Course on Two-dimensional Bose gases
Lecture 1: Bose-Einstein condensation, interactions and superfluidity

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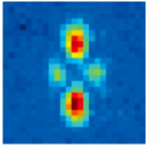
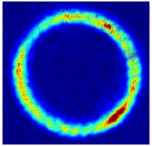
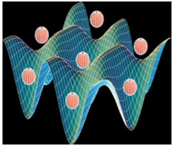
Quantum gases

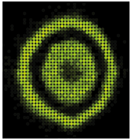
Quantum gases benefit from a high degree of control of their external and internal degrees of freedom:



- control of the temperature 1 nK – 1 μK
- interaction strength: scattering length a
- confinement geometry
- periodic potentials (optical lattices)
- low dimensional systems (1D, 2D)
- several internal states, bosons or fermions
- easy optical detection







$\Rightarrow$ an ideal system for the study of superfluid dynamics in 2D.

Outline of the course

- Lecture 1: Bose-Einstein condensation, interactions and superfluidity
- Lecture 2: Two-dimensional Bose gases (static)
- Lecture 3: Rotating two-dimensional Bose gases

Outline of the lecture

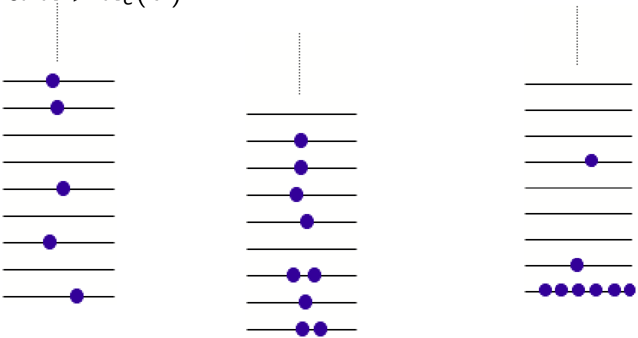
1. BEC in non interacting Bose gases
2. Interacting Bose gases: GPE
3. Hydrodynamics of Bose gases

References for the lecture

1. F. Dalfovo, S. Giorgini, L. Pitaevskii and S. Stringari, *Theory of Bose-Einstein condensation in trapped gases*, Rev. Mod. Phys. **71**, 463 (1999)
2. C. J. Pethick and H. Smith, *Bose–Einstein Condensation in Dilute Gases*, Cambridge (2008)
3. Lev Pitaevskii and Sandro Stringari, *Bose-Einstein condensation*, Oxford (2003)
4. Lev Pitaevskii and Sandro Stringari, *Bose-Einstein Condensation and Superfluidity*, Oxford (2016)
5. Lectures at Collège de France by Jean Dalibard (in French), academic year 2015-2016

Bosons at low temperature: Bose-Einstein condensation

Bose-Einstein condensation: saturation of the number of particles in excited states N' and macroscopic accumulation $N_0 \sim N$ of particles in the ground state for $T < T_c(N)$ or $N > N_c(T)$



$T > T_c$ $\qquad$ $T \sim T_c$ $\qquad$ $T < T_c$

$N_0 \ll N$ $\qquad$ $N_0 \sim N$

A mechanism linked to Bose-Einstein statistics:

$$\boxed{f(E_i) = \frac{1}{e^{\beta(E_i-\mu)} - 1}}$$

with $\beta = 1/k_B T$

A mechanism linked to the Bose statistics

Assume single-particle Hamiltonian with non degenerate ground state, energy E_0.

- Bose-Einstein distribution: occupation number $f(E_i) = \dfrac{1}{e^{\beta(E_i-\mu)} - 1}$, $\beta = 1/k_B T$
- $f > 0 \Rightarrow$ chemical potential $\mu < E_i$ for all $i \Rightarrow \mu < E_0$
  $\Rightarrow f(E_i) < 1/[e^{\beta(E_i - E_0)} - 1]$
- Saturation of the number of particles in excited states N':

$$N' = \sum_{i \neq 0} f(E_i) = \sum_{i \neq 0} \frac{1}{e^{\beta(E_i-\mu)} - 1} < N'_{\max}(T) = \sum_{i \neq 0} \frac{1}{e^{\beta(E_i - E_0)} - 1}$$

- BEC occurs if $N'_{\max}(T)$ is finite: $N_0 = N - N' > N - N'_{\max}(T)$ macroscopic
- Critical atom number $N > N_c(T) \simeq N'_{\max}(T)$ or temperature $T_c(N)$ defined by $N_c(T_c) = N$

Does Bose-Einstein condensation occur?

- Semi-classical description if level spacing $\ll k_B T$: use the density of states $\rho(\varepsilon)$
- The convergence of the sum on N_i' depends on the behavior of $\rho(\varepsilon)$:

$$N_c(T) = \int_{E_0}^{\infty} \rho(\varepsilon) \frac{1}{e^{\beta(\varepsilon - E_0)} - 1} d\varepsilon = \int_0^{\infty} \rho(\varepsilon) \frac{e^{-\beta\varepsilon}}{1 - e^{-\beta\varepsilon}} d\varepsilon$$

- Integral converges near $\varepsilon \to \infty$ if $\rho(\varepsilon)$ doesn't increase exponentially
- Near $\varepsilon \to 0$, integrand $\sim \rho(\varepsilon)/(\beta\varepsilon) \Rightarrow$ converges if $\rho(\varepsilon) \to 0$.

Does Bose-Einstein condensation occur?

- An important particular case: power law density of state $\rho(\varepsilon) \propto (\varepsilon - \varepsilon_0)^q$ with $\varepsilon > \varepsilon_0 \Rightarrow$ converges for $q > 0$.

$$N_c(T) \propto \sum_{n=1}^{\infty} \int_0^{\infty} \varepsilon^q e^{-n\beta\varepsilon} d\varepsilon \quad \propto \quad (k_B T)^{q+1} \sum_{n=1}^{\infty} \frac{1}{n^{q+1}} = (k_B T)^{q+1} g_{q+1}(1)$$

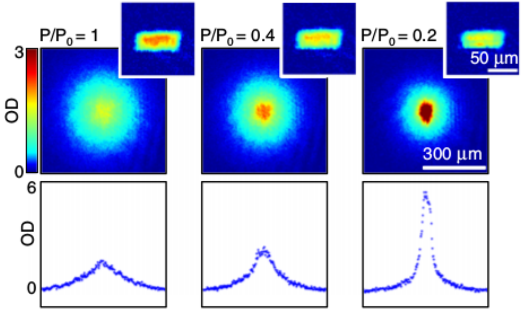
- Fraction of condensed particles:

$$N_c(T_c) = N \quad \Longrightarrow \quad \frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{q+1}$$

Bose-Einstein condensation in a 3D box

DOS in a box in dimension D: $\rho(E) \propto E^{D/2-1} \Rightarrow$ only $D = 3$!

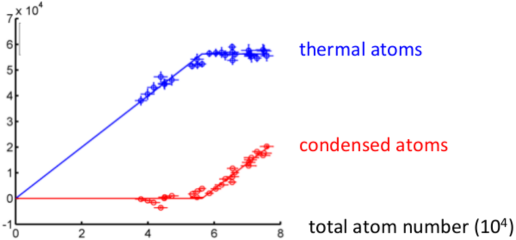
In a 3D box BEC for $n\lambda_T^3 > 2.6$ with $\lambda_T = \frac{h}{\sqrt{2\pi m k_B T}}$: $N_c \propto T^{3/2}$



measure N_0/N, 3D box potential

^{39}K, tunable interactions

PRL **110**, 200406 (2013)

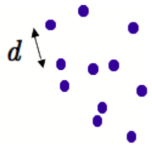


Saturation of thermal particles

Hadzibabic's group, 2013

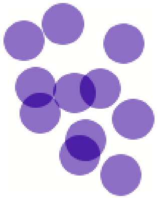
Interpretation of critical temperature

Degeneracy criterion: more than one particle per quantum state.
Size of a state: $\lambda_T = \frac{h}{\sqrt{2\pi m k_B T}}$ thermal de Broglie wavelength



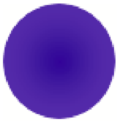
$T > T_c$

$\lambda_T \ll d$



$T \sim T_c$

$\lambda_T \sim d$



$T < T_c$

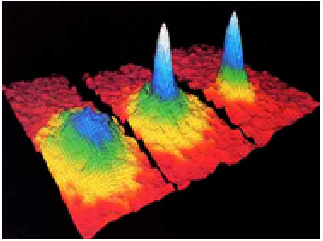
$\lambda_T > d$

a single wavefunction for all particles

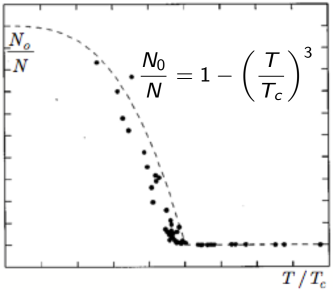
Bose-Einstein condensation in a harmonic trap

- In a harmonic trap ω_0 in dimension D, $\rho(E) \propto E^{D-1}$
  $\Rightarrow$ power law with $k = D - 1$
- BEC occurs for $D > 1$ i.e. $D = 2$ or $D = 3$ [Bagnato 1987]

3D harmonic trap:
$$k_B T_c = \hbar\omega_0 N^{1/3} \gg \hbar\omega_0$$



Cornell & Wieman
Anderson et al., Science **269**, 198 (1995)

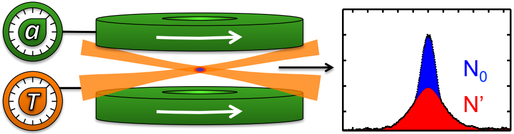


Ensher et al. PRL **77**, 4984 (1996)

Bose-Einstein condensation in a harmonic trap

- Shift in condensation temperature? T_c lower than expected.
- Do the population in excited states saturate?

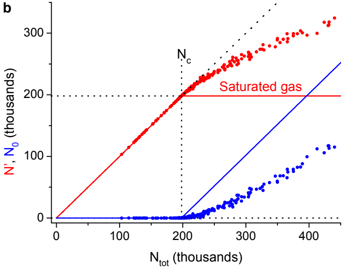
3D harmonic trap:



condensation in momentum AND in space

Tammuz et al., PRL **106**, 230401 (2011)

N' doesn't saturate!

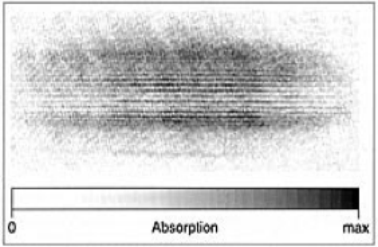


$\Rightarrow$ interactions play a role and prevent the density to further increase in the trap center

Role of interactions

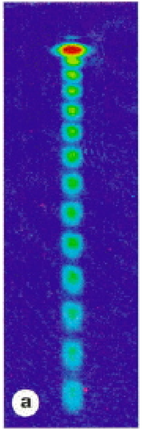
Problem: how much do interactions modify this picture of accumulation in a single-particle state? $H = \sum_{i=1}^{N} h_0^{(i)} + \sum_{i<j} V(r_i - r_j)$

Test: interferences between condensates



Andrews et al., Science **275**, 637 (1997)

- the coherence length is the cloud size
- weak interactions (dilute gas)
- a mean field description is appropriate



Bloch et al., Nature **403**, 166 (2000)

Role of interactions

How to find the ground state of $H = \sum_{i=1}^{N} h_0^{(i)} + \sum_{i<j} V(r_i - r_j)$?

- assume weak interactions (dilute gas)
- mean field description: assume the same wavefunction $\psi(\mathbf{r})$ for all atoms i.e. $|\Psi\rangle = |\psi\rangle_1 \otimes \cdots \otimes |\psi\rangle_N$, $\psi(\mathbf{r})$ to be found
- minimize the energy functional $\langle\Psi|H|\Psi\rangle$ to find ψ
- this yields the Gross-Pitaevskii equation (GPE) for ψ

$$\boxed{\left(\underbrace{-\frac{\hbar^2\nabla^2}{2m}}_{E_{\mathrm{kin}}} + \underbrace{V_{\mathrm{ext}}(\mathbf{r})}_{E_{\mathrm{pot}}} + \underbrace{g\,|\psi|^2}_{E_{\mathrm{int}}}\right)\psi = \mu\psi}$$

$g = \frac{4\pi\hbar^2 a}{m}$ interaction coupling constant, we assume here $g > 0$

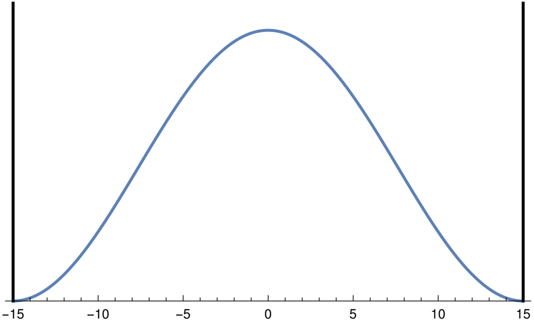
a scattering length

μ chemical potential = cost to add a particle

Solution of GPE in a box

Consider a box potential of size L: $V_{\mathrm{ext}}(\mathbf{r}) = 0$ + hard wall b.c.

- vanishing interactions (Schrödinger): $-\dfrac{\hbar^2\nabla^2\psi}{2m} = \mu\psi$

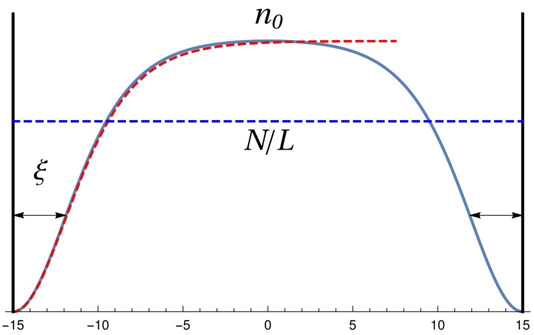


$\psi(x) \sim \sin(\pi x/L)$ ground state of $h^{(0)}$
$\Rightarrow$ density $\propto \sin^2(\pi x/L)$

Solution of GPE in a box

Consider a box potential of size L: $V_{\rm ext}(\mathbf{r}) = 0$ + hard wall b.c.

- increasing interactions: $-\frac{\hbar^2\nabla^2\psi}{2m} + g\,|\psi|^2\psi = \mu\psi$



density flattens

healing length to recover from the edge

Estimation of healing length:

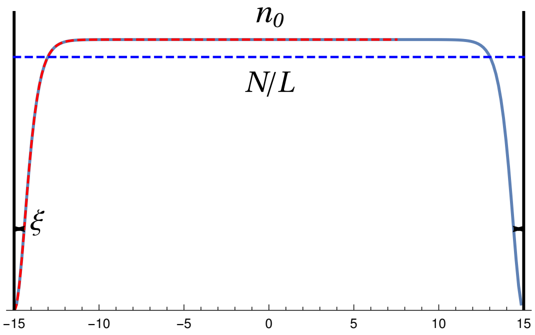
$$\frac{\hbar^2}{2m\xi^2} \simeq \mu$$

$$\Rightarrow \boxed{\xi = \frac{\hbar}{\sqrt{2m\mu}}}$$

Solution of GPE in a box

Consider a box potential of size L: $V_{\rm ext}(\mathbf{r}) = 0$ + hard wall b.c.

- large interactions: $g\,|\psi|^2\psi \simeq \mu\psi \Rightarrow n_0 = \mu/g \simeq N/L$ nearly uniform gas except at the edges within the healing length



Close to the edge:

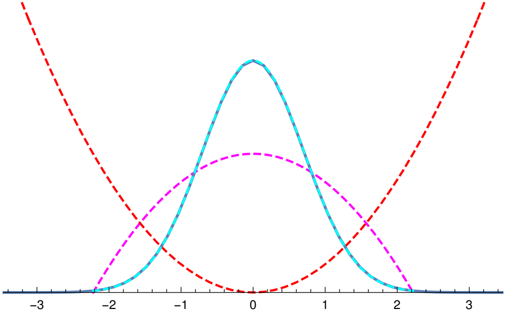
$$\psi(x) \simeq \sqrt{n_0}\tanh\left(\frac{x}{\xi\sqrt{2}}\right)$$

$$\boxed{\xi = \frac{\hbar}{\sqrt{2m\mu}}}$$

Solution of GPE in a harmonic trap

Consider now a harmonic potential $V_{\mathrm{ext}}(\mathbf{r}) = \frac{1}{2}m\omega_0^2 r^2$.

- weak interactions: Gaussian ground state of $h^{(0)}$ with size a_0
  $-\frac{\hbar^2\nabla^2\psi}{2m} + V_{\mathrm{ext}}(\mathbf{r})\psi = \mu\psi$

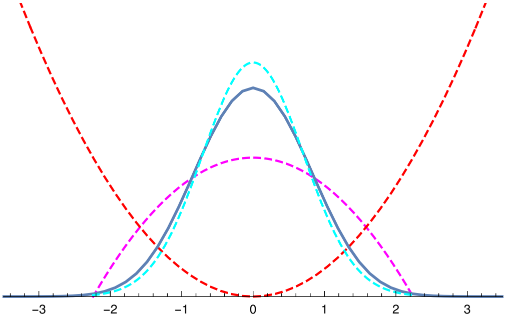


$$a_0 = \sqrt{\frac{\hbar}{m\omega_0}}$$

Solution of GPE in a harmonic trap

Consider now a harmonic potential $V_{\rm ext}(\mathbf{r}) = \frac{1}{2}m\omega_0^2 r^2$.

- increase interactions: deformed Gaussian state with size $> a_0$
  $-\frac{\hbar^2\nabla^2\psi}{2m} + V_{\rm ext}(\mathbf{r})\psi + g\,|\psi|^2\psi = \mu\psi$

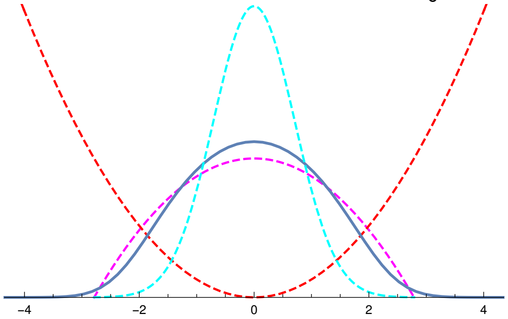


$$a_0 = \sqrt{\frac{\hbar}{m\omega_0}}$$

Solution of GPE in a harmonic trap

Consider now a harmonic potential $V_{\rm ext}(\mathbf{r}) = \frac{1}{2}m\omega_0^2 r^2$.

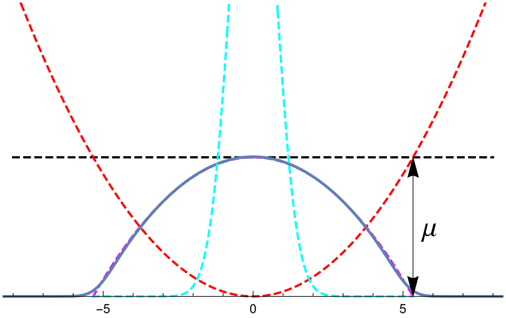
- increase interactions: non-Gaussian state with size $\gg a_0$



Solution of GPE in a harmonic trap

Consider now a harmonic potential $V_{\rm ext}(\mathbf{r}) = \frac{1}{2}m\omega_0^2 r^2$.

- large interactions: Thomas-Fermi profile of radius R_{TF}
  $V_{\rm ext}(\mathbf{r})\psi + g\,|\psi|^2\psi \simeq \mu\psi \Rightarrow n(r) = [\mu - V_{\rm ext}(\mathbf{r})]/g$



Inverted parabola of radius $R_{TF} = \dfrac{1}{\omega}\sqrt{\dfrac{2\mu}{m}}$

Time-dependent GPE

Study of the dynamics: out of equilibrium dynamics away from the ground state.
Described by the time-dependent Gross-Pitaevskii equation

$$\boxed{-i\hbar\partial_t\psi = -\frac{\hbar^2\nabla^2\psi}{2m} + V_{\rm ext}(\mathbf{r})\psi + g\,|\psi|^2\psi}$$

Hydrodynamics

Equivalent formulation of GPE with hydrodynamics equations:
$\psi(\mathbf{r},t)=\sqrt{n(\mathbf{r},t)}\,e^{i\theta(\mathbf{r},t)}$

$$(1) \qquad \partial_t n + \nabla\cdot(n\mathbf{v}) = 0 \qquad \text{continuity equation}$$

$$(2) \qquad m\partial_t \mathbf{v} = -\nabla\left(-\frac{\hbar^2}{2m}\frac{\Delta\left(\sqrt{n}\right)}{\sqrt{n}} + \frac{1}{2}mv^2 + V_{\mathrm{ext}} + g\,n\right)$$

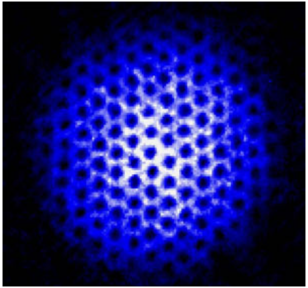
(2) Euler equation

$\mathbf{v}=\frac{\hbar}{m}\nabla\theta$ fluid velocity $\Rightarrow \nabla\times v = 0$ irrotational flow

Hydrodynamics and superfluidty

The hydrodynamics equations describe

- The condensate expansion in a time-of-flight
- The collective modes (breathing mode, quadrupolar mode. . .)
- More generally the excitations: phonons, solitons, free particles, quantized vortices
- The formation of vortices in a rotating fluid



Cornell's group

A signature of superfluidity: a vortex lattice in a rotating condensate.

Homogeneous gas: the Bogolubov spectrum

Consider $V_{\rm ext} = 0$. What is the small amplitude excitation spectrum around equilibrium ($n_0 = \mu/g$)?
Write $n = n_0 + \delta n$ and linearize for δn and $\mathbf{v}$

$$(1) \qquad \partial_t \delta n + n_0 \nabla \cdot (\mathbf{v}) = 0$$

$$(2) \qquad m\partial_t \mathbf{v} = -\nabla \left(-\frac{\hbar^2}{2m} \frac{\Delta(\delta n)}{2n_0} + g\delta n \right)$$

Look for a plane wave solution $\delta n = \delta n_0 e^{ikz - i\omega t}$, same for v:

$$(1) \qquad -m\omega\delta n + mn_0 \mathbf{k} \cdot (\mathbf{v}) = 0$$

$$(2) \qquad -\omega m n_0 \mathbf{v} = -\mathbf{k} \left(\frac{\hbar^2 k^2}{4m} \delta n + g n_0 \delta n \right)$$

$$\Rightarrow \boxed{\omega^2 = \frac{\hbar^2 k^4}{4m^2} + g n_0 \frac{k^2}{m}} \quad \text{with} \quad g n_0 = \mu$$

LPL

Bogolubov spectrum
Sound and particles

$$\boxed{\omega = \sqrt{\frac{\hbar^2 k^4}{4m^2} + \frac{\mu}{m} k^2}}$$

Two relevant limits:

- $k \to 0$: $\omega \simeq kc \Rightarrow$ sound waves with the speed of sound

$$\boxed{c = \sqrt{\frac{\mu}{m}}} \quad \text{i.e.} \quad \mu = mc^2$$

- $k \to \infty$: $\hbar\omega \simeq \mu + \frac{\hbar^2 k^2}{2m} \Rightarrow$ particle-like excitations on top of the condensed particles bringing an energy μ

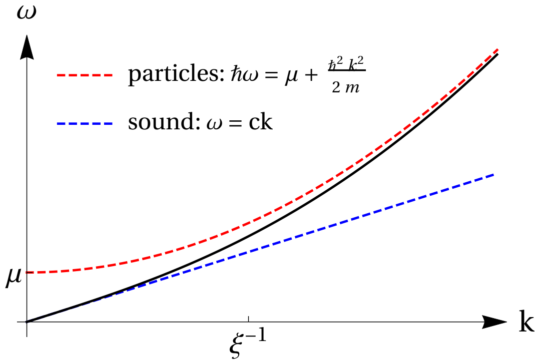
Boundary between the two regimes: $k \simeq \xi^{-1}$ with

$$\boxed{\xi = \frac{\hbar}{\sqrt{2m\mu}} = \frac{1}{\sqrt{2}} \frac{\hbar}{mc}}$$

Bogolubov spectrum

Sound and particles

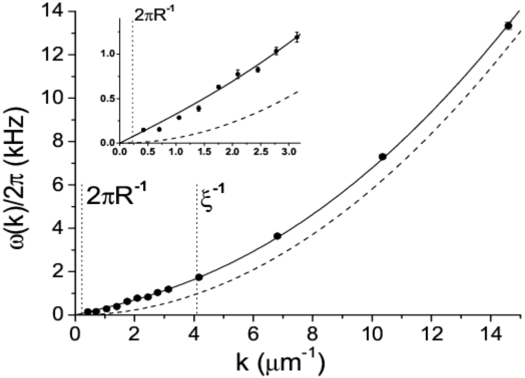
$$\boxed{\omega = \sqrt{\frac{\hbar^2 k^4}{4m^2} + \frac{\mu}{m}k^2}}$$



Bogolubov spectrum

Experimental observation

Selective excitation at (k, ω) with Bragg diffraction



linear spectrum (phonons) at small k:
$\omega(k) = ck$

quadratic spectrum at large k

Steinhauer et al., PRL **88**, 120407 (2002)

Warning: Linear at small k for interacting gases only!

Signatures of superfluidity

(1) Critical velocity

Consequence of Bogolubov dispersion $E(p) \geq cp$: Landau criterion

- Consider an **object** of mass M and momentum P dragged into the fluid at a speed $v = P/M$: can its motion be damped by creating an excitation of momentum p^* in the condensate?
- **Momentum and energy conservation**:
  Before: P, $E = P^2/2M + 0$
  After: $P' + p^*$, $E = P'^2/2M + E(p^*)$
- From $P' = P - p^*$ it follows that

$$\frac{P \cdot p^*}{M} = v \cdot p^* = E(p^*) + \frac{p^{*2}}{2M} \geq E(p^*) \geq cp^*$$

- Excitations are created only if $v \geq c$: existence of a critical velocity (Warning: c vanishes if $g \to 0$)

Signatures of superfluidity

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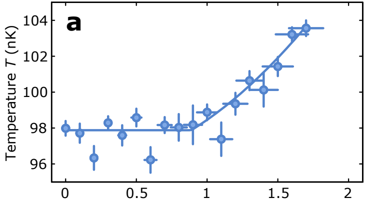


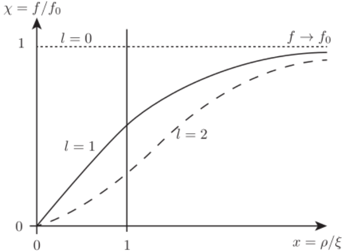
Desbuquois et al., Nat. Phys. **8**, 645 (2012)

- Excitations are created only if $v \geq c$: existence of a critical velocity (Warning: c vanishes if $g \to 0$)

Signatures of superfluidity

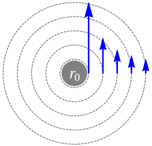
(2) Quantized vortices

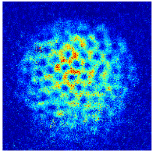
- $\nabla \times v = 0$ unless at positions $\mathbf{r_v}$ such that $n(\mathbf{r_v}) = 0$
- Around such a point $\mathbf{r_v} = 0$, $\psi(\mathbf{r}) \simeq \sqrt{n_0} e^{i\ell\theta}$
  $\Rightarrow$ rotation with $1/r$ velocity field $\mathbf{v} = \frac{\hbar}{m}\frac{\ell}{r}\mathbf{e}_\theta$
- ψ is uniquely defined $\Rightarrow \ell \in \mathbb{Z}$
- $\mathcal{C} = \oint v \cdot ds = \ell \dfrac{h}{M}, \quad \ell \in \mathbb{Z}$
  the fluid rotates with a quantized circulation
- Size of the hole: $v > c$ for $r < \frac{\hbar}{mc} = \sqrt{2}\xi$
  $\Rightarrow \sim$healing length ξ
- Kinetic energy of a vortex: $E_{\mathrm{kin}} \propto \ell^2$
- Multiply charged vortices $|\ell| > 1$ are unstable
  $\Rightarrow$ Vortices arrange into an Abrikosov lattice



vortex wavefunction (modulus)

S. Huber's course, ETHZ





Dynamics of the trapped Bose gas

Collective modes in a harmonic trap

Derivation of quantized, low energy modes in a trap (large scale). We start from the hydrodynamics equations:

$$(1) \qquad \partial_t n + \nabla \cdot (n\mathbf{v}) = 0$$

$$(2) \qquad M\partial_t \mathbf{v} = -\nabla \left(-\frac{\hbar^2}{2M}\frac{\Delta\left(\sqrt{n}\right)}{\sqrt{n}} + \frac{1}{2}Mv^2 + V(\mathbf{r}) + g\,n - \mu \right)$$

- Neglect quantum pressure
- Linearize around the Thomas-Fermi solution $n(\mathbf{r}) = [\mu - V(\mathbf{r})]/g$, we get

$$\partial_t^2 \delta n = -\omega^2 \delta n = \nabla \cdot \left[\frac{\mu - V(\mathbf{r})}{M} \nabla \delta n \right] = \nabla \cdot \left[c^2(\mathbf{r}) \nabla \delta n \right]$$

- 3D spherical harmonic trap: $V(\mathbf{r}) = \frac{1}{2}m\omega_0^2 r^2$. n_r, ℓ, m quantum numbers: $\delta n(\mathbf{r}) = P_\ell^{(2n_r)}(r/R)r^\ell Y_{\ell m}(\theta,\phi)$
- Get $\omega(n_r,\ell) = \omega_0 \left[2n_r^2 + 2n_r\ell + 3n_r + \ell\right]^{1/2}$ [Stringari, PRL **77**, 2360 (1996)]

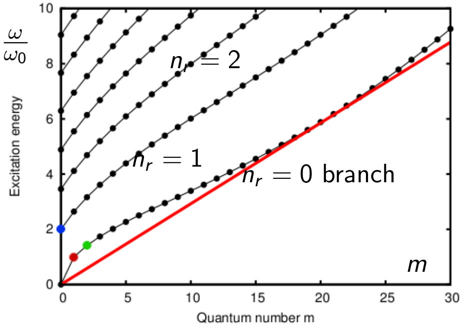
Dynamics of the trapped Bose gas

Excitation spectrum and collective modes

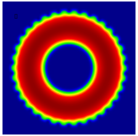
- Quantized excitation spectrum
- Example for the isotropic 2D gas: n_r, m are good quantum numbers:

$$\omega(n_r, m) = \omega_0 \left[2n_r^2 + 2n_r|m| + 2n_r + |m|\right]^{1/2}$$ [Stringari, PRA **58**, 2385 (1998)]

- Full spectrum:



red line: gives the critical velocity, related to surface modes [Anglin, PRL **87**, 240401 (2001)]



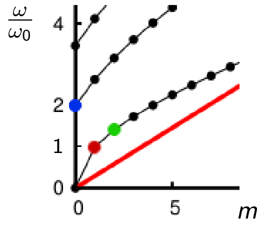
Ex: surface modes at the outer edge of an annular Bose gas
[Dubessy et al., PRA **86**, 011602 (2012)]

Dynamics of the trapped Bose gas

Excitation spectrum and collective modes

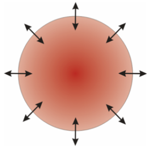
- Quantized excitation spectrum
- Example for the isotropic 2D gas: n_r, m are good quantum numbers:

$$\omega(n_r, m) = \omega_0 \left[2n_r^2 + 2n_r|m| + 2n_r + |m|\right]^{1/2}$$

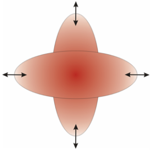


- dipole mode $n_r = 0, m = 1$, both superfluid and thermal: centre of mass oscillation

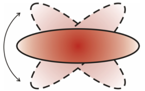
- monopole $n_r = 1, m = 0$: superfluid and thermal **signature of the EOS**



- quadrupole $n_r = 0, m = \pm 2$ **superfluid only**



- **scissors** for $\omega_x \neq \omega_y$ **superfluid only**

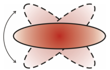


Signatures of superfluidty

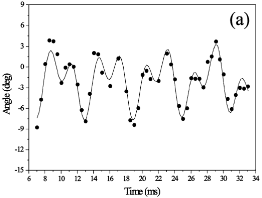
(3) Specific collective modes

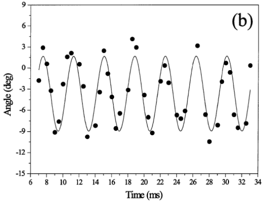
Specific modes of a superfluid gas: quadrupole mode, scissors mode: oscillation of $\langle xy \rangle \propto \theta$ in an **anisotropic** harmonic trap $\omega_x \neq \omega_y \Rightarrow$ Use the scissors mode to characterize a superfluid dilute gas

- $T > T_c$: no scissors mode in the thermal phase in the **collisionless** regime, only beat notes of harmonic modes $\omega_\pm = \omega_x \pm \omega_y$



- $T < T_c$: scissors mode expected at $\omega_{\text{sc}} = \sqrt{\omega_x^2 + \omega_y^2}$ for a **superfluid**





Theory: Guéry-Odelin & Stringari, PRL **83**, 4452 (1999);

Experiment: Maragò et al., PRL **84**, 2056 (2000)

Summary

- BEC occurs below T_c (or above N_c) in a 3D box or in a harmonic trap in 2D or 3D
- The dynamics of the condensate is captured in the mean-field regime by GPE or the hydrodynamics equations
- A weakly interacting BEC is a superfluid
- Superfluidity should be probed dynamically
- Signatures include critical velocity, vortices, collective modes

Next lecture: two-dimensional Bose gases and the BKT transition.