2$ in the 3D case), oscillate at frequency $\sqrt{2}\omega_0$. They are specific of a superfluid.
- The first mode with $n_r = 1$ (and $m = 0$ of $\ell = 0$) is the monopole or breathing mode. Its frequency is $\sqrt{5}\omega_0$ in 3D and $2\omega_0$ in 2D. In this latter case, it is the same as the monopole frequency of a thermal gas. We will see in Lecture 2 that this mode is not damped in 2D, due to the underlying scaling symmetry [18, 19].

In anisotropic traps, another mode can also exist: an oscillation along a trap axis, called the scissors mode [20]. This mode is also a signature of superfluidity and can be used to probe the superfluid state [21, 22].

3.6 Another kind of excitation: vortices

The velocity of the fluid, given by Eq. (64), is proportional to the gradient of the phase, which is defined everywhere except at the points where ψ vanishes. Outside these singularities, $\mathbf{v}$ is well-defined, and its rotational is equal to zero:

$$\nabla \times \mathbf{v} = 0. \tag{86}$$

As a consequence of the expression of $\mathbf{v}$, we will see that the circulation $\Gamma_{\mathcal{C}}$ of the velocity along a close contour $\mathcal{C}$ that does not cross a singularity is quantized. Consider first the 2D case (see Fig. 6, left):

$$\Gamma_{\mathcal{C}} = \oint_{\mathcal{C}} \mathbf{v} \cdot \mathbf{d}\ell = \frac{\hbar}{M} \oint_{\mathcal{C}} \nabla\theta \cdot \mathbf{d}\ell = \frac{\hbar}{M}\Delta\theta, \tag{87}$$

where $\Delta\theta$ is the phase difference of the wave function ψ after a close loop. ψ is singly-valued in a given point while its phase θ is defined modulo 2π, which means that $\Delta\theta$ should be a multiple of 2π, say $q2\pi$ with $q \in \mathbb{Z}$. We find finally

$$\Gamma_{\mathcal{C}} = q\frac{h}{M}, \qquad q \in \mathbb{Z}. \tag{88}$$

The circulation of the velocity is quantized in units of h/M, the *quantum of circulation.*

A singularity around which the circulation has a value qh/M, such as the point O in Fig. 6, is called a *vortex of charge q*. The density vanishes on this point and recovers

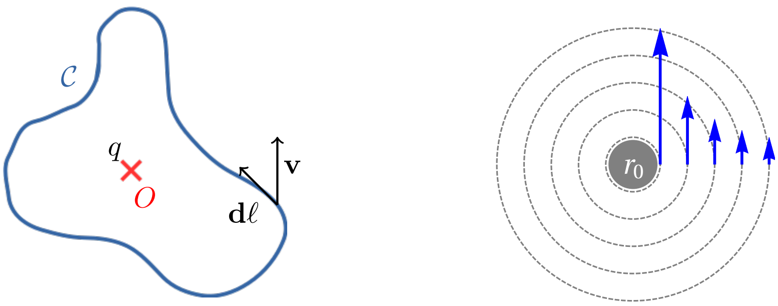

Figure 6: **Left**: Circulation of the velocity field around a contour $\mathcal{C}$. The integration is done on a path where the velocity is well-defined, while there may be points inside the contour, here point O, where the density vanishes and the velocity is not defined, which may be a vortex of charge q. The circulation is an integer number of h/M, qh/M in this latter case. **Right**: Typical velocity field around a vortex. The central region of radius $r_0 \sim \xi$ is depleted and the density vanishes in the center of the core.

its background value on a characteristic size R_v, the vortex radius. Around a vortex, the velocity field is rotating. If we assume that, at least locally around the vortex, the velocity is tangential with a modulus that only depends to the distance to the vortex core $\mathbf{v}(\mathbf{r}) = v(r)\mathbf{e}_\varphi$ in polar coordinates (r, φ), see Fig. 6, right, we can deduce the value of $v(r)$ for a given circulation. Let us compute the circulation over a circle of radius r:

$$\begin{aligned} \Gamma_{\mathcal{C}} &= \int_0^{2\pi} d\varphi v(r) r = 2\pi r v(r) = q\frac{h}{M} \\ \Rightarrow \quad v(r) &= q\frac{\hbar}{Mr}. \end{aligned} \tag{89}$$

If the fluid is in 3D and not in 2D, singular vortex points are replaced by vortex lines along which the density vanishes and around which the fluid rotates. To estimate the size of the vortex core, we write the wave function ψ as a modulus that depends only on the distance r to the vortex core, and a phase winding $q\varphi$ that leads to the correct velocity given at Eq. (89):

$$\psi = \sqrt{\frac{\mu}{g}}\chi(r)e^{iq\varphi} \tag{90}$$

where we have introduced the density μ/g far from the vortex core, linked to the healing length through $\mu = \hbar^2/2M\xi^2$, see Eq. (58). We write the GPE in cylindrical coordinates[4]

$$\mu\sqrt{\frac{\mu}{g}}\chi e^{iq\varphi} = -\frac{\hbar^2}{2M}\left(\chi'' + \frac{1}{r}\chi' - \frac{q^2}{r^2}\chi\right)\sqrt{\frac{\mu}{g}}e^{iq\varphi} + g\frac{\mu}{g}\sqrt{\frac{\mu}{g}}\chi^3 e^{iq\varphi} \tag{91}$$

where the term in q^2 comes from the second derivative along φ, and χ' and χ'' are the derivatives of $\chi(r)$. We simplify by $\sqrt{\mu/g}e^{iq\varphi}$ and replace μ by $\hbar^2/2M\xi^2$ to get

$$\chi = -\xi^2\left(\chi'' + \frac{1}{r}\chi' - \frac{q^2}{r^2}\chi\right) + \chi^3. \tag{92}$$

[4]The Laplacian of a function $f(r, \varphi, z)$ in cylindrical coordinates writes $\nabla^2 f = \partial_r^2 F + \frac{1}{r}\partial_r f + \frac{1}{r^2}\partial_\varphi^2 f + \partial_z^2 f$.

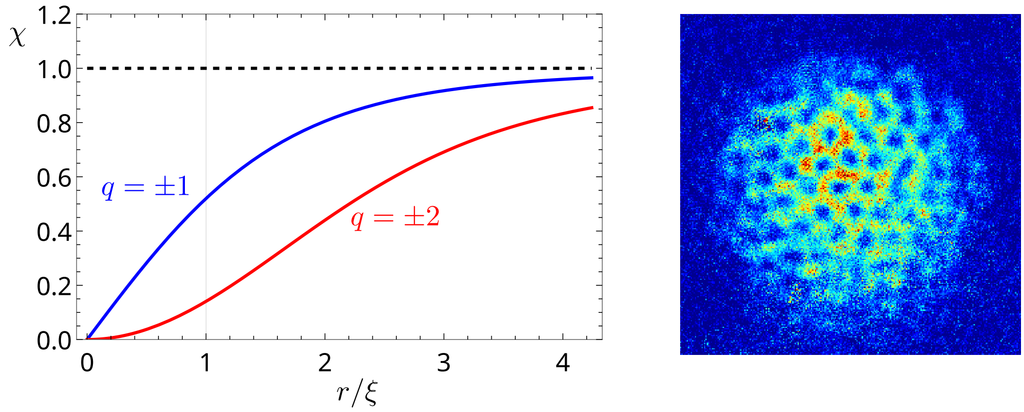

Figure 7: **Left**: Modulus of the wave function $\chi(r)$ of a vortex of charge $q = \pm 1$ (blue line) and of charge $q = \pm 2$ (red line), as a function of the distance from the core in units of ξ. **Right**: Vortex lattice with approximately 50 singly-charged vortices obtained by rotating a quasi two-dimensional gas in a harmonic trap. Figure from LPL, see also Ref. [23].

We see that the natural length appearing to describe the shape of a vortex is ξ, the healing length. The solution is shown in Fig. 7, left, for two values of $|q|$.

Let use finally estimate the energy of a cylindrical condensate of radius R with a vortex of charge q in its center. The density is homogeneous almost everywhere, $n_0 \simeq N/(\pi R^2 L)$, except in the core where t drops to zero. We will model the density profile as a step function, with zero density in a central cylinder of radius ξ and n_0 elsewhere.

The dominant energy is the kinetic energy, which integrated over the cylinder is

$$E_q = n_0 L \int_\xi^R 2\pi r\, dr\, \frac{1}{2} M v(r)^2 = \pi n_0 L \int_\xi^R dr\, \frac{q^2 \hbar^2}{Mr} = q^2 \frac{\pi \hbar^2 n_0 L}{M} \log\left(\frac{R}{\xi}\right) = q^2 E_1 \quad (93)$$

where E_1 is the energy of the condensate with a vortex of charge 1.

We can compare this energy with a system with q independent vortices, which has the same total circulation. Its energy is now qE_1 instead of $q^2 E_1$. We see here that it is energetically more favorable to break a multiply-charged vortex into several singly-charged vortices. A system placed in fast rotation, which tends to accommodate vortices, will indeed result in a vortex lattice with many vortices of charge unity, as can be seen in Fig. 7, right.

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