Course on Two-dimensional Bose gases
Lecture 2: Two-dimensional Bose gases at equilibrium

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Outline of the course

- Lecture 1: Bose-Einstein condensation, interactions and superfluidity
- Lecture 2: Two-dimensional Bose gases at equilibrium
- Lecture 3: Rotating two-dimensional Bose gases

Introduction

The role of dimensionality in physics

Physics is qualitatively changed when dimension is reduced. Topology is not the same as in 3D.

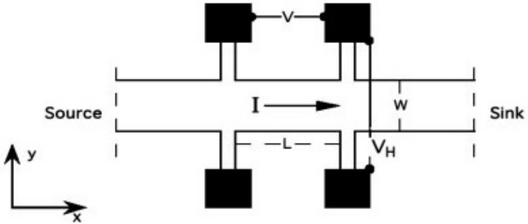
Examples include:

- in 1D: absence of thermalisation of a 1D gas, ‘fermionization’ of an interacting Bose gas, renormalization of the interactions, role of solitons...
- in 2D: absence of long-range order, (fractional) quantum Hall effect, Kosterlitz-Thouless transition, renormalization of the interactions, role of vortices...

Introduction

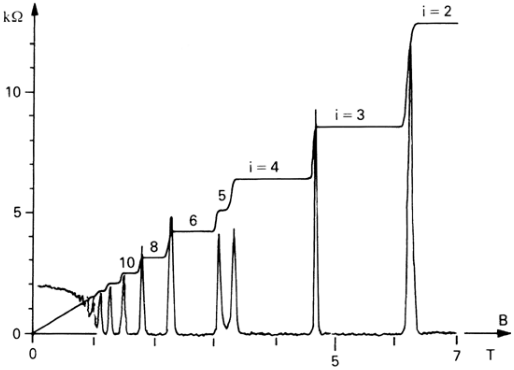
Example in 2D: the Quantum Hall Effect

- 2D electron gas at the interface of a semiconductor heterojunction
- longitudinal current I_x, high perpendicular magnetic field B_z
- measure the transverse voltage $V_H = V_y$



Introduction

Example in 2D: the Quantum Hall Effect



- plateaux of Hall resistance $R = \frac{V_y}{I_x} = \frac{h}{\nu e^2}, \quad \nu \in \mathbb{N}^*$
- longitudinal resistance $R_x = \frac{V_x}{I_x} = 0$

2D: A marginal dimension

Scaling symmetry, topology, quasi long-range order... and lots of logs

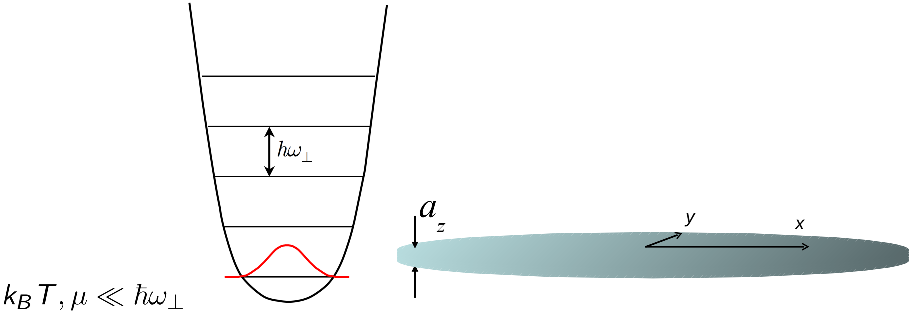
2D is a very special case!

- **Condensation and superfluidity**
  - No BEC at $T > 0$ for the homogeneous ideal gas
  - BEC recovered in a trap
  - Interactions induce a quasi long-range order...
  - ...and enable a transition to a superfluid state
- **Scaling invariance**
  - (almost) no length scale, dimensionless interaction strength
  - Equation of states depending only on $\alpha = \mu/k_B T$
  - Undamped monopole mode
- **Topology**
  - Role of vortices in the superfluid transition
  - Analogy with Quantum Hall effect for the rotating gas
  - A KT(HNY) transition for the vortex lattice

Production of 2D gases

General idea

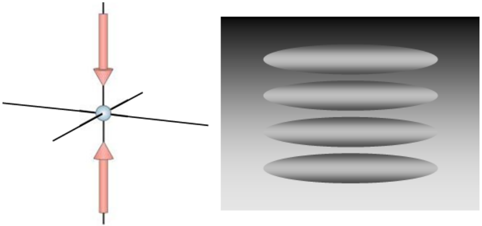
Experimental realization of 2D gases: strongly confine the transverse direction $(k_B T, \mu \ll \hbar\omega_\perp)$



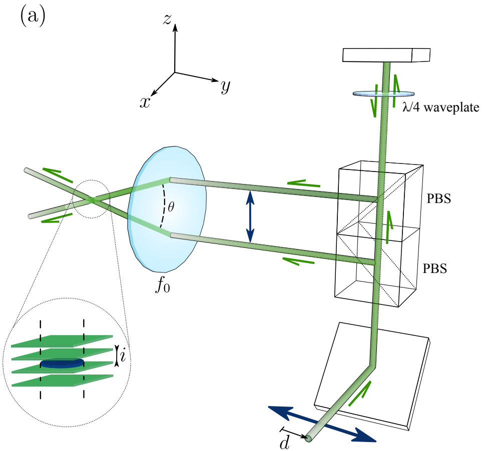
Production of 2D gases
Optical lattices

Experimental realization of 2D gases: strongly confine the transverse direction ($k_B T, \mu \ll \hbar\omega_\perp$)

1. optical lattices along 1 axis



series of parallel 2D gases



Bloch, Nat. Phys. **1**, 23 (2005)

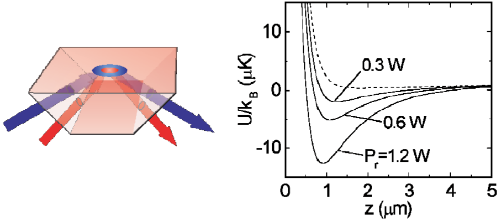
Ville et al., PRA **95**, 013632 (2017)

Production of 2D gases

Optical lattices

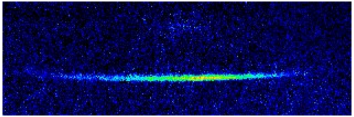
Experimental realization of 2D gases: strongly confine the transverse direction ($k_B T, \mu \ll \hbar\omega_\perp$)

1. optical lattices along 1 axis
2. 2D optical surface traps / rf-dressed magnetic traps



First 2D BEC in R. Grimm's group

Rychtarik et al., PRL **92**, 173003 (2004)



2D Bose gases in adiabatic potentials @ LPL

Colombe et al., EPL **67**, 593 (2004)

Merloti et al., NJP **15**, 033007 (2013)

Outline of the lecture

1. Introduction to 2D quantum gases
2. Quasi long-range order in 2D
3. The Berezinskii-Kosterlitz-Thouless mechanism
4. Scaling symmetry in 2D: monopole mode and equation of state

References

General references:

- *Quantum Gases in Low Dimensions*, edited by L. Pricoupenko, H. Perrin and M. Olshanii, J. Phys IV **116** (2004)
  Les Houches lectures by Shlyapnikov, Castin, Olshanii, Stringari, Cirac and Douçot.
- *Many body physics with ultra cold gases*, I. Bloch, J. Dalibard and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008)
- Lectures at Collège de France by Jean Dalibard (in French), academic year 2016-2017
- Nobel lectures by J. Michael Kosterlitz and F. Duncan M. Haldane, Rev. Mod. Phys. **89**, 040501 & 040502 (2017)

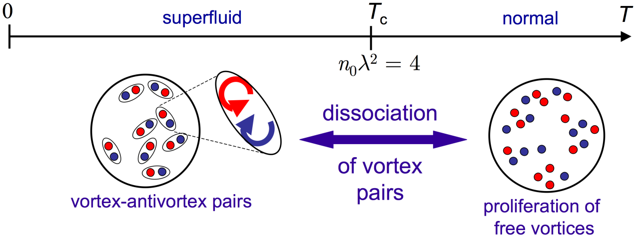
... and (many) references therein.

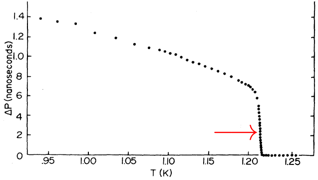
The Berezinskii-Kosterlitz-Thouless mechanism

KT transition in the homogeneous two-dimensional Bose fluid

2D is a very special case! Logs and topological phase transitions

• **2D homogeneous case** No long range order/BEC (Hohenberg–Mermin–Wagner theorem), but a Kosterlitz–Thouless transition to a superfluid state below T_{BKT}, relying on vortex-antivortex pairing. Universal jump of the superfluid density.





Bishop and Reppy, PRL **40**, 1727 (1978)

[ENS-CdF, NIST, Chicago, Palaiseau, Seoul, Cambridge, Villetaneuse, Oxford...]

2016 Nobel prize in physics to Haldane, Kosterlitz and Thouless

The two-dimensional Bose gas

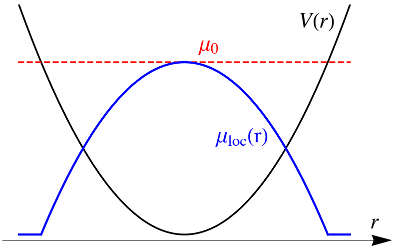
2D: A marginal dimension

- **trapped gas** $V(\mathbf{r})$:
  - **BEC** recovered in a harmonic trap (finite size helps)
  - **BKT** still relevant within local density approximation (LDA).

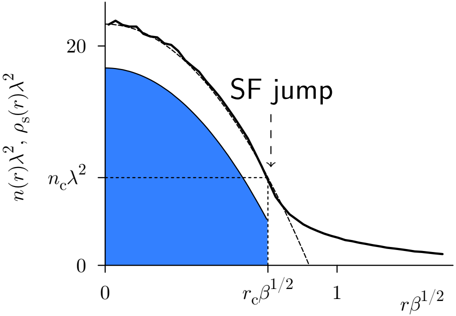
replace

μ by $\mu_{\rm loc}(\mathbf{r}) = \mu_0 - V(\mathbf{r})$,

$\alpha = \frac{\mu}{k_B T}$ by $\alpha_{\rm loc}(\mathbf{r}) = \alpha_0 - V(\mathbf{r})/k_B T$



BKT superfluid phase within LDA

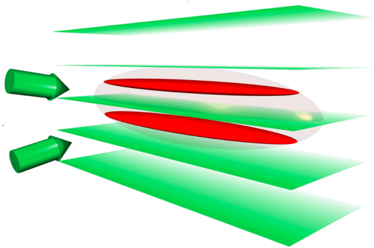


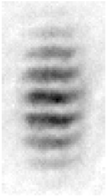
Holzmann & Krauth, PRL **100**, 190402 (2008)

The Berezinskii-Kosterlitz-Thouless mechanism

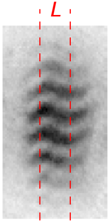
ENS experiment: observation of the transition and correlation function

ENS experiment: measure $G_1(x)$ decay by interferometry

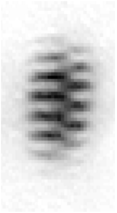




low T



phase fluc.



vortex

- measurement of the integrated contrast:

$$\frac{1}{L}\int_{-L/2}^{L/2} |G_1(x)|^2\, dx \propto \frac{1}{L^{2\alpha}}$$

exponential decay: $\alpha = \frac{1}{2}$

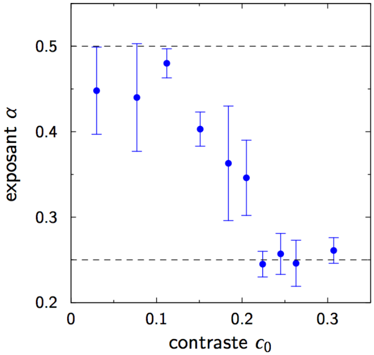
algebraic decay: $\alpha = \frac{1}{4}$

- statistics on phase defects (free vortices)

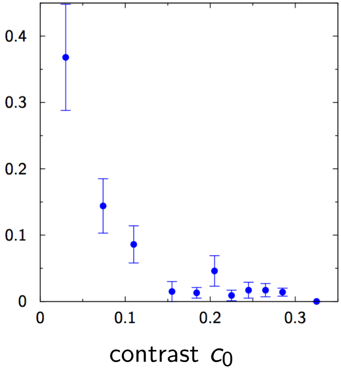
The Berezinskii-Kosterlitz-Thouless mechanism

ENS experiment: observation of the transition and correlation function

Results: Hadzibabic et al., Nature **441**, 1118 (2006)



Fraction of pictures with vortices

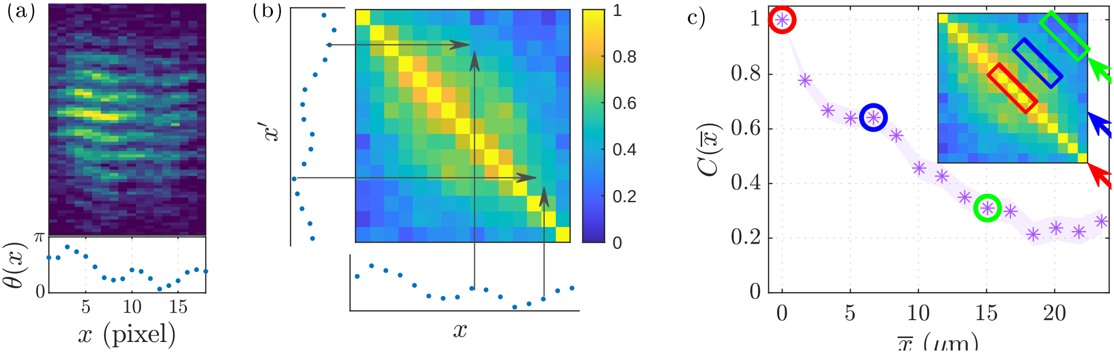


- contrast c_0 is a measure of temperature
- BKT transition evidenced by a step in exponent α and apparition of vortices

The Berezinskii-Kosterlitz-Thouless mechanism

Oxford experiment: decay of correlation function and vortices

Chris Foot's group, Sunami et al., PRL **128**, 250402 (2022)



- (a) prepare two parallel 2D quantum gases, release and select a slice around $y = 0$
- (a) recover the local phase at each x from the fringes observed in time-of-flight
- (b) compute the (x, x') correlation, (c) plot as a function of distance $|x - x'|$

The Berezinskii-Kosterlitz-Thouless mechanism

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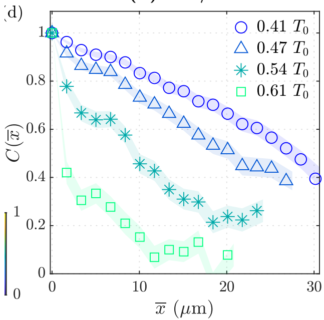
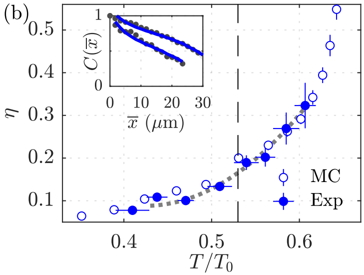
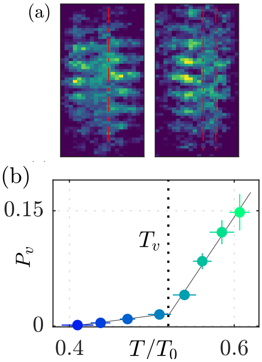
fit $C(x) \propto 1/L^{2\eta}$



exponent η



vortex probability P_v



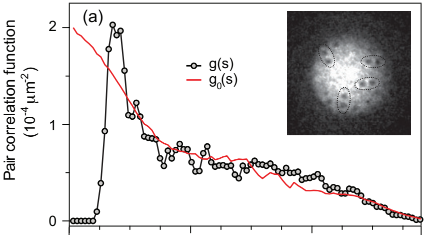
Recover the main features, more quantitative comparison to Monte-Carlo calculations.

The Berezinskii-Kosterlitz-Thouless mechanism

Seoul experiment: pairing of vortices

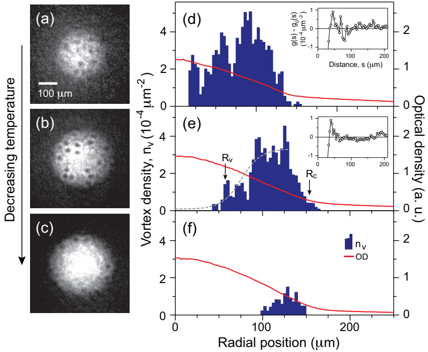
Yong-il Shin's group, Choi et al., PRL **110**, 175302 (2013)

pair correlation



distance between vortices

vortices pushed to the edges



Most probable distance between vortices depends on T/T_{BKT}.

The Berezinskii-Kosterlitz-Thouless mechanism

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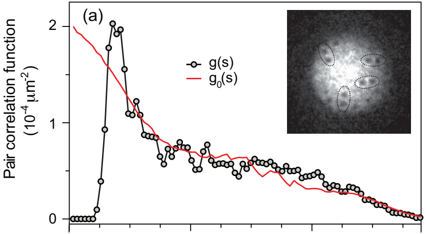
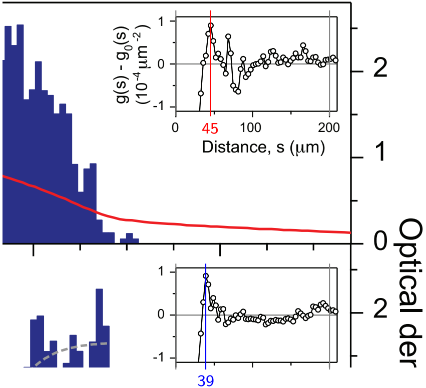
pair correlation



distance between vortices

vortices pushed to the edges



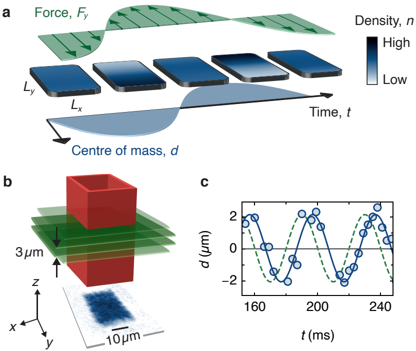
Most probable distance between vortices depends on T/T_{BKT}.

KT transition in the homogeneous 2D Bose gas

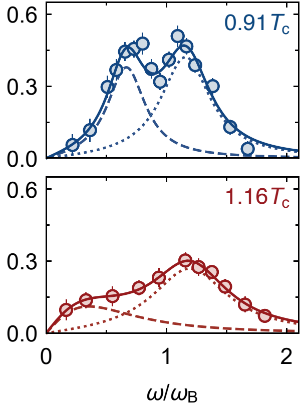
The Cambridge experiment: superfluid jump observed from first and second sound

2D homogeneous ^{39}K gas, $\tilde{g} = 0.64$

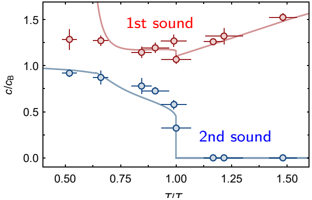
Shake the gas along y axis, record center-of-mass motion



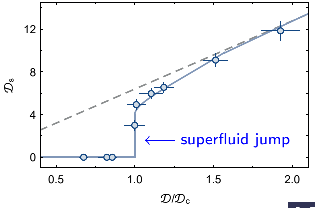
Spectroscopy of sound modes



Speed of sound



Superfluid phase-space density



Recover $n_s\lambda_{\rm dB}^2 = 4$ jump at the transition [Christodolou et al., Nature **594**, 191 (2021)]

Summary: BEC vs BKT in the two-dimensional Bose gas

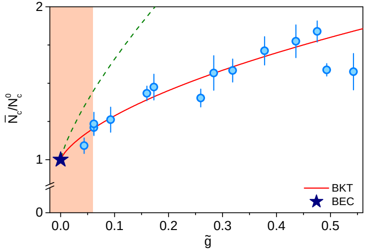
BEC or BKT depends on trapping and interactions

Summary:

	ideal	interacting
homogeneous	no BEC, no SF	BKT SF [ENS-CdF]
trapped	BEC, no SF $\longleftrightarrow$	BEC + BKT within LDA

BEC-BKT interplay in a harmonic trap

Fletcher et al., PRL **114**, 255302 (2015)

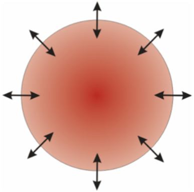


2D: A marginal dimension

Scaling symmetry and monopole mode of the 2D Bose gas

2D is a very special case! **Scaling symmetry**:

- scaling invariance $r \to \lambda r$: $E_K \to \frac{1}{\lambda^2} E_K$, $E_{\rm int} \to \frac{1}{\lambda^2} E_{\rm int}$
- (almost) no length scale: dimensionless interaction strength $\tilde{g}$: $g_{2D} = \frac{\hbar^2}{M}\tilde{g}$
- Pitaevskii-Rosch monopole mode **in an isotropic 2D harmonic trap**:



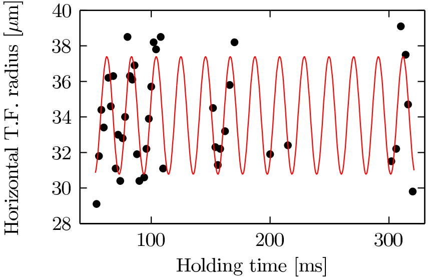
- no damping [Chevy et al., PRL **88**, 250402 (2002), in a cigar]
- monopole probes the compressibility $\Rightarrow \Omega_M$ is related to the 2D EOS $\mu(n)$: $\Omega_M = \sqrt{2(2+\epsilon)}\,\omega$ with $\epsilon = n\mu''(n)/\mu'(n)$
- 3D oblate: $\mu(n_{2D}) \propto n_{2D}^{2/3} \Rightarrow \Omega_M = \sqrt{\frac{10}{3}}\omega$ for small amplitudes
- strict 2D: $\mu(n) = g_{2D}\, n \Rightarrow \Omega_M = 2\omega$ for all amplitudes, linked to scaling symmetry
- effect of transverse confinement: Quantum anomaly: expected positive shift of Ω_M at the 0.5% level [Olshanii et al., PRL **105**, 095302 (2010)]

2D: A marginal dimension

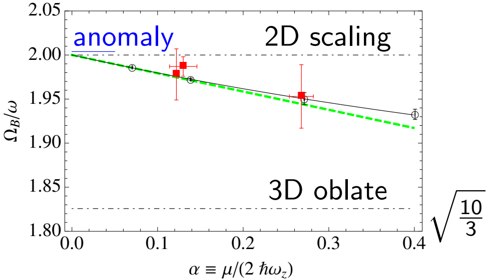
Scaling symmetry and monopole mode of the 2D Bose gas

Experiment: Karina Merloti (LPL) [thesis, NPJ **15**, 033007 & PRA **88**, 061603(R) (2013)]

no visible damping



Frequency sensitive to the EOS.



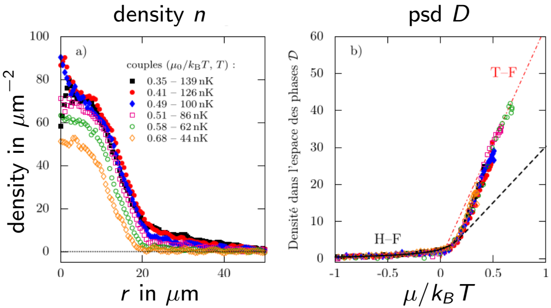
Observe a shift due to the interactions and progressive occupation of transverse modes as μ increases $\Rightarrow$ transition from $\Omega_B = 2\omega_r$ (2D) to $\Omega_B = \sqrt{\frac{10}{3}}\omega_r$ (3D oblate).

Scaling symmetry and universality

Equation of state

2D is a very special case! Logs and topological phase transitions

- **Scaling symmetry and universality**
  - kinetic energy $\propto k^2$, interactions $\propto 1/r^2$, integrand $k\,dk \Rightarrow$ critical dimension with Log divergences
  - no length scale: dimensionless interaction strength $g = \frac{\hbar^2}{M}\tilde{g}$
  - EOS depends only on $\alpha = \mu/k_B T$: $D = f(\alpha, \tilde{g})$ [ENS,Chicago]
  - $\tilde{g} \simeq 0.1$: critical psd for BKT $D_c \simeq 8$, critical $\alpha_c \simeq 0.16$



picture from T. Yefsah's PhD thesis [Yefsah et al., PRL **107**, 130401 (2011)]

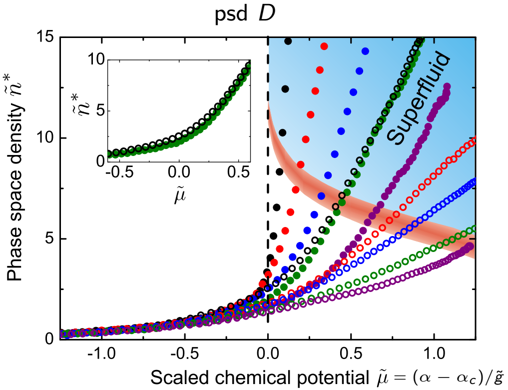
Scaling symmetry and universality

Equation of state

2D is a very special case! Logs and topological phase transitions

- **Scaling symmetry and universality**
  - kinetic energy $\propto k^2$, interactions $\propto 1/r^2$, integrand $k\,dk \Rightarrow$ critical dimension with Log divergences
  - no length scale: dimensionless interaction strength $g = \frac{\hbar^2}{M}\tilde{g}$
  - EOS depends only on $\alpha = \mu/k_B T$: $D = f(\alpha, \tilde{g})$ [ENS,Chicago]
  - large $\tilde{g}$: universal law near D_c

$$D - D_c = f\left(\frac{1}{\tilde{g}}\left[\frac{\mu}{k_B T} - \left(\frac{\mu}{k_B T}\right)_c\right]\right)$$



[Chin group, Nature **470**, 236 (2011) & PRL **110**, 145302 (2013)]