# Basics on light-Matter Interaction, atomic cooling and trapping

Raul Celistrino Teixeira
Patrícia Christina Marques Castilho
Felipe Carvalho Brambila

## 1. Semiclassical Model

### 1.1. Classical formalism for charges interacting with electromagnetic fields

The classical Hamiltonian for an electrical charge $q$ of position $\mathbf{r}$, conjugate Hamilton momentum $\mathbf{p}$ and mass $m$ is given by

$$\mathcal{H} = \frac{1}{2m}(\mathbf{p} - q\mathbf{A}(\mathbf{r}))^2 + q\phi(\mathbf{r})\,. \tag{1}$$

In the equation above, $\phi(\mathbf{r})$ is the scalar potential and $\mathbf{A}(\mathbf{r})$ is the vector potential of the electromagnetic radiation. The electromagnetic fields are related to them via

$$\mathbf{E}(\mathbf{r},t) = -\boldsymbol{\nabla}\phi(\mathbf{r},t) - \frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t}\,, \tag{2}$$

$$\mathbf{B}(\mathbf{r},t) = \boldsymbol{\nabla}\times\mathbf{A}(\mathbf{r},t)\,. \tag{3}$$

In eq. (1), the potentials are not dynamical variables; the only dynamical variables are the position and momentum of the particle. We can obtain the classical equations of motion for them within the Hamilton formalism:

$$\frac{\mathrm{d}x_j}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial p_j} = \frac{p_j - qA_j(\mathbf{r})}{m}\,, \tag{4}$$

$$\frac{\mathrm{d}p_j}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial x_j} = \frac{q}{m}(\mathbf{p} - q\mathbf{A}(\mathbf{r}))\cdot\frac{\partial \mathbf{A}}{\partial x_j} - q\frac{\partial \phi}{\partial x_j}\,, \tag{5}$$

with $j \in \{1,2,3\}$ and $x_j$ (resp. $p_j$) the components of the position vector $\mathbf{r}$ (resp. $\mathbf{p}$) in each direction of space. The first equation states that the relation between the speed and

the conjugate momentum is different from the usual:

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathbf{p} - q\mathbf{A}(\mathbf{r})}{m} \ . \tag{6}$$

This shows that the Hamiltonian above is actually just what we expect,

$$\mathcal{H} = \frac{m\mathbf{v}^2}{2} + q\phi(\mathbf{r}) \, . \tag{7}$$

To obtain the force acting of the particle, we can derive again the equation above. Writing the equation for each coordinate, we get

$$\begin{aligned} F_j = m\frac{\mathrm{d}v_j}{\mathrm{d}t} &= \frac{\mathrm{d}p_j}{\mathrm{d}t} - q\frac{\mathrm{d}}{\mathrm{d}t}A_j(\mathbf{r},t) \\ &= \frac{q}{m}(\mathbf{p} - q\mathbf{A}(\mathbf{r})) \cdot \frac{\partial \mathbf{A}}{\partial x_j} - q\frac{\partial \phi}{\partial x_j} - q\boldsymbol{\nabla}A_j \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} - q\frac{\partial A_j}{\partial t} \\ &= q\left(-\frac{\partial \phi}{\partial x_j} - \frac{\partial A_j}{\partial t}\right) + q\left(\mathbf{v} \cdot \frac{\partial \mathbf{A}}{\partial x_j} - (\mathbf{v}\cdot\boldsymbol{\nabla})A_j\right) \ . \end{aligned} \tag{8}$$

In the first term of the last line, we recognize the $j$ component of the electric field; The second term can be shown to be

$$\mathbf{v} \cdot \frac{\partial \mathbf{A}}{\partial x_j} - (\mathbf{v}\cdot\boldsymbol{\nabla})A_j = (\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A}))_j \ , \tag{9}$$

and we finally find the Lorentz force

$$\mathbf{F} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \ . \tag{10}$$

## 1.2. A QUANTUM CHARGE INTERACTING WITH CLASSICAL LIGHT

We can quantize the degrees of freedom of the charge (its position and momentum). This is done via the canonical quantization

$$\mathbf{r} \ \rightarrow \ \hat{\mathbf{r}} \tag{11}$$

$$\mathbf{p} \ \rightarrow \ \hat{\mathbf{p}} \tag{12}$$

Accordingly, the new Hamiltonian becomes the operator

$$\hat{\mathcal{H}} = \frac{1}{2m}(\hat{\mathbf{p}} - q\mathbf{A}(\hat{\mathbf{r}}))^2 + q\phi(\hat{\mathbf{r}}) \, . \tag{13}$$

As usually in Quantum Mechanics, the momentum operator is represented as $\hat{\mathbf{p}} = -i\hbar\boldsymbol{\nabla}$ in the space representation, in order to conserve the canonical commutation relation

$$[x_j, p_l] = i\hbar\delta_{jl} \ . \tag{14}$$

On the other hand, the velocity operator must be quantized from Eq. (6), and becomes

$$\hat{\mathbf{v}} = \frac{\hat{\mathbf{p}} - q\mathbf{A}(\hat{\mathbf{r}})}{m} \ , \tag{15}$$

such as

$$\hat{\mathcal{H}} = \frac{m}{2}\hat{\mathbf{v}}^2 + q\phi(\hat{\mathbf{r}}) \, . \tag{16}$$

It is instructive to calculate the average force felt by the charge. For that, we will use Ehrenfest's theorem:

$$\frac{d\langle \mathbf{v} \rangle}{dt} = \frac{1}{i\hbar}[\hat{\mathbf{v}}, \hat{\mathcal{H}}] + \langle \frac{\partial \mathbf{v}}{\partial t} \rangle \tag{17}$$

This is a long calculation. Some intermediate steps are outlined below:

- $[\hat{v}_x, \hat{v}_y] = (-q/m^2)[\hat{p}_x, \mathrm{A}_y(\mathbf{r})] + (q/m^2)[\hat{p}_y, \mathrm{A}_x(\mathbf{r})]$
- $[\hat{p}_x, \mathrm{A}_y(\mathbf{r})]\psi = -i\hbar\dfrac{\partial \mathrm{A}_y}{\partial x}\psi \ \rightarrow \ [\hat{v}_x, \hat{v}_y] = \dfrac{i\hbar q}{m^2}B_z(\hat{\mathbf{r}})$
- $[\hat{v}_x, \hat{v}_y^2] = \dfrac{i\hbar q}{m^2}(\hat{v}_y B_z(\hat{\mathbf{r}}) + B_z(\hat{\mathbf{r}})\hat{v}_y) \ \rightarrow \ [\hat{v}_x, m\hat{v}^2/2] = \dfrac{i\hbar q}{2m}(\hat{\mathbf{v}} \times \mathbf{B}(\mathbf{r}) - \mathbf{B}(\mathbf{r}) \times \hat{\mathbf{v}})_x$
- $[\hat{v}_x, q\phi(\hat{\mathbf{r}})] = -i\hbar q\dfrac{d\phi(\hat{\mathbf{r}})}{dx}$
- $\langle \dfrac{\partial \mathbf{v}}{\partial t} \rangle = \langle \dfrac{q}{m}\dfrac{\partial \mathbf{A}_x}{\partial t} \rangle$

Plugging all the results into 17, we get the **quantum analog of the Lorentz force**, applied to the average value of the velocity operator:

$$m\frac{d\langle \hat{\mathbf{v}} \rangle}{dt} = q\langle \mathbf{E}(\hat{\mathbf{r}}) \rangle + \frac{q}{2}\langle \hat{\mathbf{v}} \times \mathbf{B}(\hat{\mathbf{r}}) - \mathbf{B}(\hat{\mathbf{r}}) \times \hat{\mathbf{v}} \rangle \, . \tag{18}$$

We note that the symmetrized version of $\hat{\mathbf{v}} \times \mathbf{B}(\hat{\mathbf{r}})$ appears at the force expression. This is ubiquitous in the canonical quantization: Only symmetrized versions of operators which are function of both $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$ are observables.

## 1.3. A QUANTUM ATOM INTERACTING WITH A CLASSICAL PLANE WAVE

We will now use those results to describe a Hydrogen atom interacting with incoming light. This atom will have a positive nucleus, and an electron in the valence shell orbiting

it, of charge $q = -e$, with $e = +1,602\,176\,634.10^{-19}C$ the modulus of the elementary charge. The charge we described in the previous subsection is now just the electron orbiting around the positive core. The degrees of freedom of the electronic structure of the atom are called the *internal degrees of freedom* of the atom, and this is what interest us now. We will not include as a degree of freedom here the center of mass of the atom (the *external degrees of freedom* of the atom); we will treat the consequences to those degrees of freedom in a next section.

Given an incident plane wave with a wavevector $\mathbf{k}$, we can write the electric and magnetic fields:

$$\mathbf{E} = \mathbf{E}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t) \ , \tag{19}$$

$$\mathbf{B} = \frac{\mathbf{k}\times\mathbf{E}}{\omega} . \tag{20}$$

We must also have $\mathbf{E}_0 \cdot \mathbf{k} = 0$. We will place ourselves now in the **Coulomb gauge**, and choose a scalar and vector potentials that satisfy $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$. In this gauge, we write $\mathbf{A} = \mathbf{A}_\perp$. We have

$$\mathbf{A} = \mathbf{A}_\perp(\mathbf{r},t) = \frac{\mathbf{E}_0}{\omega}\sin(\mathbf{k}\cdot\mathbf{r} - \omega t) \ , \tag{21}$$

$$\phi(\mathbf{r},t) = 0 \tag{22}$$

We must also include in the scalar potential the Coulomb interaction potential with the positive nucleus. The Hamiltonian becomes

$$\hat{\mathcal{H}} = \frac{1}{2m}(\hat{\mathbf{p}} - q\mathbf{A}_\perp(\hat{\mathbf{r}}))^2 + V_{\text{Coul}}(\hat{r}) = \underbrace{\frac{\hat{\mathbf{p}}^2}{2m} + V_{\text{Coul}}(\hat{r})}_{\hat{\mathcal{H}}_0} \ \underbrace{-\frac{q}{m}\hat{\mathbf{p}}\cdot\mathbf{A}_\perp(\hat{\mathbf{r}})}_{\hat{\mathcal{H}}_{I1}} + \underbrace{\frac{q^2}{2m}{\mathbf{A}_\perp}^2(\hat{\mathbf{r}})}_{\hat{\mathcal{H}}_{I2}} \tag{23}$$

with

$$V_{\text{Coul}}(\hat{r}) = -\frac{e^2}{4\pi\varepsilon_0\hat{r}} \ . \tag{24}$$

For deducing the Hamiltonian above, we used the fact that, in Coulomb gauge,

$$\hat{\mathbf{p}}\cdot\mathbf{A}_\perp(\hat{\mathbf{r}}) = \mathbf{A}_\perp(\hat{\mathbf{r}})\cdot\hat{\mathbf{p}} - i\hbar\boldsymbol{\nabla}\cdot\mathbf{A}_\perp = \mathbf{A}_\perp(\hat{\mathbf{r}})\cdot\hat{\mathbf{p}} \ . \tag{25}$$

We can separate the Hamiltonian of Eq. (23) in three main terms. First,

$$\hat{\mathcal{H}}_0 = \frac{\hat{\mathbf{p}}^2}{2m} + V_{\text{Coul}}(\hat{r}) \tag{26}$$

is the Hamiltonian of the Hydrogen atom. Then we have the two last terms of Eq. (23), respectively $\hat{\mathcal{H}}_{I1}$ and $\hat{\mathcal{H}}_{I2}$, that correspond together to the Hamiltonian of interaction of the atom with light, $\hat{\mathcal{H}}_I = \hat{\mathcal{H}}_{I1} + \hat{\mathcal{H}}_{I2}$.

### 1.3.1. Quick reminder of the Hydrogen atom model

The eigenstates of the Hydrogen atom are parametrized as $|n, \ell, m_l\rangle$, such as the action of the operators $\hat{\mathcal{H}}_0$, the squared modulus of the angular momentum $\hat{\mathbf{L}}^2$ and the $z$-component of the angular momentum $\hat{L}_z$ are simply

- $\hat{\mathcal{H}}_0|n, \ell, m_\ell\rangle = \dfrac{E_1}{n^2}$, $n \in \mathbb{N}^*$, with $E_1 = \dfrac{me^4}{32\pi^2\epsilon_0^2\hbar^2}$;
- $\hat{\mathbf{L}}^2|n, \ell, m_\ell\rangle = \ell(\ell+1)\hbar^2|n, \ell, m_\ell\rangle$, $\ell \in \mathbb{N}$, $\ell \leq n-1$;
- $\hat{\mathrm{L}}_z|n, \ell, m_\ell\rangle = m_\ell\hbar\,|n, \ell, m_\ell\rangle$, $m_\ell \in \mathbb{Z}$, $|m_\ell| \leq \ell$.

In the position representation, the eigenstates of the Hydrogen atom corresponds to the following wavefunctions:

$$\langle \mathbf{r}|n, \ell, m_\ell\rangle = \psi_{n,\ell,m_\ell}(r, \theta, \varphi) = \mathcal{R}_{n\ell}(r)\, Y_\ell^{m_\ell}(\theta, \varphi)\ . \tag{27}$$

The radial function $\mathcal{R}_{n\ell}(r)$ is given by

$$\mathcal{R}_{n\ell}(r) = \sqrt{\left(\frac{2}{na_0}\right)^3 \frac{(n-\ell-1)!}{2n(n+\ell)!}}\, \mathrm{e}^{-\frac{r}{na_0}}\, \rho^\ell\, \mathcal{L}_{n-\ell-1}^{2\ell+1}\!\left(\frac{2r}{na_0}\right)\ . \tag{28}$$

In the expression above, $a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2}$ is the Bohr radius, the typical length scale of the electronic orbitals, and $\mathcal{L}_{j_1}^{j_2}$ is the generalized Laguerre polynomial, defined by

$$\mathcal{L}_{j_1}^{j_2}(\xi) = \frac{\xi^{-j_2}}{j_1!}\left(\frac{d}{d\xi} - 1\right)^{j_1} \xi^{j_1+j_2}\ . \tag{29}$$

The angular function $Y_\ell^{m_\ell}(\theta, \varphi)$ is the spherical harmonic,

$$Y_\ell^m(\theta, \varphi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, \mathrm{e}^{im\varphi} \tag{30}$$

with the definition of the associated Legendre polynomials

$$P_\ell^m(\xi) = \frac{1}{2^\ell \ell!}(1-\xi^2)^{m/2}\, \frac{d^{\ell+m}}{d\xi^{\ell+m}}(\xi^2-1)^\ell\ . \tag{31}$$

### 1.3.2. The $\mathbf{A}\cdot\mathbf{p}$ interaction Hamiltonian

The other two terms of the Hamiltonian, $\mathcal{H}_I = \mathcal{H}_{I1} + \mathcal{H}_{I2}$, describe the interaction between the atom and the incident electric field in Coulomb's gauge. Those terms describe

how the impinging light causes electronic transitions, ionization, and energy shifts. In full generality, they are difficult to handle; but we can in most cases expand their effect in a way analogous to the multipolar expansion of the classical interaction between electric charges and electromagnetic fields.

Considering the typical size of electronic orbitals as being the Bohr radius, and comparing it to the wavelength of visible light (which sets the typical spatial scale of variation of the electromagnetic fields), $a_0/\lambda_0 \sim 10^{-4}$, so $\mathbf{k}\cdot\hat{r} \lesssim 10^{-3}$. This means that it is fair enough to consider that the whole electronic orbital interacts with the same electric field. If the center of mass of the atom is in position $\mathbf{R}$, and taking at first $\mathbf{R}$ at the origin, $\mathbf{R}=\mathbf{0}$, we can make the approximation $\mathrm{A}_\perp(\hat{r}) \approx \mathrm{A}_\perp(\mathbf{0}) = -E_0/\omega \sin(\omega t)$. This is called the **long wavelength approximation**. With this, the term $\mathcal{H}_{I2}$ becomes simply a global energy shift, without physical consequence; and the interaction term $\mathcal{H}_I = \mathcal{H}_{I1}$ becomes

$$\mathcal{H}_I = \frac{-q}{m}(\mathbf{A}_\perp(\mathbf{0})\cdot\hat{\mathbf{p}})\sin\omega t = \frac{e}{m\omega}(\mathbf{E}_0\cdot\hat{\mathbf{p}})\sin\omega t\ . \tag{32}$$

This is called **A.p Hamiltonian**. This form of the interaction Hamiltonian is particularly suitable to numerically calculate atomic ionization rates under intense electromagnetic fields. However, for treating the excitation of an atom to low-lying electronic states, it is more common to work with a different form of the interaction Hamiltonian, that we will deduce below.

### 1.3.3. The electric dipolar interaction Hamiltonian

The first thing we will need to do is to perform a gauge transformation on the electromagnetic potentials, to the Göppert Mayer gauge.

A change in gauge is described by the transformation $(\mathbf{A},\phi)\to(\mathbf{A}',\phi')$, keeping the electromagnetic fields unchanged, through

$$\begin{cases} \mathbf{A}' = \mathbf{A} + \vec{\nabla}\cdot f \\ \phi' = \phi - \frac{\partial f}{\partial t} \end{cases} \tag{33}$$

for any $f(\mathbf{r},t$. To find the potentials in the gauge of Göppert Mayer, we let $f(\mathbf{r},t) = -\mathbf{r}\cdot\mathbf{A}_\perp(\mathbf{0},t)$. In this gauge, the new potentials are written as

$$\mathbf{A}'(\mathbf{r},t) = \mathbf{A}_\perp(\mathbf{r},t) - \mathbf{A}_\perp(\mathbf{0},t)\ , \tag{34}$$

$$\phi'(\mathbf{r},t) = \phi(\mathbf{r},t) + \mathbf{r}\cdot\frac{\partial\mathbf{A}_\perp(\mathbf{0},t)}{\partial t}\ . \tag{35}$$

With these new potentials, the Hamiltonian of Eq. (13) becomes

$$\hat{\mathcal{H}} = \frac{1}{2m}(\hat{\mathbf{p}} - q\mathbf{A}'(\hat{\mathbf{r}},t))^2 + V_{\text{coulomb}}(\hat{r}) + q\hat{\mathbf{r}}\cdot\frac{\partial \mathbf{A}_\perp(\mathbf{0},t)}{\partial t} \ . \tag{36}$$

Now, if we again perform the **long wavelength approximation**,

$$\mathbf{A}'(\mathbf{r},t) \simeq \mathbf{A}'(\mathbf{0},t) = \mathbf{A}_\perp(\mathbf{0},t) - \mathbf{A}_\perp(\mathbf{0},t) = \mathbf{0} \ ; \tag{37}$$

$$\mathbf{E}(\mathbf{r},t) \simeq \mathbf{E}(\mathbf{0},t) = -\left.\frac{\partial \mathbf{A}'(\mathbf{r},t)}{\partial t}\right|_{\mathbf{r}=\mathbf{0}} = -\left.\frac{\partial \mathbf{A}_\perp(\mathbf{r},t)}{\partial t}\right|_{\mathbf{r}=\mathbf{0}} = -\frac{\partial \mathbf{A}_\perp(\mathbf{0},t)}{\partial t} \ . \tag{38}$$

We also define $\hat{\mathbf{d}} = q\hat{\mathbf{r}} = -e\hat{\mathbf{r}}$ the electric dipole moment of the atom, and we finally get

$$\hat{\mathcal{H}} = \frac{\hat{\mathbf{p}}^2}{2m} + V_{\text{coulomb}} - \hat{\mathbf{d}}\cdot\mathbf{E}(\mathbf{0},t) = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_{\text{Id}} \ , \tag{39}$$

with the interaction term in this gauge and in the long wavelength approximation given by $\hat{\mathcal{H}}_{\text{I}} \simeq \hat{\mathcal{H}}_{\text{Id}}$, the **Dipolar interaction Hamiltonian**,

$$\hat{\mathcal{H}}_{\text{Id}} = -\hat{\mathbf{d}}\cdot\mathbf{E}(\mathbf{0},t) \ . \tag{40}$$

If the center of mass of the atom is at any position $\mathbf{R}$, we write instead

$$\hat{\mathcal{H}}_{\text{Id}} = -\hat{\mathbf{d}}\cdot\mathbf{E}(\mathbf{R},t) \ . \tag{41}$$

It is important to note that the electric field in the dipolar approximation is taken at the center of mass of the atom; as such, it is no longer an operator. The only operator at $\hat{\mathcal{H}}_{\text{Id}}$ is the electric dipole $\hat{\mathbf{d}}$. Interestingly, in the Göppert Mayer gauge and within the long wavelength approximation, by use of Eq. (37) we have

$$\hat{\mathbf{v}} = \frac{\hat{\mathbf{p}} - q\mathbf{A}'(\hat{\mathbf{r}},t)}{m} \simeq \frac{\hat{\mathbf{p}}}{m} \ ; \tag{42}$$

i.e., in the electric dipolar approximation, we recover the simple relation between the velocity and the momentum operator.

A final comment on the two different forms of the interaction Hamiltonian that we have found, the $\mathbf{A}\cdot\mathbf{p}$ and the $\mathbf{d}\cdot\mathbf{E}$: It can be surprising that a change in gauge created different Hamiltonians, since electromagnetism should be gauge-invariant (and it is). What happens here is that, since we have made the long wavelength approximation after the change of gauge, those Hamiltonians agree *only to order 0 in* $a_0/\lambda$ - which is precisely the order of precision of what is obtained when we perform the long wavelength approximation. This is why it can be important to choose a proper form for the interaction Hamiltonian when treating specific physical situations. In what follows, we will use the

electric dipolar form.

## 1.4. The electric dipole operator

The interaction Hamiltonian in the electric dipolar approximation, $\hat{\mathcal{H}}_{Id}$, is not diagonal in the basis of the eigenstates of the Hamiltonian $\hat{\mathcal{H}}_0$ of the Hydrogen atom. In first order, the interaction term can cause transitions from a level $|n, \ell, m_\ell\rangle$ to another level $|n, \ell', m'_\ell\rangle$ whenever the off-diagonal matrix element

$$\langle n', \ell', m'_\ell | \hat{\mathcal{H}}_{Id} | n, \ell, m_\ell \rangle \tag{43}$$

is not zero. But we will see that most of those non-diagonal matrix elements are actually zero: This implies that there are only a few electronic transitions allowed by the electric dipolar in first order: Those that respect the **selection rules**. We will deduce the selection rules after discussing the impact of the dipolar coupling in more general terms. But before we proceed, let us just deduce the fact that the diagonal elements of the electric dipolar operator are all zero. The diagonal elements of the electric dipolar operator are of the form

$$\langle n, \ell, m_\ell | \hat{\mathcal{H}}_{Id} | n, \ell, m_\ell \rangle \ . \tag{44}$$

The diagonal terms will change the energy of the electronic level in first order, with respect to the unperturbed energy of the Harmonic oscillator. We can cast the matrix element above as an integral,

$$\begin{aligned} \langle n, \ell, m_\ell | \hat{\mathcal{H}}_{Id} | n, \ell, m_\ell \rangle = \langle n, \ell, m_\ell | \left( e \hat{\mathbf{r}} \cdot \mathbf{E}(\mathbf{R}, t) \right) | n, \ell, m_\ell \rangle = \\ e \langle n, \ell, m_\ell | \hat{\mathbf{r}}(\mathbf{R}, t) | n, \ell, m_\ell \rangle \cdot \mathbf{E}(\mathbf{R}, t) = e \left[ \int_{\mathbb{R}^3} \mathrm{d}^3 \mathbf{r} \ \left| \psi^*_{n,\ell,m_\ell}(\mathbf{r}) \right|^2 \mathbf{r} \right] \cdot \mathbf{E}(\mathbf{R}, t) \ . \end{aligned} \tag{45}$$

Each eigenstate $\psi_{n,\ell,m_\ell}(\mathbf{r})$ is proportional to the spherical harmonic $Y_\ell^{m_\ell}$, which is in turn a polynomial of order $\ell$ on the normalized coordinates $x/r$, $y/r$ and $z/r$. This means that each eigenstate has definite parity: $\psi_{n,\ell,m_\ell}(-\mathbf{r}) = (-1)^\ell \psi_{n,\ell,m_\ell}(\mathbf{r})$, and then $\left|\psi^*_{n,\ell,m_\ell}(-\mathbf{r})\right|^2 = \left|\psi^*_{n,\ell,m_\ell}(\mathbf{r})\right|^2$: The squared modulus of the eigenstates is an even function. This, in turn, implies that the $r_i$ component of the integrand is of the form $\left|\psi^*_{n,\ell,m_\ell}(\mathbf{r})\right|^2 r_i$, with $\mathbf{r} = (r_1, r_2, r_3) = (x, y, z)$. Those integrands are odd, whose integral over all $\mathbb{R}^3$ is equal to zero:

$$\langle n, \ell, m_\ell | \hat{\mathcal{H}}_{Id} | n, \ell, m_\ell \rangle = 0 \ . \tag{46}$$

The dipolar operator on the basis of the eigenstates of the free Hydrogen atom is

thus purely non-diagonal. In what follows, we will operate a very strong simplification of the model we are developing. We will consider that the atom is initially found in the electronic level $|g\rangle$, which is an eigenstate of $\mathcal{H}_0$, and is coupled to another bare eigenstate $|e\rangle$ by the electric dipolar matrix:

$$\langle g|\hat{\mathbf{d}}|e\rangle = \mathbf{d} \ , \tag{47}$$

and the dipolar Hamiltonian matrix element is

$$\langle g|\hat{\mathcal{H}}_{Id}|e\rangle = -\mathbf{d}\cdot\mathbf{E}(\mathbf{R},t) \ . \tag{48}$$

For a classical charge distribution with no net dipole prior to the action of the external electric field, the electric dipole is in the direction of the electric field. In our semiclassical model, the same will happen, but we will only be able to fully verify it afterwards (see the discussion on selection rules at the end of this section). So, by now, we will accept the fact that the electric dipole induced by the incident electric field is in the same direction as the electric field. If $|\mathbf{d}| = \mathrm{d} \in \mathbb{R}$ (we can always make it real by a proper choice of the phases of the eigenstates) and $|\mathbf{E}(\mathbf{R},t)| = \mathrm{E}(\mathbf{R},t)$, we write

$$\langle g|\hat{\mathcal{H}}_{Id}|e\rangle = -\mathrm{d}\,\mathrm{E}(\mathbf{R},t) = -\mathrm{d}\,\mathrm{E}_0\cos(\mathbf{k}\cdot\mathbf{R}-\omega_0 t) = \hbar\Omega_0\cos(\mathbf{k}\cdot\mathbf{R}-\omega_0 t) \ , \tag{49}$$

with $\Omega_0 = -\mathrm{d}\,\mathrm{E}_0/\hbar \in \mathbb{R}$ the **Rabi frequency** of the coupling between light and the electronic transition. We obtain the total hermitian electric dipolar Hamiltonian as

$$\hat{\mathcal{H}}_{\mathrm{Id}} = \hbar\Omega_0\cos\left(\mathbf{k}\cdot\mathbf{R}-\omega t\right)\left(|g\rangle\langle e| + |e\rangle\langle g|\right) \ . \tag{50}$$

If we consider the eigenstate $|e\rangle$ to be of higher energy than $|g\rangle$, and suppose the energy of $|g\rangle$ to be zero, then we write $\langle e|\mathcal{H}_0|e\rangle = E_e = \hbar\omega_0$, and the Hamiltonian $\mathcal{H}_0$ reduced to those two-levels can be written as $\mathcal{H}_0 = \hbar\omega_0|e\rangle\langle e|$. The total Hamiltonian in the base $|g\rangle, |e\rangle$ is written

$$\mathcal{H} = \hbar\begin{bmatrix} 0 & \Omega_0\cos\left(\phi_{\mathrm{L}}(\mathbf{R})-\omega t\right) \\ \Omega_0\cos\left(\phi_{\mathrm{L}}(\mathbf{R})-\omega t\right) & \omega_0 \end{bmatrix} \ . \tag{51}$$

In the equation above, we have replaced $\mathbf{k}\cdot\mathbf{R} = \phi_{\mathrm{L}}(\mathbf{R})$, the spatial phase of light.

Before we move to the discussion of the dynamics entailed by this Hamiltonian, it is instructive to define the raising $\hat{\sigma}_+$ and lowering $\hat{\sigma}_-$ operators for this two-level system:

$$\hat{\sigma}_- = |g\rangle\langle e| \tag{52}$$

$$\hat{\sigma}_+ = (\hat{\sigma}_-)^\dagger = |e\rangle\langle g| \tag{53}$$

The names of the operators come from their action on the internal state of the atom: The raising operator acts in the fundamental state as $\hat{\sigma}_+|g\rangle \rightarrow |e\rangle$, while the lowering operator acts in the excited state as $\hat{\sigma}_-|e\rangle \rightarrow |g\rangle$. The electric dipole operator can be written as a function of those operators as

$$\hat{\mathrm{d}} = \mathrm{d}\left(\hat{\sigma}_+ + \hat{\sigma}_-\right) \ , \tag{54}$$

and the interaction Hamiltonian becomes

$$\hat{\mathcal{H}}_{\mathrm{Id}} = \hbar\Omega_0 \cos\left(\phi_{\mathrm{L}}(\mathbf{R}) - \omega t\right)\left(\hat{\sigma}_+ + \hat{\sigma}_-\right) \ . \tag{55}$$

Is it valid to reduce the total atom to this two-level system? For several situations, yes, but we will be able to discuss it fully after deducing the consequences of this simple model. So, let us do the math!

# 2. Rabi oscillations and Bloch optical equations

## 2.1. Coherent Rabi oscillations

In this section, we will keep the center of mass of the atom in the origin of the coordinate system, that is, $\mathbf{R} = \mathbf{0}$, which makes $\phi_{\mathrm{L}}(\mathbf{R}) = 0$. The Schrödinger equation applied to the general state $|\psi\rangle = c_g(t)|g\rangle + c_e(t)|e\rangle$ in the two-level approximation, with $|c_g|^2 + |c_e|^2 = 1$, is written as (using $\cos\omega t = \left(\mathrm{e}^{i\omega t} + \mathrm{e}^{-i\omega t}\right)/2$)

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} c_g \\ c_e \end{pmatrix} = \hbar\begin{bmatrix} 0 & \frac{\Omega_0}{2}\left(\mathrm{e}^{i\omega t} + \mathrm{e}^{-i\omega t}\right) \\ \frac{\Omega_0}{2}\left(\mathrm{e}^{i\omega t} + \mathrm{e}^{-i\omega t}\right) & \omega_0 \end{bmatrix}\begin{pmatrix} c_g \\ c_e \end{pmatrix} \ . \tag{56}$$

This gives two coupled equations:

$$\begin{cases} \dfrac{\mathrm{d}c_g}{\mathrm{d}t} = -i\frac{\Omega_0}{2}\left\{\mathrm{e}^{i\omega t} + \mathrm{e}^{-i\omega t}\right\} c_e \\ \dfrac{\mathrm{d}c_e}{\mathrm{d}t} = -i\frac{\Omega_0}{2}\left\{\mathrm{e}^{i\omega t} + \mathrm{e}^{+i\omega t}\right\} c_g - i\omega_0 c_e \end{cases} \ . \tag{57}$$

### 2.1.1. Rotating Wave Approximation (RWA)

Some terms of the equation above are highly non-resonant. Intending to highlight them, let's make a change of reference frame. Let's write our wavefunction as:

$$|\psi\rangle = c_g(t)|g\rangle \, + \, \mathrm{e}^{-i\omega t} c'_e(t)|e\rangle \, , \tag{58}$$

by doing the transformation $c'_e = c_e \, \mathrm{e}^{i\omega t}$. The equations become

now, we say that we are put in the rotating frame with frequency $\omega$ of the incident light. The previous equations become

$$\begin{cases} \dfrac{\mathrm{d}c_g}{\mathrm{d}t} = -i\frac{\Omega_0}{2}\{1 + \mathrm{e}^{-2i\omega t}\}\, c_e \\ \dfrac{\mathrm{d}c'_e}{\mathrm{d}t} = -i\frac{\Omega_0}{2}\{1 + \mathrm{e}^{+2i\omega t}\}\, c_g + i(\omega - \omega_0)c'_e \end{cases} \tag{59}$$

If the light detuning $\Delta = \omega - \omega_0$ satisfies $|\Delta| \ll \omega, \omega_0$, then the terms $\mathrm{e}^{\pm 2i\omega t}$ are clearly much faster than the other frequencies of the system, $\Omega_0$ and $\Delta$. Then, we can substitute these exponentials by their time-averaged value: $\mathrm{e}^{-2i\omega t} \to \langle \mathrm{e}^{-2i\omega t} \rangle = 0$, $\mathrm{e}^{2i\omega t} \to \langle \mathrm{e}^{2i\omega t} \rangle = 0$. This approximation is known as the Rotating Wave Approximation. Finally, this allow us to write the coupled equations in a simpler and nicer form:

$$\frac{d}{dt}\begin{pmatrix} c_g \\ c'_e \end{pmatrix} = \begin{bmatrix} 0 & -i\dfrac{\Omega_0}{2} \\ -i\dfrac{\Omega_0}{2} & i\Delta \end{bmatrix} \begin{pmatrix} c_g \\ c'_e \end{pmatrix} \tag{60}$$

which configures a first order set of linear and homogeneous ODE's in the form

$$\frac{d}{dt}\begin{pmatrix} c_g \\ c'_e \end{pmatrix} = \hat{M} \begin{pmatrix} c_g \\ c'_e \end{pmatrix} \, . \tag{61}$$

The eigenvalues of $\hat{M}$ are:

$$\lambda_\pm = \frac{i[\Delta \pm \Omega]}{2} \, ; \, \Omega = \sqrt{\Delta^2 + |\Omega_0|^2} \, . \tag{62}$$

Considering the time evolution of the eigenvectors, the eigenvalues must satisfy $|\psi(t)\rangle = a_+ \exp(\lambda_+ t) + a_- \exp(\lambda_- t)$. Therefore

$$\begin{cases} c_g(t) = A\mathrm{e}^{\lambda_+ t} + B\mathrm{e}^{\lambda_- t} \\ c'_e(t) = C\mathrm{e}^{\lambda_+ t} + D\mathrm{e}^{\lambda_- t} \end{cases} \tag{63}$$

If the atomic initial state is the ground state, then $|\psi\rangle(0) = |g\rangle$, and we get

$$\begin{cases} A + B = 1 \\ C + D = 0 \end{cases} \tag{64}$$

We get

$$c_g(t) = Ae^{(\lambda_+ t)} + (1-A)e^{\lambda_- t} = e^{i\Delta t/2}\left[\cos\frac{\Omega t}{2} + i(2A-1)\sin\frac{\Omega t}{2}\right] \quad (65)$$

$$c_e'(t) = C\left(e^{\lambda_+ t} - e^{\lambda_- t}\right) = 2iC\, e^{i\Delta t/2}\sin\frac{\Omega t}{2} \quad (66)$$

$$(67)$$

The temporal evolution in Eq. (60) gives two relations between A and C:

$$\lambda_+ A = -i\frac{\Omega_0}{2}C \quad (68)$$

$$\lambda_-(1-A) = +i\frac{\Omega_0}{2}C \quad (69)$$

Solving both, we find

$$A = \frac{\Omega - \Delta}{2\Delta} \quad (70)$$

$$C = -\frac{\Omega_0}{2\Omega} \quad (71)$$

Finally, the state in a time $t$ and in the rotating light wave (of frequency $\omega$) reference frame is given by:

$$|\psi(t)\rangle = e^{i\Delta t/2}\left[\left(\cos\left(\frac{\Omega t}{2}\right) + \frac{i\Delta}{\Omega}\sin\left(\frac{\Omega t}{2}\right)\right)|g\rangle - \frac{i\Omega_0}{\Omega}\sin\left(\frac{\Omega t}{2}\right)|e\rangle\right]. \quad (72)$$

We find that the ground and excited states are coupled by light and are not stationary states of the system. The excited state population is

$$P_e = |c_e|^2 = \frac{\Omega_0^2}{\Omega_0^2 + \Delta^2}\sin^2\left(\frac{\Omega t}{2}\right). \quad (73)$$

From the sinusoidal behavior of the excited population comes the name "Rabi Oscillations". Using the concepts shown, we can use them to manipulate the atomic system. There are two kinds of light pulses that are extremely useful and very famous:

- $\pi$ pulse: Resonant excitation ($\Delta = 0$) with a pulse duration $\Delta t = \pi/\Omega_0$. This changes the state of the system, $|g\rangle \to |e\rangle$.
- $\pi/2$ pulse: Resonant excitation, $\Delta t = \pi/(2\Omega_0)$. If $|\psi(0)\rangle = |g\rangle$, then $|\psi(t = \pi/2\Omega_0)\rangle = \frac{|g\rangle - i|e\rangle}{\sqrt{2}}$, which is a coherent superposition of the ground and excited states.

### 2.1.2. Bloch Sphere

The most general state of a two-level system can be written as:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|g\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|e\rangle \tag{74}$$

We are not considering here a global phase; this do not change the physical properties of the system that we are concerned in this context. It is possible to make a bijection between the state and points of a sphere:

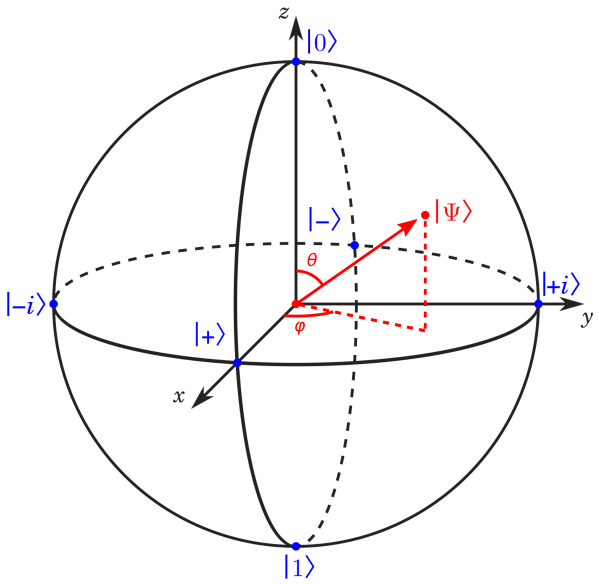



Figure 1: Bloch Sphere Representation.

In this representation, a Rabi oscillation is just a rotation around an axis of the sphere.

## 2.2. Including spontaneous emission: Optical Bloch equations

At this point, we want to add an important phenomena concerning the atomic system dynamics: spontaneous decay of the excited state. It is an **incoherent** process that occurs with **random** timing. This process will increase the entropy of our system, inducing loss of information and irreversibility. The state of the system will not be described anymore by a pure state represented by a ket $|\psi(t)\rangle$. We will need a generalized representation of the state of a system, called density matrix, which can account for states which are statistical mixtures of pure states.

### 2.2.1. The density matrix

For a physical system in a state $|\psi\rangle$, the density matrix is defined as

$$\hat{\rho} = |\psi\rangle\langle\psi| \ . \tag{75}$$

For our 2-level system, the density matrix is a 2x2 matrix, which we write as

$$\hat{\rho} = \begin{bmatrix} \sigma_{\mathrm{gg}} & \sigma_{\mathrm{ge}} \\ \sigma_{\mathrm{eg}} & \sigma_{\mathrm{ee}} \end{bmatrix} \ . \tag{76}$$

From this density matrix, we can get an expression for the average value of an operator:

$$\left\langle \hat{A} \right\rangle = \mathrm{Tr}[\hat{\rho}\hat{A}] \ . \tag{77}$$

Indeed, if we write the state in the basis of the eigenvectors of $\hat{A}$, $|\psi\rangle = \sum_i c_i |A_i\rangle$, where $\hat{A}|A_i\rangle = A_i|A_i\rangle$, we get

$$\begin{aligned} \mathrm{Tr}[\hat{\rho}\hat{A}] = \mathrm{Tr}[|\psi\rangle\langle\psi|\hat{A}] &= \mathrm{Tr}\left[\left(\sum_i c_i|A_i\rangle\right)\left(\sum_i c_i^*\langle A_i|\right)\hat{A}\right] \\ &= \mathrm{Tr}\left[\sum_i A_i|c_i|^2|A_i\rangle\langle A_i|\right] = \sum_i A_i|c_i|^2 = \left\langle \hat{A} \right\rangle \ . \end{aligned} \tag{78}$$

The main advantage of the density matrix formalism is that we can also represent states with entropy, that is, with a lack of information on the exact state of the system. As an example, let us suppose that of a physical system, we know that with probability $p_j$, it is found in state $|\psi_j\rangle$. The probabilities must sum, of course, $\sum_j p_j = 1$. Note that these probabilities are different from the probabilistic outcomes of a measurement in a quantum system, which are all contained in the state $|\psi\rangle$ of the system. Here, we have a *classical* lack of information about the system. Its density matrix will now be written

$$\hat{\rho} = \sum_j p_j |\psi_j\rangle\langle\psi_j| \ . \tag{79}$$

This state clearly represents a statistical mixture of states. To be convinced, we note that any physical prediction on the system will be just a weighted combination of the physical predictions on each state:

$$\left\langle \hat{A} \right\rangle = \mathrm{Tr}[\hat{\rho}\hat{A}] = \mathrm{Tr}[\sum_j p_j|\psi_j\rangle\langle\psi_j|\hat{A}] = \sum_j p_j \mathrm{Tr}[|\psi_j\rangle\langle\psi_j|\hat{A}] = \sum_j p_j \left\langle \hat{A} \right\rangle_j \ . \tag{80}$$

We see that the probabilistic outcomes add up without interference. This is another signature of the *incoherent* combination of possible states of the system.

The temporal evolution of the density matrix is given by the von Neumann equation:

$$\frac{d\hat{\rho}}{dt} = \frac{1}{i\hbar}[\hat{\mathcal{H}}, \hat{\rho}]\,. \tag{81}$$

We can verify rightaway that, for a pure state where $\hat{\rho} = |\psi\rangle\langle\psi|$, this equation implies the Schrödinger equation for $|\psi\rangle$.

The diagonal elements $\sigma_{\rm gg}$ and $\sigma_{\rm ee}$ are called the **populations**. They give the probability of finding the system in the state $|g\rangle$ or $|e\rangle$, respectively. Indeed, since the probability of finding the state in $|g\rangle$ is given by $\langle|g\rangle\langle g|\rangle$,

$$P_g = \langle|g\rangle\langle g|\rangle = {\rm Tr}[\hat{\rho}|g\rangle\langle g|] = \sigma_{\rm gg}\ . \tag{82}$$

The same can be verified for $P_e = \sigma_{\rm ee}$. This entails a general property of the density matrix:

$${\rm Tr}[\hat{\rho}] = 1\ . \tag{83}$$

For our two-level system, the equation above becomes

$$\sigma_{\rm gg} + \sigma_{\rm ee} = 1\ . \tag{84}$$

The nondiagonal elements are called **coherences**. They measure the degree of coherence of the superposition of states represented by the density matrix. The closer to zero, the more incoherent is the superposition. For a pure state $|\psi\rangle = c_g|g\rangle + c_e|e\rangle$, $\sigma_{\rm gg} = |c_g|^2$; $\sigma_{\rm ee} = |c_e|^2$; $\sigma_{\rm ge} = c_g c_e^*$; $\sigma_{\rm eg} = \sigma_{\rm ge}^*$. In general, the definition of the density matrix of Eq. (79) shows that $\hat{\rho}^\dagger = \hat{\rho}$; for our two-level system,

$$\sigma_{\rm eg} = \sigma_{\rm ge}^* \tag{85}$$

even when the state is in a statistical mixture.

### 2.2.2. Temporal evolution of the density matrix. The optical Bloch equations

We can use the von Neumann equation and the Hamiltonian of Eq. (51) to obtain the temporal evolution of the coefficients of the density matrix

$$\frac{\mathrm{d}\sigma_{\mathrm{ge}}}{\mathrm{d}t} = -i\frac{\Omega_0}{2}\left(\mathrm{e}^{-i(\phi_{\mathrm{L}}(\mathbf{R})-\omega t)} + \mathrm{e}^{i(\phi_{\mathrm{L}}(\mathbf{R})-\omega t)}\right)(\sigma_{\mathrm{ee}} - \sigma_{\mathrm{gg}}) + i\omega_0\sigma_{\mathrm{ge}} \tag{86a}$$

$$\frac{\mathrm{d}\sigma_{\mathrm{ee}}}{\mathrm{d}t} = -i\frac{\Omega_0}{2}\left(\mathrm{e}^{-i(\phi_{\mathrm{L}}(\mathbf{R})-\omega t)} + \mathrm{e}^{i(\phi_{\mathrm{L}}(\mathbf{R})-\omega t)}\right)(\sigma_{\mathrm{ge}} - \sigma_{\mathrm{eg}}) \tag{86b}$$

Due to the relation between the coefficients of the density matrix, we can write the other equations as a function of those two: $\frac{\mathrm{d}\sigma_{\mathrm{eg}}}{\mathrm{d}t} = \left(\frac{\mathrm{d}\sigma_{\mathrm{ge}}}{\mathrm{d}t}\right)^*$, $\frac{\mathrm{d}\sigma_{\mathrm{gg}}}{\mathrm{d}t} = -\frac{\mathrm{d}\sigma_{\mathrm{ee}}}{\mathrm{d}t}$.

Those equations must reproduce the same results as what was found in the previous section. But now, we can also add the effect of the spontaneous emission. In order to deduce the effect of spontaneous emission, we must make full model where the light is also quantized, which is beyond this description. You will thus trust me on what I will say next.

Essentially, spontaneous emission causes 2 effects: loss of excited population and loss of coherence. The loss of excited population is described by a decay rate independent of time, which is described by an additional decay term in the equation for the time derivative of the excited population, of the form $\left(\frac{\mathrm{d}\sigma_{\mathrm{ee}}}{\mathrm{d}t}\right)_{sp} - \Gamma\sigma_{\mathrm{ee}}$. At the same time, the coherences will also decay, but with a different rate, which we call $\gamma$, such that $\left(\frac{\mathrm{d}\sigma_{\mathrm{ge}}}{\mathrm{d}t}\right)_{sp} - \gamma\sigma_{\mathrm{ge}}$. With a full quantum model, one can show that the rate of coherence loss due to spontaneous emission is $\gamma_{sp} = \Gamma/2$. Other effects, such as atomic collisions, magnetic field fluctuations or inhomogeneities, can increase $\gamma$, such as in general we have $\gamma \geq \Gamma/2$. The full evolution equations, including spontaneous decay and other dephasing mechanisms, are written as

$$\frac{\mathrm{d}\sigma_{\mathrm{ge}}}{\mathrm{d}t} = -i\frac{\Omega_0}{2}\left(\mathrm{e}^{-i(\phi_{\mathrm{L}}(\mathbf{R})-\omega t)} + \mathrm{e}^{i(\phi_{\mathrm{L}}(\mathbf{R})-\omega t)}\right)(\sigma_{\mathrm{ee}} - \sigma_{\mathrm{gg}}) + i\omega_0\sigma_{\mathrm{ge}} - \gamma\,\sigma_{\mathrm{ge}} \tag{87a}$$

$$\frac{\mathrm{d}\sigma_{\mathrm{ee}}}{\mathrm{d}t} = -i\frac{\Omega_0}{2}\left(\mathrm{e}^{-i(\phi_{\mathrm{L}}(\mathbf{R})-\omega t)} + \mathrm{e}^{i(\phi_{\mathrm{L}}(\mathbf{R})-\omega t)}\right)(\sigma_{\mathrm{ge}} - \sigma_{\mathrm{eg}}) - \Gamma\,\sigma_{\mathrm{ee}} \tag{87b}$$

### 2.2.3. Rotating wave approximation

Just as in the coherent case, we put ourselves in the light rotating reference frame, pursuing the identification of the non-resonant terms. Doing the following transformation:

$$\begin{cases} \sigma_{\mathrm{ge}} \to \sigma'_{\mathrm{ge}} = \mathrm{e}^{-i\omega t}\sigma_{\mathrm{ge}} \\ \sigma_{\mathrm{eg}} \to \sigma'_{\mathrm{eg}} = \mathrm{e}^{+i\omega t}\sigma_{\mathrm{eg}} \end{cases} \tag{88}$$

we obtain the new equations

$$\frac{\mathrm{d}\sigma'_{\mathrm{ge}}}{\mathrm{d}t} = -i\frac{\Omega_0}{2}\left(\mathrm{e}^{-i(\phi_{\mathrm{L}}(\mathbf{R})-\omega t)} + \mathrm{e}^{i(\phi_{\mathrm{L}}(\mathbf{R})-\omega t)}\right)(\sigma_{\mathrm{ee}} - \sigma_{\mathrm{gg}})\,\mathrm{e}^{-i\omega t} + i(\omega_0 - \omega)\sigma_{\mathrm{ge}'} - \gamma\,\sigma_{\mathrm{ge}}$$

$$\frac{\mathrm{d}\sigma_{\mathrm{ee}}}{\mathrm{d}t} = -i\frac{\Omega_0}{2}\left(\mathrm{e}^{-i(\phi_{\mathrm{L}}(\mathbf{R})-\omega t)} + \mathrm{e}^{i(\phi_{\mathrm{L}}(\mathbf{R})-\omega t)}\right)(\mathrm{e}^{i\omega t}\sigma'_{\mathrm{ge}} - \mathrm{e}^{-i\omega t}\sigma'_{\mathrm{eg}}) - \Gamma\,\sigma_{\mathrm{ee}}$$

Doing the RWA after this modification: $e^{\pm 2i\omega t} \approx \langle e^{\pm 2i\omega t} \rangle = 0$ , we get the so-called **Bloch Optical Equations**:

$$\begin{cases} \dfrac{d\sigma'_{ge}}{dt} = -i\dfrac{\Omega_0}{2} e^{-i\phi_L(\mathbf{R})} (\sigma_{ee} - \sigma_{gg}) - (i\Delta + \gamma)\sigma'_{ge} \\ \dfrac{d\sigma_{ee}}{dt} = -i\dfrac{\Omega_0}{2} \left( e^{i\phi_L(\mathbf{R})} \sigma'_{ge} - e^{-i\phi_L(\mathbf{R})} \sigma'_{eg} \right) - \Gamma \sigma'_{ee} \end{cases} \tag{89}$$

In the absence of other forms of dephasing, $\gamma = \Gamma/2$. Notice that $\dfrac{d\sigma_{gg}}{dt} = -\dfrac{d\sigma_{ee}}{dt}$, also $\left(\dfrac{d\sigma_{ge}}{dt}\right) = \left(\dfrac{d\sigma_{eg}}{dt}\right)^*$.

### 2.2.4. Evolution in absence of laser excitation

The set of equations above depends on 4 frequencies ($\Omega_0$, $\Gamma$, $\omega$, $\gamma$), so one can imagine how messy the general solution is. Although, a notable case is an isolated atom ($\Omega_0 = 0$) initially in the excited state. Solving the Optical Bloch Equations results in:

$$\begin{cases} \sigma_{ee}(t) = e^{-\Gamma t} \\ \sigma_{gg}(t) = 1 - e^{-\Gamma t} \end{cases} \tag{90}$$

In this case, the lifetime of the excited state is given by $t_{sp} = 1/\Gamma$. Spontaneous emission does not create coherence; the density matrix for this system is diagonal. For any given time $t$, the state of the system is a statistical mixture.

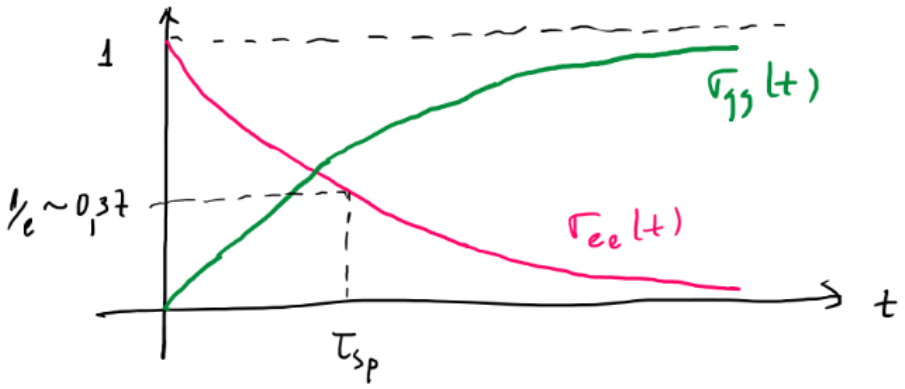



Figure 2: Evolution of the population in absence of incident light

### 2.2.5. Stationary state

The spontaneous emission induces irreversibility in the system. Because of irreversibility, the atomic state will evolve to a stationary state in timescales $t \gg \Gamma^{-1}$; Rabi oscillations will prevail only for a short, transient time. The dynamics of the excited and ground state populations are graphed above in Fig. 2. We can see it in the evolution of the excited population shown at Fig. 3

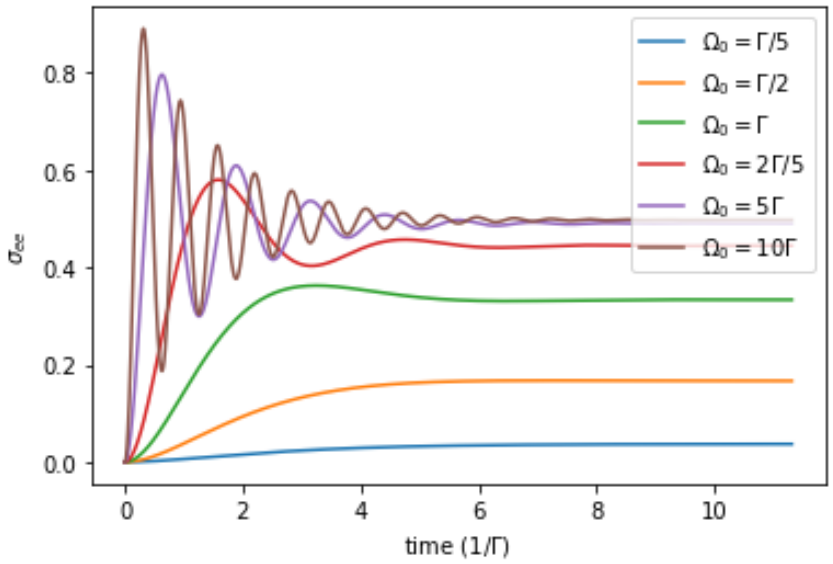



Figure 3: Evolution of the excited population for different Rabi frequencies

To obtain the stationary state in the general case (i.e., for any value of $\Omega(t)$), we set the temporal derivatives at Eq. ( 89) to 0 in the case $\gamma = \Gamma/2$. Writing the saturation parameter s as

$$s = \frac{1}{2}\frac{|\Omega_0|^2}{\Delta^2 + \Gamma^2/4}; \tag{91}$$

we obtain the population of the excited state in the stationary regime:

$$\sigma_{\mathrm{ee,st}} = \frac{1}{2}\frac{s}{1+s}. \tag{92}$$

In resonance, the saturation parameter becomes:

$$s_0 = s(\Delta = 0) = \frac{2\Omega_0^2}{\Gamma^2}. \tag{93}$$

We can write it as a function of the incident intensity. The incident intensity if the modulus of the Poynting vector,

$$I = \frac{\mathrm{E}_0^2}{2\mu_0 c}. \tag{94}$$

Replacing $\mathrm{E}_0 = \hbar\Omega_0/\mathrm{d}$ at the equation above, we find $I$ as a function of $\Omega_0$. Then, replacing in the expression for $s_0$,

$$s_0 = \frac{4\mu_0 c \mathrm{d}^2 I}{\hbar^2\Gamma^2} = \frac{I}{I_{\mathrm{sat}}}, \tag{95}$$

with

$$I_{\text{sat}} = \frac{(\hbar\Gamma)^2}{4\mu_0 c \mathrm{d}^2} \tag{96}$$

the saturation intensity of the transition. We can deduce from a full quantum model that the decay rate $\Gamma$ and the dipole moment d are related by $\mathrm{d}^2 = 3\pi\varepsilon_0\hbar\Gamma c^3/\omega_0^3$. Therefore,

$$I_{\text{sat}} = \frac{\hbar\Gamma\omega_0^3}{12\pi c^2} = \frac{\pi}{3}\frac{hc\Gamma}{\lambda_0^3}\,, \tag{97}$$

with $\lambda_0 = 2\pi c/\omega_0$ the wavelength of the atomic resonance. Typically, the order of magnitude of the saturation intensity is [mW/cm$^2$]. A new useful expression for $\sigma_{\text{ee,st}}$ can be written:

$$\sigma_{\text{ee,st}} = \frac{1}{2}\frac{s_0}{1+s_0}\frac{1}{1+\dfrac{4\Delta^2}{(1+s_0)\Gamma^2}}\,, \tag{98}$$

which is a Lorentzian curve with it's maximum value set in resonance. The minimum measurable width possible is $\Gamma$ - the reason why it is called the natural decay rate. As shown in Fig. 4, the FWHM (Full Width at Half Maximum) of the curve is

$$\Delta_{1/2} = \sqrt{\Gamma^2 + 2|\Omega_0|^2}\,. \tag{99}$$

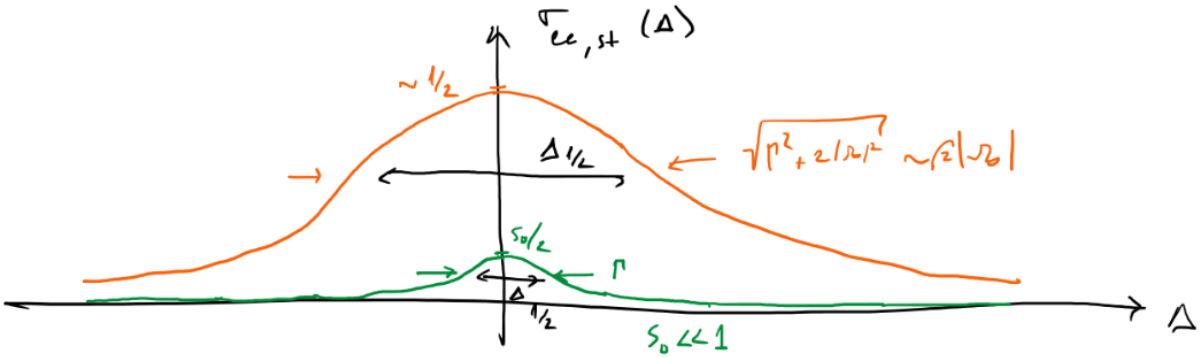



Figure 4: Stationary excited state population as a function of the detuning, for different resonant saturations

If $s_0 \ll 1$, $|\Gamma_0| \ll \Gamma \to \Delta_{1/2} \sim \Gamma$ (linear regime). But, if $s_0 \gg 1$, $|\Gamma_0| \gg \Gamma$, then $\Delta_{1/2} \sim \sqrt{2}|\Omega_0|$. This phenomenon is called **power broadening**.

Besides the expression obtained in equation 92, doing the same procedure results in the expression for the stationary coherence

$$\sigma'_{\text{ge,st}} = \frac{1}{1+s}\frac{i\Omega_0\,\mathrm{e}^{-i\phi_{\text{L}}(\mathbf{R})}}{2}\frac{1}{\Gamma/2 + i\Delta}\,. \tag{100}$$

From that, we can calculate the stationary value of the raising and lowering operators:

$$\langle\hat{\sigma}_+\rangle = \mathrm{Tr}[\hat{\rho}\,\hat{\sigma}_+] = \sigma_{ge,st} = \mathrm{e}^{i\omega t}\sigma'_{ge,st} = \frac{1}{1+s}\,\frac{\Omega_0\,\mathrm{e}^{-i(\phi_\mathrm{L}(\mathbf{R})-\omega t)}}{2}\,\frac{1}{\Delta - i\Gamma/2} \quad (101)$$

$$\langle\hat{\sigma}_-\rangle = \mathrm{Tr}[\hat{\rho}\,\hat{\sigma}_+] = \sigma_{eg,st} = \mathrm{e}^{-i\omega t}\sigma'_{eg,st} = \frac{1}{1+s}\,\frac{\Omega_0\,\mathrm{e}^{i(\phi_\mathrm{L}(\mathbf{R})-\omega t)}}{2}\,\frac{1}{\Delta + i\Gamma/2} \quad (102)$$

They are used to calculate the average value of the stationary atomic dipole moment:

$$\langle\hat{\mathrm{d}}\rangle_\mathrm{st} = \mathrm{d}\,(\langle\hat{\sigma}_+\rangle + \langle\hat{\sigma}_-\rangle) = \frac{\mathrm{d}\sqrt{s}}{1+s}\left[\Delta\cos\left(\phi_\mathrm{L}(\mathbf{R}) - \omega t\right) + \frac{\Gamma}{2}\sin\left(\phi_\mathrm{L}(\mathbf{R}) - \omega t\right)\right] . \quad (103)$$

For $s \ll 1$, $\langle\hat{\mathrm{d}}\rangle_\mathrm{st} \propto \mathrm{E}_0$, and for $s \gg 1$, $\langle\hat{\mathrm{d}}\rangle_\mathrm{st} \to 0$. This does not mean that the dipole is not being excited; but it's phase is lost in the high saturation regime.

## 2.3. The semiclassical Hamiltonian in the RWA

We see from the expressions for $\langle\hat{\sigma}_+\rangle$ and $\langle\hat{\sigma}_-\rangle$ that they evolve with the frequencies $\pm\omega$. In order to obtain an expression of the interaction Hamiltonian that eliminates the fast rotating terms, we rewrite the Hamiltonian of Eq. (55) as

$$\hat{\mathcal{H}}_\mathrm{Id} = \frac{\hbar\Omega_0}{2}\left(\mathrm{e}^{i(\phi_\mathrm{L}(\mathbf{R})-\omega t)} + \mathrm{e}^{-i(\phi_\mathrm{L}(\mathbf{R})-\omega t)}\right)(\hat{\sigma}_+ + \hat{\sigma}_-) . \quad (104)$$

It is now easy to see that, from the four terms above, two of them will evolve with a rapid phase term of the form $\mathrm{e}^{\pm 2i\omega t}$, due to the dependence of the operators themselves on time. We can perform the RWA directly on the Hamiltonian, eliminating exactly the terms that give rise to the terms that were eliminated while calculating the dynamics in the last section. We get the interaction Hamiltonian in the RWA:

$$\hat{\mathcal{H}}_\mathrm{Id,RWA} = \frac{\hbar\Omega_0(\hat{\mathbf{R}})}{2}\left(\mathrm{e}^{i(\phi_\mathrm{L}(\hat{\mathbf{R}})-\omega t)}\,\hat{\sigma}_+ + \mathrm{e}^{-i(\phi_\mathrm{L}(\hat{\mathbf{R}})-\omega t)}\,\hat{\sigma}_-\right) . \quad (105)$$

## 2.4. Selection rules for electric dipolar transitions

We will now consider in detail which transitions are allowed by the dipolar Hamiltonian.

In first order, the interaction Hamiltonian in the electric dipolar approximation, $\hat{\mathcal{H}}_{Id}$, can cause transitions from a level $|n,\ell,m_\ell\rangle$ to another level $|n,\ell',m'_\ell\rangle$ whenever the off-diagonal matrix element

$$\langle n',\ell',m'_\ell|\hat{\mathcal{H}}_{Id}|n,\ell,m_\ell\rangle \quad (106)$$

is not zero. But most of those non-diagonal matrix elements are actually zero: This

implies that there are only a few electronic transitions allowed by the electric dipolar in first order: Those that respect the **selection rules**. We will now explicitly work out these selection rules.

The atomic eigenstates defined by Eq. (27) take the $z$ axis as the quantization direction (i.e., the direction of the angular momentum that defines the $m_\ell$ quantum number). Within this definition, we define three orthonormal directions for the electric field of the incoming light, in the complex representation for the electric field:

$$\boldsymbol{\sigma}_+ = \frac{\mathbf{e}_x + i\mathbf{e}_y}{\sqrt{2}} \; ; \tag{107}$$

$$\boldsymbol{\sigma}_- = \frac{\mathbf{e}_x - i\mathbf{e}_y}{\sqrt{2}} \; ; \tag{108}$$

$$\boldsymbol{\pi} = \mathbf{e}_z \; . \tag{109}$$

In the expressions above, $\mathbf{e}_x$ (resp. $\mathbf{e}_y$, $\mathbf{e}_z$) stands for the unitary vector in the $x$ (resp. $y$, $z$) direction. We can decompose the amplitude of the electric field on this basis as

$$\mathbf{E}_0 = \mathrm{E}_+ \boldsymbol{\sigma}_+ + \mathrm{E}_- \boldsymbol{\sigma}_- + \mathrm{E}_z \boldsymbol{\pi} = \mathbf{E}_+ + \mathbf{E}_- + \mathbf{E}_z \; . \tag{110}$$

The total electric field in the complex configuration is written as

$$\mathbf{E}(\mathbf{0}, \mathbf{t}) = \mathrm{Re}[\mathbf{E}_0 \mathrm{e}^{-i\omega t)}] \; . \tag{111}$$

We will explicitly verify how the electric field for the complex polarizations $\sigma_\pm$ behaves:

$$\mathbf{E}(\mathbf{0}, \mathbf{t}) = \mathrm{Re}[\mathbf{E}_\pm \mathrm{e}^{-i\omega t}] = \mathrm{Re}[\mathrm{E}_\pm \boldsymbol{\sigma}_\pm \mathrm{e}^{-i\omega t}] = \frac{\mathrm{E}_\pm}{\sqrt{2}} \left[\cos(\omega t)\, \mathbf{e}_x \pm \sin(\omega t)\, \mathbf{e}_y\right] \; . \tag{112}$$

This means that the polarizations $\sigma_\pm$ represent the two orthogonal circular polarizations on the plane $xy$.

We note that not all polarizations are allowed for a light field with a definite $\mathbf{k}$. For example, for light propagating in $z$ direction, $\mathbf{k} = k\,\mathbf{e}_z$, the electric field direction must be contained in the $xy$ plane, and only the $\boldsymbol{\sigma}_+$ and $\boldsymbol{\sigma}_-$ polarizations are allowed. In any case, only 2 orthogonal polarizations are allowed for definite $\mathbf{k}$, which will be a linear combination of the three polarizations defined above. It is also important to note that these polarizations do not correspond directly to the **helicity** of light. Helicity corresponds to the polarization of light (or to the spin of its constituent photons) projected in its propagation direction. This is different from the definition above, rather related to

a fixed quantization direction. In particular, light of positive helicity and propagating with $\mathbf{k} = +k\,\mathbf{e}_z$ has $\boldsymbol{\sigma}_+$ polarization on the basis defined above; light of positive helicity propagating at $\mathbf{k} = -k\,\mathbf{e}_z$ has $\boldsymbol{\sigma}_-$ polarization on the same basis; and light of positive helicity propagating at $\mathbf{k} = +k\,\mathbf{e}_x$ will have a combination of all three polarizations.

Let us work out the selection rules for each one of the different polarizations of light:

### 2.4.1. Selection rules for $\boldsymbol{\sigma}_+$ polarization

For $\boldsymbol{\sigma}_+$ polarization, the dipolar Hamiltonian is written as

$$\hat{\mathcal{H}}_{\mathrm{Id}}^{(\boldsymbol{\sigma}_+)} = -\hat{\mathbf{d}}\cdot\mathbf{E}_+(\mathbf{0},t) = e\frac{\mathrm{E}_+}{\sqrt{2}}\left(\hat{x}\,\cos(\omega t) + \hat{y}\,\sin(\omega t)\right)$$
$$= \frac{e\mathrm{E}_+}{2}\left[\mathrm{e}^{-i\omega t}\left(\frac{\hat{x}+i\hat{y}}{\sqrt{2}}\right) + \mathrm{e}^{i\omega t}\left(\frac{\hat{x}-i\hat{y}}{\sqrt{2}}\right)\right]. \quad (113)$$

We have learned before (see Eq. (105)) that the term that rotates with $\mathrm{e}^{-i\omega t}$ is quasi-resonant to transitions from the ground to the excited state; and the term that rotates with $\mathrm{e}^{i\omega t}$ is quasi-resonant to transitions from the excited to the ground state. Now, we need to see what is the effect of each one of the terms $\frac{\hat{x}\pm i\hat{y}}{\sqrt{2}}$ on the electronic wavefunctions. For that, let us calculate the nondiagonal matrix elements of those operators:

$$\langle n',\ell',m'_\ell|\left(\frac{\hat{x}\pm i\hat{y}}{\sqrt{2}}\right)|n,\ell,m_\ell\rangle = \int_{\mathbb{R}^3}\mathrm{d}^3\mathbf{r}\,\psi^*_{n',\ell',m'_\ell}(\mathbf{r})\,\frac{(x\pm iy)}{\sqrt{2}}\,\psi_{n,\ell,m_\ell}(\mathbf{r})$$
$$= \int_0^\infty \mathrm{d}r\,r^2\,R^*_{n'\ell'}(r)R_{n\ell}(r)\int_0^\pi \mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\varphi\,\sin\theta[Y_{\ell'}^{m'_\ell}(\theta,\varphi)]^*Y_\ell^{m_\ell}(\theta,\varphi)\,\frac{r\sin\theta\,\mathrm{e}^{\pm i\varphi}}{\sqrt{2}}\,. \quad (114)$$

We can write part of the integrand as a function of spherical harmonics, as

$$\frac{(x\pm iy)}{\sqrt{2}} = \frac{r\sin\theta\,\mathrm{e}^{\pm i\varphi}}{\sqrt{2}} = \mp\sqrt{\frac{4\pi}{3}}\,r\,Y_1^{\pm1}\,. \quad (115)$$

Replacing it in Eq. (114), we get

$$\langle n',\ell',m'_\ell|\left(\frac{\hat{x}\pm i\hat{y}}{\sqrt{2}}\right)|n,\ell,m_\ell\rangle =$$
$$\mp\sqrt{\frac{4\pi}{3}}\underbrace{\int_0^\infty \mathrm{d}r\,r^3\,R^*_{n'\ell'}(r)R_{n\ell}(r)}_{\text{Radial}}\;\underbrace{\int_0^\pi\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\varphi\,\sin\theta[Y_{\ell'}^{m'_\ell}(\theta,\varphi)]^*\,Y^1_{\pm1}(\theta,\varphi)\,Y_\ell^{m_\ell}(\theta,\varphi)}_{\text{Angular}}\,. \quad (116)$$

The angular part of this integral is of the form

$$I_{l\,1\,l'}^{m_l\,m\,m_{l'}} = \int_0^{\pi} \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\varphi [Y_l^{m_l}(\theta,\varphi)]^* Y_1^m(\theta,\varphi) Y_{l'}^{m_l'}(\theta,\varphi)\,. \tag{117}$$

This integral has the following property: it is different from 0 only if

$$\begin{cases} |\ell - \ell'| = 1 \\ m'_\ell = m_\ell + m \end{cases} \tag{118}$$

This means that the polarization $\boldsymbol{\sigma}_+$ will only cause a transition between two levels with $|\Delta\ell| = 1$; and it will increase $m_\ell$ by 1 if the transition goes from a level of lower energy to a level of higher energy (or decrease $m_\ell$ by 1 if the transition goes from a level of higher energy to a level of lower energy).

### 2.4.2. Selection rules for $\boldsymbol{\sigma}_-$ polarization

For $\boldsymbol{\sigma}_-$ polarization, the dipolar Hamiltonian is written as

$$\begin{aligned} \hat{\mathcal{H}}_{\mathrm{Id}}^{(\boldsymbol{\sigma}_-)} = -\hat{\mathbf{d}} \cdot \mathbf{E}_-(\mathbf{0},t) = e\frac{\mathrm{E}_+}{\sqrt{2}} \left(\hat{x}\,\cos(\omega t) - \hat{y}\,\sin(\omega t)\right) \\ = e\frac{\mathrm{E}_+}{2}\left[\mathrm{e}^{-i\omega t}\left(\frac{\hat{x}-i\hat{y}}{\sqrt{2}}\right) + \mathrm{e}^{i\omega t}\left(\frac{\hat{x}+i\hat{y}}{\sqrt{2}}\right)\right]. \end{aligned} \tag{119}$$

We see that the role of the terms $\left(\frac{\hat{x}-i\hat{y}}{\sqrt{2}}\right)$ is inverted with respect to the $\boldsymbol{\sigma}_+$ polarization. We can thus jump to the conclusions: the polarization $\boldsymbol{\sigma}_-$ will only cause a transition between two levels with $|\Delta\ell| = 1$; and it will decrease $m_\ell$ by 1 if the transition goes from a level of lower energy to a level of higher energy (or increase $m_\ell$ by 1 if the transition goes from a level of higher energy to a level of lower energy).

### 2.4.3. Selection rules for $\boldsymbol{\pi}$ polarization

For $\boldsymbol{\pi}$ polarization, the dipolar Hamiltonian is written as

$$\hat{\mathcal{H}}_{\mathrm{Id}}^{(\boldsymbol{\pi})} = -\hat{\mathbf{d}} \cdot \mathbf{E}_z(\mathbf{0},t) = e\hat{z}\,\mathrm{E}_z\cos(\omega t) = e\,\hat{z}\,\frac{\mathrm{E}_z}{2}\left(\mathrm{e}^{i\omega t} + \mathrm{e}^{-i\omega t}\right)\,. \tag{120}$$

Now, both the term quasi-resonant with the excitation to an upper level and the term quasi-resonant with the excitation to a lower level are proportional to the same operator $\hat{z}$. We can write $z$ in spherical coordinates as

$$\frac{z}{r} = \sqrt{\frac{4\pi}{3}} r\, Y_1^0 \tag{121}$$

The matrix element will now be proportional to the term

$$\langle n', \ell', m'_\ell | \hat{\mathcal{H}}_{Id}^{(\boldsymbol{\pi})} | n, \ell, m_\ell \rangle =$$
$$- \sqrt{\frac{4\pi}{3}} \underbrace{\int_0^\infty \mathrm{d}r\, r^3\, R^*_{n'\ell'}(r) R_{n\ell}(r)}_{\text{Radial}} \underbrace{\int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi \sin\theta [Y_{\ell'}^{m'_\ell}(\theta,\varphi)]^*\, Y_1^0(\theta,\varphi)\, Y_\ell^{m_\ell}(\theta,\varphi)}_{\text{Angular}} . \tag{122}$$

We can also apply the result for the integral $I_{\ell 1 \ell'}^{m_\ell m m'_\ell}$, and we find that for $\boldsymbol{\pi}$ polarization, the nondiagonal element will be nonzero if $|\Delta\ell| = |\ell - \ell'| = 1$, and if $m'_\ell = m_\ell$.

We can conclude that, in general, the electric dipolar transition only causes transitions between levels separated by $|\Delta\ell| = |\ell - \ell'| = 1$, regardless of the light polarization: A level S can only be coupled to a level P, a level P to a level S or D, and so on. As for the selection rule on the $m_\ell$ quantum number, it shows that the change in $m_\ell$ depends on the incoming light polarization, as shown in Fig. (5).

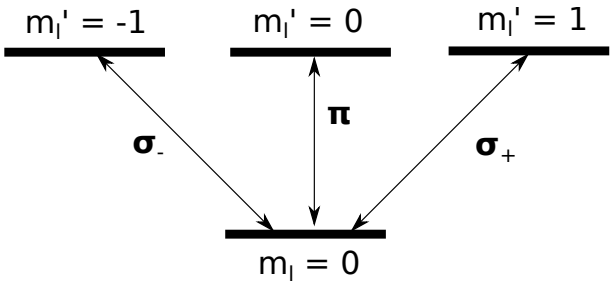



Figure 5: Transitions allowed by polarizations $\boldsymbol{\sigma}_+$, $\boldsymbol{\sigma}_-$ and $\boldsymbol{\pi}$.

These selection rules are directly related to the **conservation of angular momentum**. A photon with polarization $\boldsymbol{\sigma}_+$ has angular momentum of $+\hbar$ in the $+\mathbf{e}_z$ direction, while a photon with polarization $\boldsymbol{\sigma}_-$ has angular momentum of $-\hbar$ in the $+\mathbf{e}_z$ direction. This angular momentum is absorbed by the electron of the atom. This means that the only allowed transitions for the electron are S $\leftrightarrow$ P $\leftrightarrow$ D $\leftrightarrow$ F. The electron absorbs the photon and it's angular momentum, so these properties are directly related to angular momentum conservation.

# 3. Light forces

In the previous section, we calculated the effect of incident light on the electronic degrees of freedom (also called internal degrees of freedom) of the atom; here, we will ask what effect the incident light has on the position $\mathbf{R}$ and momentum $\mathbf{P}$ of the atom's center of mass, also called external degrees of freedom. To do this, we must now let $\mathbf{R}$ to be a new dynamical variable of the system. If the atom is treated in quantum mechanics, then $\mathbf{R} \to \hat{\mathbf{R}}$. We must also consider its canonical momentum $\hat{\mathbf{P}}$, and add to the Hamiltonian

of eq. (51) the kinetic energy of the center of mass of the atom of mass $m$, so that the total Hamiltonian becomes

$$\hat{\mathcal{H}}_{RWA} = \hbar\omega_0|e\rangle\langle e| + \frac{\hbar\Omega_0(\hat{\mathbf{R}})}{2}\left(\mathrm{e}^{i(\phi_\mathrm{L}(\hat{\mathbf{R}})-\omega t)}\,\hat{\sigma}_+ + \mathrm{e}^{-i(\phi_\mathrm{L}(\hat{\mathbf{R}})-\omega t)}\,\hat{\sigma}_-\right) + \frac{\hat{\mathbf{P}}^2}{2M}\,. \tag{123}$$

We note also that now, we allow the Rabi frequency to depend on the position of the center of mass. This accounts for incoming light fields whose amplitude depend on the position:

$$\mathrm{E}(\mathbf{R},t) = \mathrm{E}_0(\mathbf{R})\,\cos(\phi_\mathrm{L}(\mathbf{R}) - \omega_0 t)\;, \tag{124}$$

and then $\Omega_0(\mathbf{R}) = \mathrm{d}\,\mathrm{E}_0(\mathbf{R})/\hbar$. For the theoretical ideal case of plane waves, the amplitude is constant; but for the experimentally relevant case of an incoming laser beam, described by a Gaussian profile, we have

$$\mathrm{E}(\mathbf{R},t) = \mathrm{E}_\mathrm{max}\,\frac{w_0}{w(Z)}\mathrm{e}^{\frac{-\rho^2}{w(Z)^2}}\mathrm{e}^{\,i\left(kZ+k\frac{\rho^2}{2R_f(Z)}-\psi_G(Z)\right)-i\omega t}\,. \tag{125}$$

In the above equation, we suppose that the beam is propagating in the $z$ direction. The parameter $w_0$ is the waist of the beam in the focal position $Z = 0$, and the functions

$$w(Z) = w_0\sqrt{1+\left(\frac{Z}{Z_R}\right)^2} \tag{126}$$

$$R_f(Z) = Z\left[1-\left(\frac{Z_R}{Z}\right)^2\right] \tag{127}$$

$$\psi_G(Z) = \arctan\left(Z/Z_R\right) \tag{128}$$

are the waist of the beam, the radius of the wavefront, and the Gouy phase, respectively, with $Z_R = \pi w_0^2/\lambda$ the Rayleigh length. We identify from the equation above

$$\mathrm{E}_0(\mathbf{R}) = \mathrm{E}_\mathrm{max}\,\frac{w_0}{w(Z)}\mathrm{e}^{\frac{-\rho^2}{w(Z)^2}} \tag{129}$$

$$\phi_\mathrm{L}(\mathbf{R}) = kZ + k\tfrac{\rho^2}{2R(Z)} - \psi_G(Z)\;. \tag{130}$$

More more complex laser beam configurations, the phase and amplitude can still vary; in general, we will describe any light field impinging on the atom by the expression of Eq. (124), leading to the Hamiltonian (123).

The dynamics of the atom determined by the above Hamiltonian is, in the general

case, complicated because it couples the internal and external degrees of freedom of the atom, and because the atom can be delocalized in time and space. We will place ourselves in a simplified configuration of **localized atoms in space and momentum**.

- **Localization in space:** The typical length scale of radiation is $\lambda$. We consider an atom localized in space if the typical extension of its wavefunction $\Delta R$ is much smaller than $\lambda$: $\Delta R$
  $\lambda$. This guarantees that the whole wavefunction of the atom sees the same phase of light.

- **Localization in momentum space:** Spectrally, the atoms are considered localized in momentum if the whole atomic wavefunction sees approximately the light with same detuning. This means that the momentum spread, $\Delta P$, must be smaller than the momentum spread that causes a Doppler shift of the resonance equal to the resonance width $\Gamma$: $\Delta_D = k\Delta v = k\Delta P/M \ll \Gamma$, or still $\Delta P \ll M\Gamma/k$.

  We can combine both conditions via the uncertainty principle:

$$\frac{\hbar}{2} \leq \Delta R\, \Delta P \ll \frac{M\Gamma\lambda}{k} = \frac{2\pi M\Gamma}{k^2} \ . \tag{131}$$

  Rearranging the condition above, we get, apart from a numerical factor of order 1,

$$\frac{M}{\hbar k^2} \ll \Gamma^{-1} \ . \tag{132}$$

This is called the **condition of broad resonance**. It is a hierarchy of two different timescales. The timescale on the left is the timescale for the atomic velocity to change by the Doppler quantity, $k\Delta v \sim \Gamma$, which can be considered to be a typical timescale for the external dynamics to have an impact on the internal dynamics. Indeed, if each photon carries a momentum $\hbar k$, and if the typical rate of photon absorption is $\Gamma$, then in a time $\Delta t$ the atom has absorbed $\Gamma\,\Delta t$ photons with a total momentum $\Gamma\,\Delta t\,\hbar k$ and a change in velocity $\Gamma\,\Delta t\hbar k/M$, establishing a Doppler timescale $k(\Gamma\,\Delta t_{ext}\,\hbar k/M) \sim \Gamma$, or still $\Delta t_{ext} \sim M/(\hbar k^2)$. On the other hand, the tmescale for the internal dynamics is given by the lifetime $\tau \sim \Gamma^{-1}$ of the internal levels. Thus, the condition of broad resonance can be reinterpreted as

$$t_{ext} \gg \tau \ . \tag{133}$$

This entails a hierarchy in the dynamics of the system. In this regime, the functions of $\mathbf{R}$ present in the Bloch equations (89) are practically constant during times of the order of $\Gamma^{-1}$, so we can solve first the internal dynamics, by considering the external variables constant, and then solve for the external dynamics, supposing that the internal state is always quasi-stationary, adiabatically following the external state dynamics. Moreover,

the conditions for atom localization in space and momentum allow to replace the operators $\hat{\mathbf{R}}$ and $\hat{\mathbf{P}}$ by their average values while solving for the internal dynamics. This means that the procedure used to obtain the stationary behavior of the internal degrees of freedom is still valid, but now the steady state obtained will be a slow function of the external variables.

It is important to note that not all atomic transitions verify the decoupling of timescales. Intuitively, if a transition is very narrow, that is, with a small $\Gamma$, the internal timescale $\Gamma^{-1}$ will be slow, and we may experience a breakdown of decoupling.

Since we have already solved for the internal dynamics in the previous section, we now turn to the external dynamics, considering the position and momentum of the center of mass as operators again, in order to obtain their dynamics.

We calculate the time derivative of the average value of the position via the Ehrenfest theorem:

$$\frac{\mathrm{d}\langle\mathbf{R}\rangle}{\mathrm{d}t} = \frac{1}{i\hbar}\langle[\hat{\mathbf{R}}, \hat{\mathrm{H}}_{\mathrm{RWA}}]\rangle + \cancelto{0}{\left\langle\frac{\partial\hat{\mathbf{R}}}{\partial t}\right\rangle} = \frac{\langle\hat{\mathbf{P}}\rangle}{M} \ .$$

This means that the speed is a simple function of the average momentum. The first derivative of the momentum is thus the net force:

$$\mathbf{F} = \frac{\mathrm{d}\langle\mathbf{P}\rangle}{\mathrm{d}t} \ . \tag{134}$$

The time derivative of the momentum is found via the Ehrenfest theorem:

$$\mathbf{F} = \frac{1}{i\hbar}\langle[\hat{\mathbf{P}}, \hat{\mathrm{H}}_{\mathrm{RWA}}]\rangle + \cancelto{0}{\left\langle\frac{\partial\hat{\mathbf{P}}}{\partial t}\right\rangle} = \frac{1}{i\hbar}\langle[\hat{\mathbf{P}}, \hat{\mathrm{H}}_{\mathrm{d,RWA}}]\rangle \ .$$

In the position representation, $\hat{\mathbf{P}} = -i\hbar\boldsymbol{\nabla}_{\mathbf{R}}$, and we obtain

$$\mathbf{F} = -\langle\boldsymbol{\nabla}_{\mathbf{R}}\,\hat{\mathrm{H}}_{\mathrm{Id,RWA}}\rangle \ . \tag{135}$$

We obtain directly

$$\begin{aligned}\mathbf{F} = -\frac{\hbar}{2}\Big\langle\, \boldsymbol{\nabla}_{\mathbf{R}}\Omega_0(\hat{\mathbf{R}})\left(\hat{\sigma}_+\,\mathrm{e}^{i\left(\phi_{\mathrm{L}}(\hat{\mathbf{R}})-\omega_{\mathrm{L}}t\right)} + \hat{\sigma}_-\,\mathrm{e}^{-i\left(\phi_{\mathrm{L}}(\hat{\mathbf{R}})-\omega_{\mathrm{L}}t\right)}\right)\Big\rangle \\ - \frac{i\hbar}{2}\left\langle\Omega_0(\hat{\mathbf{R}})\left(\hat{\sigma}_+\,\mathrm{e}^{i\left(\phi_{\mathrm{L}}(\hat{\mathbf{R}})-\omega_{\mathrm{L}}t\right)} - \hat{\sigma}_-\,\mathrm{e}^{-i\left(\phi_{\mathrm{L}}(\hat{\mathbf{R}})-\omega_{\mathrm{L}}t\right)}\right)\,\boldsymbol{\nabla}_{\mathbf{R}}\phi_{\mathrm{L}}(\hat{\mathbf{R}})\right\rangle \ .\end{aligned}$$

Considering that the atom is localized, we replace $\hat{\mathbf{R}}$ by $\mathbf{R}$ and we find

$$\mathbf{F} = -\frac{\hbar\,\boldsymbol{\nabla}_{\mathbf{R}}\Omega_0(\mathbf{R})}{2}\left(\langle\hat{\sigma}_+\rangle\,\mathrm{e}^{i(\phi_{\mathrm{L}}(\mathbf{R})-\omega_{\mathrm{L}}t)} + \langle\hat{\sigma}_-\rangle\,\mathrm{e}^{-i(\phi_{\mathrm{L}}(\mathbf{R})-\omega_{\mathrm{L}}t)}\right) - \frac{i\hbar\,\Omega_0(\mathbf{R})}{2}\left(\langle\hat{\sigma}_+\rangle\,\mathrm{e}^{i(\phi_{\mathrm{L}}(\mathbf{R})-\omega_{\mathrm{L}}t)} - \langle\hat{\sigma}_-\rangle\,\mathrm{e}^{-i(\phi_{\mathrm{L}}(\mathbf{R})-\omega_{\mathrm{L}}t)}\right)\boldsymbol{\nabla}_{\mathbf{R}}\phi_{\mathrm{L}}(\mathbf{R})\ .$$

We know already the average values of the raising and lowering operators, and we can work out the expressions in parenthesis:

$$\left(\langle\hat{\sigma}_+\rangle\,\mathrm{e}^{i(\phi_{\mathrm{L}}(\mathbf{R})-\omega_{\mathrm{L}}t)} + \langle\hat{\sigma}_-\rangle\,\mathrm{e}^{-i(\phi_{\mathrm{L}}(\mathbf{R})-\omega_{\mathrm{L}}t)}\right) = \frac{1}{1+s}\frac{\Omega_0\Delta}{\Delta^2+\left(\frac{\Gamma}{2}\right)^2} \tag{136}$$

$$\left(\langle\hat{\sigma}_+\rangle\,\mathrm{e}^{i(\phi_{\mathrm{L}}(\mathbf{R})-\omega_{\mathrm{L}}t)} - \langle\hat{\sigma}_-\rangle\,\mathrm{e}^{-i(\phi_{\mathrm{L}}(\mathbf{R})-\omega_{\mathrm{L}}t)}\right) = \frac{i}{1+s}\frac{\Omega_0\frac{\Gamma}{2}}{\Delta^2+\left(\frac{\Gamma}{2}\right)^2} \tag{137}$$

Based on Eq. (92), we can also write

$$\sigma_{\mathrm{ee,st}} = \frac{1}{2}\frac{s}{1+s} = \frac{1}{2}\,\frac{1}{1+s}\,\frac{\Omega_0^2}{\Delta^2+\left(\frac{\Gamma}{2}\right)^2}\ , \tag{138}$$

And we get the total force

$$\mathbf{F} = -2\hbar\Delta\sigma_{\mathrm{ee,st}}\,\frac{\boldsymbol{\nabla}_{\mathbf{R}}\Omega_0(\mathbf{R})}{\Omega_0(\mathbf{R})} + \hbar\Gamma\sigma_{\mathrm{ee,st}}\,\boldsymbol{\nabla}_{\mathbf{R}}\phi(\mathbf{R}) = \mathbf{F}_d + \mathbf{F}_{\mathrm{RP}}\ , \tag{139}$$

where $\mathbf{F}_{\mathrm{d}}$ is the dipolar force and $\mathbf{F}_{\mathrm{RP}}$ the radiative pressure force. Let us discuss them separately.

## 3.1. The radiation pressure force

The radiation pressure force is given by

$$\mathbf{F}_{\mathrm{RP}} = \hbar\Gamma\sigma_{\mathrm{ee,st}}\,\boldsymbol{\nabla}_{\mathbf{R}}\phi_{\mathrm{L}}(\mathbf{R})\ . \tag{140}$$

In order to interpret the expression for the radiation pressure, we note that the spatial phase of light $\phi_{\mathrm{L}}(\mathbf{R})$ is equal to $\mathbf{k}\cdot\mathbf{R}$ for a plane wave, or even $kz$ for a wave vector in the $z$ direction, $\mathbf{k} = k\,\mathbf{e}_z$; while in the case of the Gaussian beam (see eq. (125)), it has two additional terms. However, these two additional terms have a typical spatial variation determined by the typical distance $Z_R \gg \lambda$, whereas the term $\mathbf{k}\cdot\mathbf{R}$ varies on the spatial scale of the wavelength of light, $\lambda$; thus, in both cases (and, in general, for any light source),

$$\boldsymbol{\nabla}_{\mathbf{R}}\phi_{\mathrm{L}} \simeq \boldsymbol{\nabla}_{\mathbf{R}}\,(\mathbf{k}\cdot\mathbf{R}) = \mathbf{k}\ ,$$

which means that the pressure force becomes

$$\mathbf{F}_{\mathrm{RP}} = \Gamma\sigma_{\mathrm{ee}} \cdot \hbar\mathbf{k} = \frac{1}{2}\frac{s_0}{1+s_0+4\Delta^2/\Gamma^2}\,\Gamma\hbar\mathbf{k}\ . \tag{141}$$

This expression has a simple interpretation. We first note that the term $\hbar\mathbf{k}$ is the linear momentum of a photon with wave vector $\mathbf{k}$. Next, $\Gamma$ represents the decay rate of the excited state, that is, the temporal rate of re-emission of a photon of light that has been absorbed by the atom to become excited; so that $\sigma_{\mathrm{ee,st}}\Gamma$ represents the average rate of absorption and re-emission of photons. The radiation pressure force, then, is nothing more than the time-averaged momentum received by the atom from the absorption of laser photons. The subsequent re-emission of absorbed photons does not contribute to the average force, since the direction of spontaneous emission is random and the acquired average vector momentum is zero. On the other hand, despite being zero on average, it exhibits non-zero fluctuations, since photons are constantly being re-emitted, and spontaneous emission therefore causes heating of the atom by causing a random evolution of its momentum state. It is important to note that the radiation pressure force is non-conservative, that is, it does not derive from a potential. The dissipative nature of this force is inherited from the dissipative nature of spontaneous emission, discussed in the previous section.

For high intensity, resonant light, the acceleration caused by radiation pressure in the atom can reach high values. Indeed, for incoming light $s_0 \gg 1$ and $\Delta = 0$ with the rubidium 780 nm resonance of natural width $\Gamma = 2\pi \times 6$ MHz, the acceleration caused on the rubidium-87 atom with mass $m_{\mathrm{Rb87}}$ will be equal to $\Gamma/2\,\hbar k/m_{\mathrm{Rb87}} = 1.1.10^5$ m.s$^{-2}$. This is $10^4$ times more intense than gravity!

## 3.2. The dipolar force

The dipolar force is written as

$$\mathbf{F}_{\mathrm{d}} = -2\hbar\Delta\sigma_{\mathrm{ee,st}}\,\frac{\boldsymbol{\nabla}_{\mathbf{R}}\Omega_0(\mathbf{R})}{\Omega_0(\mathbf{R})} = -\hbar\Delta\frac{s}{1+s}\frac{\boldsymbol{\nabla}_{\mathbf{R}}\Omega_0(\mathbf{R})}{\Omega_0(\mathbf{R})}\ . \tag{142}$$

This force derives from the potential

$$U_{\mathrm{d}}(\mathbf{R}) = \frac{\hbar\Delta}{2}\ln\left(1+s(\mathbf{R})\right)\ , \tag{143}$$

where the dependence of $s$ on $\mathbf{R}$ comes from the dependence of $\Omega_0$ on $\mathbf{R}$. Indeed,

$$\begin{aligned}\mathbf{F}_{\mathrm{d}} = -\boldsymbol{\nabla}_{\mathbf{R}}U_{\mathrm{d}}(\mathbf{R}) &= -\frac{\hbar\Delta}{2}\,\boldsymbol{\nabla}_{\mathbf{R}}\left[\ln\left(1+s(\mathbf{R})\right)\right] = -\frac{\hbar\Delta}{2}\,\frac{1}{1+s}\,\boldsymbol{\nabla}_{\mathbf{R}}s(\mathbf{R}) \\ &= -\frac{\hbar\Delta}{2}\,\frac{1}{1+s}\,\frac{2\Omega_0(\mathbf{R})\boldsymbol{\nabla}_{\mathbf{R}}\Omega_0(\mathbf{R})}{\Delta^2+\left(\frac{\Gamma}{2}\right)^2} = -\hbar\Delta\frac{s}{1+s}\frac{\boldsymbol{\nabla}_{\mathbf{R}}\Omega_0(\mathbf{R})}{\Omega_0(\mathbf{R})}\ .\end{aligned} \tag{144}$$

Near resonance, the radiation pressure force is much stronger than the dipolar force; on the other hand, when the light becomes highly detuned, we see from Eq. (141) that the radiation pressure tends to zero as $1/\Delta^2$, while the dipolar force decays as $1/\Delta$. This is the conservative regime where the dipolar force is usually relevant. In this regime, for which $\Delta \gg \Omega, \Gamma$, the saturation parameter satisfies $s \ll 1$, so that $\ln(1+s) \simeq s$, and we can simplify the above expression as

$$\mathrm{U_d} \simeq \frac{\hbar\Delta}{2} s(\mathbf{R}) = \frac{\hbar\Delta}{2} \frac{2\Omega_0^2(\mathbf{R})}{\Gamma^2 + 4\Delta^2} \simeq \frac{\hbar\Delta}{2} \frac{2\Omega_0^2(\mathbf{R})}{4\Delta^2} = \frac{\hbar\Omega_0^2(\mathbf{R})}{4\Delta} \ .$$

From the definition of the saturation intensity (Eq. (97)) and of the saturation parameter, we can also write

$$\Omega_0^2 = \frac{\Gamma^2}{2} \frac{\mathrm{I}}{\mathrm{I_{sat}}} = \frac{6\pi c^2 \Gamma}{\hbar\omega_0^3} \mathrm{I} \ ,$$

in such a way as to write the dipolar potential in the regime $\Delta \gg \Gamma, \Omega$ as

$$\mathrm{U_d}(\mathbf{R}) \simeq \frac{3\pi c^2}{2\omega_0^3} \frac{\Gamma}{\Delta} \mathrm{I}(\mathbf{R}) \ ,$$

where we again indicate the explicit dependence of the intensity on $\mathbf{R}$ to let clear that the spatial dependence of the potential comes from the spatial dependence of the light intensity.

The expression above is derived in the context of the RWA, for which $|\Delta| = |\omega - \omega_0| \ll \omega, \omega_0$. However, atomic trapping in a potential of the above form involves the use of very high detunings, $\Delta \sim \omega, \omega_0$ (for example, when we use light of wavelength 1064 nm to trap Rb or Sr, from resonances at 780 nm and 461 nm, respectively). It can be shown that preserving the non-resonant terms would cause another term to appear in the above equation, so that the total potential for a far-detuned optical dipolar trap (ODT) is written as

$$\mathrm{U_{ODT}}(\mathbf{R}) = \frac{3\pi c^2}{2\omega_0^3} \left( \frac{\Gamma}{\omega - \omega_0} + \frac{\Gamma}{\omega + \omega_0} \right) \mathrm{I}(\mathbf{R}) \ . \tag{145}$$

The above potential can also be derived from a purely classical approach, in which the atom is treated as a classical dipole. Indeed, in the high-detuning regime, there is no evidence that the atom's behavior follows the laws of quantum mechanics. An excellent reference for the classical derivation of the above expression is given in [2].

We see in the expression above that for negative detunings, $\Delta < 0$, the minimum of the potential happens for the highest intensities, i.e. the atoms are attracted to the maximum of the intensity; while for $\Delta > 0$, the atoms are repelled by the light intensity.

# 4. Cooling and trapping neutral atoms

## 4.1. Doppler shift and Doppler broadening of the atomic resonance

The atomic velocity with respect to the laboratory reference frame will modify the frequency of the incoming light seen by the atom, due to the Doppler effect. Let us suppose an atom with velocity $\mathbf{v}$ interacting with light that has wave vector $\mathbf{k}$ in the laboratory reference frame. With $\omega = kc$ the frequency of light in the laboratory reference frame, the frequency seen by the atom will be $\omega' = \omega - \mathbf{k}\cdot\mathbf{v}$. This means that in all the expressions deduced previously, the detuning $\Delta$ must be corrected for an atom of velocity $\mathbf{v}$ in order to account for the Doppler effect, as

$$\Delta' = \omega' - \omega_0 = \omega - \mathbf{k}\cdot\mathbf{v} - \omega_0 = \Delta - \mathbf{k}\cdot\mathbf{v} \ . \tag{146}$$

This equation can be reinterpreted by saying that the atomic velocity shifts its resonance in the laboratory frame to $\omega_0' = \omega_0 + \mathbf{k}\cdot\mathbf{v}$, so that rearranging the terms, $\Delta' = \omega - \omega_0'$. When we consider an atomic vapor of temperature $T$, the Doppler shift of each individual atom shifts its resonances according to the thermal velocity distribution, which gives rise to a **Doppler broadening** of the transition. To describe the phenomenon, let us assume that the wave vector of the light incident on the vapor is in the $z$ direction, $\mathbf{k} = k\mathbf{e}_z$. The Doppler effect of each atom is given by the term $\mathbf{k}\cdot\mathbf{v} = k\,v_z$, with $v_z$ the $z$ component of the velocity $\mathbf{v}$. In an atomic vapor at temperature $T$, the $v_z$ component is a random variable of Maxwell-Boltzmann distribution

$$\rho_{v,T}(v_z) = \frac{1}{\sqrt{2\pi}v_{\mathrm{RMS}}}\mathrm{e}^{-v_z^2/(2\,v_{\mathrm{RMS}}^2)} \tag{147}$$

with

$$v_{\mathrm{RMS}} = \sqrt{\frac{k_{\mathrm{B}}T}{M}} \tag{148}$$

the mean square velocity in any direction, and $k_{\mathrm{B}}$ the Boltzmann constant. This distribution of $v_z$ implies a distribution of atomic resonances $\omega_0' = \omega_0 + k\,v_z$ around the resonance $\omega_0$ of an atom at rest,

$$\rho_{\omega,T}(\omega_0') = \frac{1}{\sqrt{2\pi}\Delta_D}\mathrm{e}^{-(\omega_0'-\omega_0)^2/(2\,\Delta_D^2)} \ , \tag{149}$$

with the Doppler width (in rad/s) $\Delta_D = kv_{\mathrm{RMS}}$, or still

$$\Delta_D = 2\pi \times \frac{1}{\lambda}\sqrt{\frac{k_{\mathrm{B}}T}{M}} \ . \tag{150}$$

For the $\lambda = 780.24$ nm transition of $^{87}$Rb, assuming a vapor at 300 K, the Doppler broadening is $\Delta_D = 2\pi \times 217$ MHz, much larger than the natural width of the transition, $\Gamma = 2\pi \times 6.07$ MHz. This is why the absorption spectra of light from atomic vapors at room temperature have broadened lines. On the other hand, a cloud of cold Rb atoms with a typical temperature of 50 μK has a Doppler broadening $\Delta_D = 2\pi \times 89$ kHz, negligible with respect to $\Gamma$: In this regime, there are no notable effects of Doppler broadening.

## 4.2. Optical molasses

One of the first cooling techniques invented with the help of the light forces discussed in the last section is called optical molasses. The basic idea of this technique is to obtain, through the clever combination of light beams, an effective dissipative force on atoms, in the form

$$\mathbf{F}_{\text{mel}} = -\beta\,\mathbf{v}\ . \tag{151}$$

A force with this dependence on the velocity, with $\beta > 0$, always decreases the energy of the system, as it always performs negative work on the atom: The power generated by the force is given by

$$P_{\text{mel}} = \mathbf{F}_{\text{mel}} \cdot \mathbf{v} = -\beta\, v^2\ . \tag{152}$$

Furthermore, the equilibrium condition for this force is $v = 0$, which suggests that with such a force we could obtain a gas that is precisely still, with zero temperature. This, however, is not true, since we are only considering the average value of the force; we will see how the fluctuations of the light forces will limit the minimum temperature achievable with this technique.

A way to obtain a dissipative force as shown above is by combining counterpropagating beams in the three orthogonal directions of space, with a frequency $\omega$ slightly lower than the atomic frequency $\omega_0$. Since the beams will be close to resonance, the radiation pressure force is much greater than the dipole force, so we will neglect the latter. Since the radiation pressure force occurs in the direction of $\mathbf{k}$, that is, the direction of beam propagation, we see that each pair of counterpropagating beams acts independently of the other pairs, modifying only the velocity component in its propagation direction. We will thus consider in the following a 1D model, with a pair of counterpropagating beams in the $\pm\mathbf{e}_z$ direction acting on the $v_z$ component of the atomic velocity.

In this 1D model, then, we consider a pair of beams incident on the atom. One of them has wave vector $\mathbf{k}_+ = k\,\mathbf{e}_z$, and the second $\mathbf{k}_- = -k\mathbf{e}_z$, both with the same detuning $\Delta = \omega - \omega_0$ with respect to an atom at rest. However, since the $z$ component of the atom velocity is given by $v_z$, the detuning seen by the atom will be corrected by

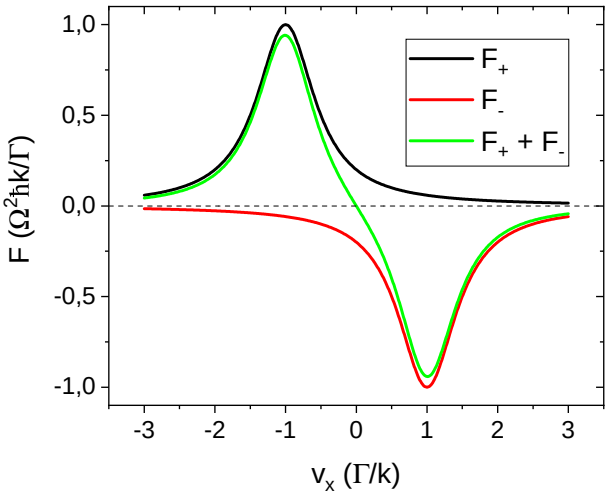



Figure 6: Plot of the forces $\mathbf{F}_\pm$ made by each counterpropagating beam in the directions $\pm\hat{\mathbf{z}}$, and the total force represented by the sum of both. The force is normalized by the maximum force $\Omega_0^2\,\hbar k/\Gamma$, and the velocity is normalized by the velocity $\Gamma/k$ required to shift the atomic resonance away from $\Gamma$. The light detuning is $\Delta = -\Gamma$.

the Doppler effect: $\Delta'_\pm = \Delta \mp k v_z$, with $\Delta'_+$ (resp. $\Delta'_-$) the detuning with respect to the beam propagating in $\mathbf{e}_z$ (resp. $-\mathbf{e}_z$) direction. Each of the beams causes a force on the atom, which we will indicate respectively by $\mathbf{F}_+$ and $\mathbf{F}_-$. In order to consider the total force acting on the atom as the sum of the individual forces, the saturation parameter $s$ must satisfy $s \ll 1$; indeed, in the opposite case where $s \gtrsim 1$, the saturation of the transition by one beam alters the absorption of photons from the opposite beam and vice versa, since the atom spends less time in the ground state. Within the regime $s \ll 1$, we obtain the expression for $\mathbf{F}_+$ and $\mathbf{F}_-$ from the expression (141) for the radiation pressure force:

$$\mathbf{F}_\pm = \frac{2\Omega_0^2}{\Gamma^2 + 4(\Delta'_\pm)^2}\,\frac{\Gamma}{2}\,\hbar\,\mathbf{k}_\pm = \pm\,\frac{\Omega_0^2\,\Gamma}{\Gamma^2 + 4\,(\Delta \mp k v_z)^2}\,\hbar\,k\mathbf{e}_z\ , \tag{153}$$

where we have used the relation, valid for $s \ll 1$,

$$\frac{s}{1+s} \simeq s = \frac{2\Omega^2}{\Gamma^2 + 4\Delta^2}\ .$$

We see in Fig. 6 the plot of the total force acting on the atom, $\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_-$, which is written as

$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- = -\frac{16\,\Omega_0^2\,\Gamma\,\Delta\,\hbar\,k^2\,v_z}{\left[\Gamma^2 + 4\,(\Delta + k v_z)^2\right]\left[\Gamma^2 + 4\,(\Delta - k v_z)^2\right]}\,\mathbf{e}_z\ . \tag{154}$$

Fig. 6 shows that the total force presents a linear behavior for low speed, as we anticipated in Eq. (151). Obtaining the limit for low speeds from the expression above, we have

$$\mathbf{F}_{v_z \to 0} \simeq \frac{16\,\Omega_0^2\,\Gamma\,\Delta\,\hbar\,k^2}{\left(\Gamma^2+4\Delta^2\right)^2}\,v_z\,\hat{\mathbf{z}} = -\beta\,v_z\,\mathbf{e}_z\ , \tag{155}$$

with

$$\beta = -16\hbar k^2 \frac{\Omega_0^2\,\Gamma\,\Delta}{\left(\Gamma^2+4\Delta^2\right)^2} = -8\hbar k^2 \frac{s\,\Gamma\,\Delta}{\Gamma^2+4\Delta^2}\ . \tag{156}$$

We see that $\beta$ is a positive quantity for $\Delta < 0$; that is, we need to have a negative detuning in order to cool the atomic sample. In contrast, a positive detuning gives $\beta < 0$, which creates a force in the same direction as the velocity, causing heating of the atomic sample.

In the above expression, the friction coefficient increases with $s$; thus, increasing the intensity of the beams ensures increasingly faster cooling. We must point out, however, that the expression is valid only in the regime $s \ll 1$. To obtain an expression for $\beta$ in the regime $s \gtrsim 1$, we would actually need to recalculate the atomic internal state subject to light from both directions, and then recalculate the force from this state. We would find that the coefficient $\beta$ does not increase indefinitely with $s$, but rather reaches a maximum and then begins to decrease. This occurs because increasing $s$ causes the transition width to widen, as shown by eq. (99). This means that the force becomes less sensitive to variations in atomic velocity due to the Doppler shift, decreasing the cooling effect.

The dissipative force shown above is related to a typical temporal rate of cooling, which we will call $\gamma_{\text{mel}}$. This rate can be simply obtained from the parameter $\beta$ in the case $\beta > 0$ as

$$\gamma_{\text{mel}} = \frac{\beta}{m}\ . \tag{157}$$

From this rate, we extract a characteristic time for the evolution of the atomic velocity until reaching the final velocity, $\tau_{\text{mel}} = 1/\gamma_{\text{mel}}$, or

$$\tau_{\text{mel}} = \frac{m}{4\hbar k^2}\frac{\Gamma^2+4\Delta^2}{s\,\Gamma\,|\Delta|}\ . \tag{158}$$

Let us calculate this time for typical parameters of an optical molasses applied to the broad transition of a $^{88}$Sr atom ($\lambda = 460.82$ nm, $\Gamma = 30.5$ MHz), of mass $1.46.10^{-25}$ kg. Let us take $\Delta = -\Gamma/2$ and $s_0 = 0.1$, so that $\tau_{\text{mel}} = 150$ µs. This justifies the typical experimental timescales that optimize the cooling of an optical molasses, of the order of the millisecond or even smaller.

If we now turn to the 3D situation, with three pairs of counterpropagating beams, we realize that in all directions we will have a similar force. If the intensity of the beams in each direction of space is the same, we will have the same Rabi frequency $\Omega_0$; and if the detuning is the same, we will also have the same $\Delta$, which guarantees the same $\beta$ for the three directions of space. In this regime, the total force for $kv \ll \Delta$ will be written

as the sum of the total forces in each direction:

$$\mathbf{F}_{\text{mel}} = -\beta\,(v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z) = -\beta\,\mathbf{v}\ , \tag{159}$$

as anticipated in the beginning of the section.

### 4.2.1. Doppler limit of optical cooling

As stated previously, although the equilibrium velocity of the above force is $v = 0$, an atomic cloud subjected to this force does not reach zero temperature (which would be prohibited, in fact, by the Third Law of Thermodynamics). This occurs for two main reasons: The first is because spontaneous emission, responsible for the re-emission of absorbed photons, causes a force that is zero on average (since the direction of photon emission is random) but presents non-zero fluctuations. These fluctuations cause a diffusion of the photon's momentum in momentum space, which implies heating. The balance between the heating effect caused by the fluctuation of the force and the cooling effect obtained by the average force gives rise to an equilibrium temperature, the so-called **Doppler limit**[3]

$$k_{\text{B}}T_{\text{D}} = \frac{\hbar\Gamma}{4}\,\frac{1 + 2Ds_0 + \left(\dfrac{2\Delta}{\Gamma}\right)^2}{\dfrac{2|\Delta|}{\Gamma}}\ , \tag{160}$$

where $D$ is the molasses dimension: 1 for a pair of counter-propagating beams, 2 for 2 mutually orthogonal counter-propagating pairs, and so on. In the low-saturation limit,

$$k_{\text{B}}T_{\text{D},s_0\ll 1} = \frac{\hbar\Gamma}{4}\,\frac{1 + \left(\dfrac{2\Delta}{\Gamma}\right)^2}{\dfrac{2|\Delta|}{\Gamma}}\ . \tag{161}$$

This expression reaches a minimum value for $\Delta = -\Gamma/2$, which gives the minimum temperature

$$T_{\text{D,min}} = \frac{\hbar\Gamma}{2k_{\text{B}}}\ . \tag{162}$$

For the $D_2$ line of Rb, the temperature limit is $T_{\text{D,min}} = 146$ μK, while for the 460.82 nm transition of Sr, $T_{\text{D,min}} = 732$ μK. A narrow transition can have an extremely low Doppler limit; for example, the 689 nm transition of Sr with $\Gamma = 2\pi \times 7.4$ kHz has $T_{\text{D,min}} = 178$ nK.

It is important to say that, when the atom has a hyperfine structure, as is the case with Rb and all other alkaline atoms, it is possible to cool it to temperatures much lower than the Doppler limit, due to the so-called **Sisyphus effect** or polarization gradient

cooling, which will not be discussed here (for those interested, consult [3] and [4]).

Another important limit is determined by the quantization of light. Indeed, the force exerted on atoms presents a fundamental "granularity", since radiation pressure involves the absorption or emission of quantized packets of momentum, the momentum of a single photon $\Delta p_{\mathrm{r}} = \hbar k$. This granularity establishes a minimum average kinetic energy of the order of the recoil kinetic energy

$$\Delta E_{\mathrm{r}} = \frac{\hbar^2 k^2}{2m} \ . \tag{163}$$

Equating this energy to $k_{\mathrm{B}}T$, we obtain the **recoil limit**:

$$T_{\mathrm{r}} = \frac{\hbar^2 k^2}{2mk_{\mathrm{B}}} \ . \tag{164}$$

This limit is generally very low, on the order of hundreds of nanoKelvins, and is irrelevant for broad transitions. However, for narrow transitions, it is higher than the Doppler limit. For example, for the 689 nm transition of Stronrium, $T_{\mathrm{r}} = 229$ nK, comparable to the Doppler temperature of 178 nK. This is expected, since the condition (132), which determines whether the resonance is broad or narrow, turns out to be exactly a comparison between the energy scales given by the Doppler and recoil temperatures (within a factor of 2).

## 4.3. Magneto-optical trap

The optical molasses presented in the previous section is capable of cooling an atom, but not trapping it, since the molasses force does not depend on position, but only on velocity. We could try to take advantage of the typical intensity distribution of a Gaussian laser beam in a scheme that guarantees a spatially confining force; however, it is possible to show that in the presence of only light beams exerting radiation pressure, it is not possible to create a confining potential in the three directions of space.

The solution frequently applied in cold atom experiments is to spatially modulate atomic resonances through the application of a position-dependent magnetic field. The presence of the magnetic field will alter the atomic resonances through the Zeeman effect, and a clever combination of the magnetic field profile with the polarization of the light will ensure a trapping potential, as we will see below. This type of trap is called a magneto-optical trap (MOT), but do not be fooled by the name: the forces involved in its operation are only due to radiation pressure. The magnetic field will act here by modulating the atomic resonances, in order to create spatial confinement.

As we did to understand the molasses mechanism, we will initially make a 1D model to understand how the magneto-optical trap works, and we will call $z$ the spatial coordinate; the center of the trap is determined by the position $z = 0$. In the region where the

trapping occurs, there is a linear magnetic field, which is 0 at the center of the trap and points in the direction $\mathbf{e}_z$:

$$\mathbf{B}(z) = b\, z\, \mathbf{e}_z \ , \tag{165}$$

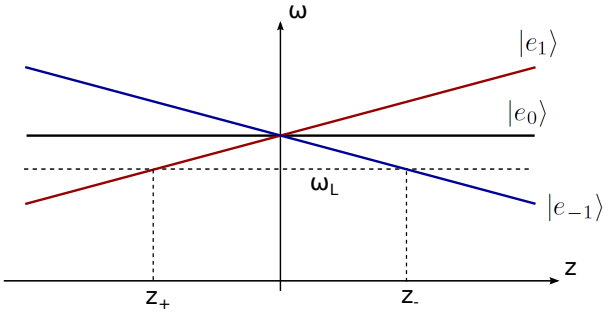
where $b$ is the magnetic field gradient. The presence of this magnetic field causes a modification of the energies of the atomic states by the Zeeman effect. Let us suppose the simplest level structure for a dipolar allowed transition, from a nondegenerate level S with $m_\ell = 0$, to a level P with three different magnetic sublevels, $m'_\ell = -1$, 0 and 1. The level structure is shown in Fig. 5. We will call those excited levels $|e_{-1}\rangle$, $|e_0\rangle$ and $|e_1\rangle$, following their magnetic number. The Zeeman effect caused by a magnetic field in the $z$ direction makes a energy shift of the atomic level which is proportional to the magnetic number,

$$E_{m'_\ell} = \mu_B\, g\, m'_\ell\, B \ . \tag{166}$$

In the expression above, $\mu_B = e\hbar/(2m)$ is the Bohr magneton, and $g$ is the Landé factor. For the simple model of the Hydrogen atom, where we do not consider the spin of the electron and of the nucleus, $g_\ell = 1$; for the general case, $g_F$ is an dimensionless number which is a function of the whole angular momentum $F$ of the atom.

Furthermore, since the magnetic field depends on the position, the energy of each level will also depend on the position. As a consequence, the transitions $\sigma_\pm$ to the levels with $m'_\ell = \pm 1$ will have a natural frequency $\omega_{0,\pm}(z)$ shifted by the electric field. This frequency that can be calculated from the expression (166) for the Zeeman effect:

$$\omega_{0,\pm}(Z) = \omega_0 \pm \mu_\mathrm{B} g_F B(Z) = \omega_0 \pm \mu_\mathrm{B} g_F\, b\, Z \ . \tag{167}$$



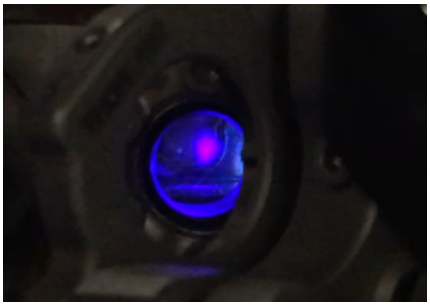



Figure 7: (a): Spatial dependence of atomic resonances in the presence of a positive magnetic field gradient. (b): Photograph of a strontium atomic cloud trapped in a MOT, re-emitting blue light at 460.82 nm that is continuously sent to the MOT for cooling and trapping. The sample contains $5.10^7$ atoms.

We shine a pair of counterpropagating laser beams on this atom in the presence of a magnetic field, just as we did in the molasses case; to maintain the dissipative effect

of the molasses, we will choose a detuning $\Delta = \omega - \omega_0 < 0$. But now, it is necessary to make the polarization of the beams explicit, since the resonance condition modified by the Zeeman effect depends on it. In the case drawn in Fig. 7(a) with $b > 0$, the intelligent choice of polarization consists of having both beams with positive helicity. This means that the beam coming from the right, propagating in the $+\mathbf{e}_z$ direction, has polarization $\boldsymbol{\sigma}_+$ and will excite the excited state $|e_+\rangle$, while the counter-propagating beam in the $-\mathbf{e}_z$ direction has polarization $\boldsymbol{\sigma}_-$ in the quantization direction $+\mathbf{e}_z$, and excites the transition that leads to the state $|e_-\rangle$. The radiation pressure force applied by each of these beams will be greater as the atoms get closer to resonance with their transition; if we call $z_\pm$ the resonance position of the beams propagating respectively in the direction $\pm\mathbf{e}_z$, we can write the condition for resonance for each one of the beams as $\omega = \omega_{0,\pm}(z_\pm)$, which gives us, using eq. (167),

$$z_\pm = \pm\frac{\Delta}{\mu_B g_F b} \ . \tag{168}$$

In Fig. (7)(a), we indicate the laser frequency in dotted lines, and we see that the positions $z_\pm$ are determined graphically by the intersection of this dotted line with the lines that identify the resonances $\omega_{0,\pm}$. Since $\Delta < 0$, we see that the beam propagating in the direction $+\mathbf{e}_z$ (coming from the right in Fig. (7)(a)) will be in resonance with the transition it excites at a point $z_+ < 0$, and the opposite occurs with the beam propagating from the left. This means that these beams will exert a restoring force, which tends to bring the atom towards the center if it moves away to the edges.

Quantitatively, we can write the total force by adapting Eqs. (153) to include the Zeeman effect in each of the forces $\mathbf{F}_\pm$ in the regime $s \ll 1$:

$$\mathbf{F}_\pm = \pm\,\frac{\Omega^2\,\Gamma}{\Gamma^2 + 4\left(\Delta \mp k v_z \mp \mu_B g_F b\, z\right)^2}\,\hbar\, k\mathbf{e}_z \ . \tag{169}$$

The total force becomes

$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- = -\frac{16\,\Omega^2\,\Gamma\,\Delta\,\hbar\,k\,(k v_z + \mu_B g_F b\, z)}{\left[\Gamma^2 + 4\left(\Delta + k v_z + \mu_B g_F b\, z\right)^2\right]\left[\Gamma^2 + 4\left(\Delta - k v_z - \mu_B g_F b\, z\right)^2\right]}\,\mathbf{e}_z \ . \tag{170}$$

For low velocities, we can again expand the expression above in first order, and we get

$$\mathbf{F} = -\beta v_z\,\mathbf{e}_z - \kappa z\,\mathbf{e}_z \ . \tag{171}$$

The value of $\beta$ is the same value given in eq. (156). Now, an additional restoring term appears in the force, identical to an elastic force with elastic constant $\kappa$ given by

$$\kappa = -8\hbar k \frac{\mu_B g_F b\, s\, \Gamma\, \Delta}{\Gamma^2 + 4\Delta^2} \ . \tag{172}$$

Again, for $\Delta < 0$, and assuming $b > 0$, $\kappa$ is positive, and the force is restoring, deriving from a confining harmonic potential. We thus see that the magneto-optical trap provides a force with two components: a dissipative component as the optical molasses, and an effectively conservative component, derived from a potential. Both forces are products of the radiation pressure of the counterpropagating beams. It is also important to emphasize that the sign of $\kappa$ is a function of the helicity chosen for the counterpropagating beams. If we chose the opposite helicity for the light, $\kappa$ would change sign and the force would be anti-trapping, expelling the atoms from the center of the trap. On the other hand, if we changed the sign of the magnetic field gradient $b$, we would also have to choose the opposite helicity of the light to ensure a restoring force.

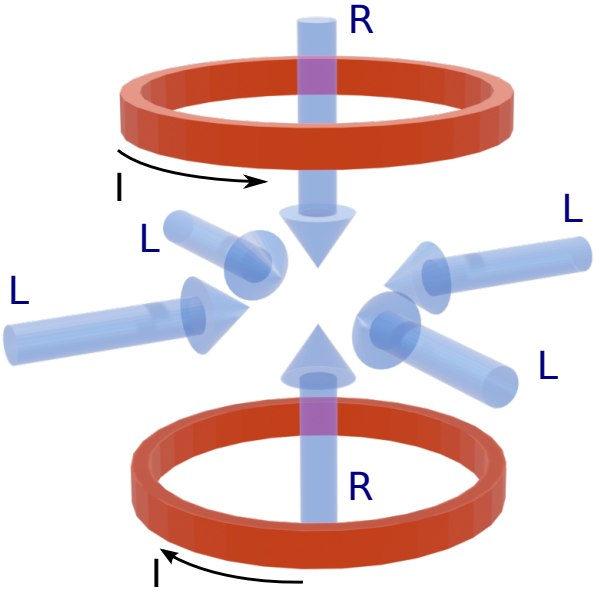



Figure 8: Typical experimental setup for obtaining a 3D MOT: A pair of coils with the same current flowing through them in opposite signs creates the quadrupole magnetic field, and three pairs of counterpropagating beams create the radiation pressure force. The helicity of each beam is indicated..

The extension of the above model to a 3D model is straightforward in the case of $s \ll 1$. However, in general the $\kappa$ in each direction of space will be different. This is because the magnetic field gradients in each direction of space, $\partial B_x/\partial x$, $\partial B_y/\partial y$, and $\partial B_z/\partial z$, cannot be equal, as they are linked through the Maxwelle equations:

$$\boldsymbol{\nabla} \cdot \mathbf{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \ , \tag{173}$$

which prevents the magnitude of the three from being equal. A very common magnetic field configuration used to obtain a MOT is the configuration created with a pair of field coils in an anti-Helmholtz arrangement. This arrangement corresponds to two identical

coils, with the same axis (assumed to be in the $z$ direction) and separated by a distance $D$; in each coil, we pass an electric current of the same magnitude and opposite direction; a schematic of the configuration can be found in Fig. 8. This arrangement assures, near the midpoint between the coils, at a distance of $D/2$ from the center of each of them, a magnetic field in the form

$$\mathbf{B}_{\mathrm{aH}} \simeq b\left(-\frac{x}{2}\,\mathbf{e}_x - \frac{y}{2}\,\mathbf{e}_y + z\,\mathbf{e}_z\right) \ . \tag{174}$$

In this case, $\dfrac{\partial B_x}{\partial x} = \dfrac{\partial B_y}{\partial y} = -b/2$ and $\dfrac{\partial B_z}{\partial z} = b$.

## 4.4. Magneto traps

In the previous sections, we discussed cooling and trapping techniques that use the radiation pressure force (Eq. 141) as one of their operating principles. This force exhibits a dissipative character that culminates in a limit to the minimum temperature achievable in these systems. Thus, to access even lower temperature regimes, on the order of hundreds of nanokelvins, or to produce a gas in a well-defined equilibrium state, it is necessary to use conservative traps. The first type of conservative trap capable of trapping neutral atoms were magnetic traps, resulting from the interaction of the atomic magnetic moment, $\boldsymbol{\mu}$, and an external magnetic field $\mathbf{B}$, given by eq. 166. Thus, if the magnitude of the magnetic field $B(\mathbf{R})$ depends on the position $\mathbf{R}$, eq. (166) shows that the energy of the atom will depend on the position. This energy appears here in fact as a magnetic potential energy, a function of the position $\mathbf{R}$:

$$U_{\mathrm{mag}}(\mathbf{R}) = \mu_B\, g_F\, m_F\, B(\mathbf{R}). \tag{175}$$

In the equation above, we have assumed that the quantization direction of the atom is the same as the direction of the magnetic field. For moving atoms, the magnetic field experienced by them has a direction that varies on time. If the rate of change of the direction of the magnetic field is not very high, the direction of the magnetic moment will follow adiabatically the direction of the magnetic field, and the above expression is valid. On the contrary, if the atom traverses a region of low magnetic field, the energy separation between the different magnetic sublevels is decreased, allowing for some mixing of the levels as the magnetic field direction seen by the moving atom is changed; the adiabatic condition is not anymore fulfilled, and there can be a leak to other magnetic states.

For an atom in an electronic sublevel with $m_F > 0$, its energy increases with the magnitude of the magnetic field $B$, so that the potential minimum occurs at the position of the minimum value of $B(\mathbf{R})$; for this reason, levels with $m_F > 0$ are called *low field seekers* (LFS). Conversely, energy levels with $m_F < 0$ are called *high field seekers*. It is possible to show from the laws of electromagnetism that it is not possible to obtain a

local maximum of the magnetic field magnitude, a result known as Earnshaw's theorem [5]. However, it is possible to obtain a field with a local minimum, so that it is possible to trap LFS levels around a local minimum of the magnetic field.

### 4.4.1. Quadrupolar magnetic trap

The quadrupolar magnetic trap, produced by a pair of coils in an anti-Helmholtz configuration (with currents in opposite directions) with a field given by eq. (174) is one of the most common configurations in ultracold atom experiments, thanks to its simplicity, large capture volume and high collision rate between the trapped atoms, essential for the implementation of other cooling processes, such as evaporative cooling[6]. From eqs. (174) and (175), the quadrupole trap potential can then be written as:

$$U_{\text{Quad}} = \mu_B \, g_F \, m_F \, b \sqrt{\frac{x^2}{4} + \frac{y^2}{4} + z^2}. \tag{176}$$

The quadrupolar trap works very well for not so cold samples, but below a few tens of microkelvins it faces a severe limitation. This is because the magnetic field is zero exactly at the center of the trap, giving rise to non-adiabatic transitions to non-magnetically trappable atomic states, resulting in losses called *Majorana losses*. It can be shown that these losses occur in the vicinity of the region of zero magnetic field, called the "Majorana hole"[7]. Some strategies exist to circumvent this problem: Applying purely magnetic traps with a magnetic field different from zero at the bottom (Ioffe-Pritchard trap, QUIC trap, other strategies for atom chips), applying an optical beam that creates a repulsive potential at the center (plug trap), or transferring the atoms to a conservative optical dipolar trap.

## 4.5. Far-from-resonance optical dipolar traps

The optical dipolar trap makes use of the dipolar force, resulting from the interaction between the electric dipole moment induced in the atom and the electric field of a laser beam with a frequency far from the atomic transition frequencies [2], such as to eliminate the radiation pressure force. The potential resulting from this interaction for a laser beam with intensity profile $I(\mathbf{R})$ is given by eq. (145).

In the case of a focused Gaussian laser beam propagating along the $\mathbf{e}_z$ axis, its intensity profile can be written from the electric field given by eq. (125) and is written as:

$$I(\mathbf{R}) = I(\rho, Z) = \frac{2P}{w_0^2} \frac{1}{1 + (Z/Z_R)^2} \mathrm{e}^{\frac{-2\rho^2}{w_0^2} \frac{1}{1+(Z/Z_R)^2}}, \tag{177}$$

The optical potential is thus given by

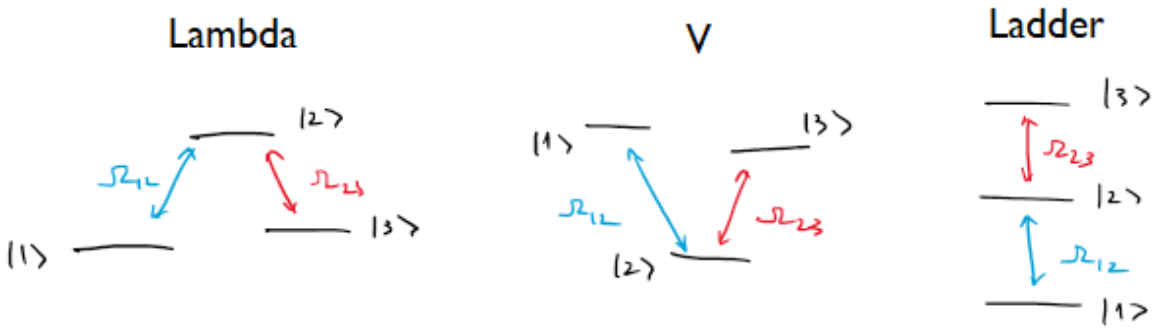



Figure 9: Different possible configurations for a 3-level system with 2 coupling light fields.

$$U_{\mathrm{ODT}} = U_0 \frac{1}{1+(Z/Z_R)^2} \mathrm{e}^{\frac{-2\rho^2}{w_0^2}\frac{1}{1+(Z/Z_R)^2}}, \tag{178}$$

with $U_0 = 3\pi c^2 \Gamma P/\Delta\omega_0^3 w_0^2$ the depth of the optical potential. It is easy to see that the trapping potential of an optical trap acquires an attractive character if $\Delta < 0$, that is, for laser beams with frequencies towards the red of the atomic transitions, or repulsive if $\Delta > 0$ and frequencies towards the blue of the atomic transition.

For a light field of any shape, we can generate light potentials of very different shapes also. In particular, uniform potentials, harmonic potentials, optical lattices, arrays of optical tweezers, rings, and other exotic potentials can be created by shaping the wavefront and intensity of the light beams sent to the atoms.

## 5. Three energy levels connected by dipolar transitions

Let us turn again to the dynamics of the internal states of the atom. But let us now include a third level, which will allow us to describe several new phenomena, such as: light shift of atomic levels, two-photon transitions, Electromagnetic-Induced transparency, dark states, ...

For a three-level system, three configurations are possible: The $\lambda$ configuration, the V configuration and the ladder configuration, depending on how the incoming electric fields will couple those levels; those are shown in Fig. 9.

We will focus here on two configurations: $\lambda$ and ladder. For both configurations, light of Rabi frequency $\Omega_{12}$ couples level 1 to level 2, and $\Omega_{23}$ couples level 2 to level 3. The levels 1 and 3 are considered stable or metastable, and do not decay.

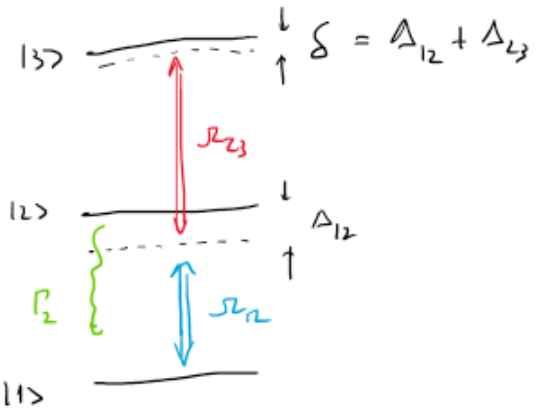



Figure 10: 3 level system in ladder configuration.

## 5.1. Ladder configuration

In the ladder configuration, as shown in Fig. 10, the level 2 can decay to 1 with rate $\Gamma_{21}$. The Hamiltonian is written

$$\hat{\mathcal{H}}_3 = \hbar\omega_{12}^{(0)}|2\rangle\langle 2| + \hbar\omega_{13}^{(0)}|3\rangle\langle 3| + \hbar\Omega_{12}\cos(\omega_{12}t)\left[|1\rangle\langle 2| + |2\rangle\langle 1|\right] + \hbar\Omega_{23}\cos(\omega_{23}t)\left[|2\rangle\langle 3| + |3\rangle\langle 2|\right] . \tag{179}$$

The frequencies $\omega_{12}$ and $\omega_{23}$ are just the frequencies of the light fields quasi-resonant with the transitions $|1\rangle \leftrightarrow |2\rangle$ and $|2\rangle \leftrightarrow |3\rangle$, respectively, with Rabi frequencies $\Omega_{12}$ and $\Omega_{23}$. For the ladder configuration, the energy difference betweel levels 2 and 3 is

$$\Delta E_{23} = \hbar(\omega_{13}^{(0)} - \omega_{12}^{(0)}) = \hbar\omega_{23}^{(0)} . \tag{180}$$

We can thus define the detunings from each transition $\Delta_{12} = \omega_{12} - \omega_{12}^{(0)}$ and $\Delta_{23} = \omega_{23} - \omega_{23}^{(0)} = \omega_{23} - (\omega_{13}^{(0)} - \omega_{12}^{(0)})$. We also define the total detuning for the two-photon excitation $\delta = \Delta_{12} + \Delta_{23} = \omega_{12} + \omega_{23} - \omega_{13}^{(0)}$.

The state of the atom is now written

$$|\psi\rangle = c_1|1\rangle + c_2|2\rangle + c_3|3\rangle . \tag{181}$$

We apply the Schrödinger equation to the evolution of this state, and we get the system of equations

$$\frac{\mathrm{d}c_1}{\mathrm{d}t} = -i\frac{\Omega_{12}}{2}\left(\mathrm{e}^{i\omega_{12}t}+\mathrm{e}^{-i\omega_{12}t}\right)c_2 \tag{182}$$
$$\frac{\mathrm{d}c_2}{\mathrm{d}t} = -i\frac{\Omega_{12}}{2}\left(\mathrm{e}^{i\omega_{12}t}+\mathrm{e}^{-i\omega_{12}t}\right)c_1 - i\omega_{12}^{(0)}c_2 - i\frac{\Omega_{23}}{2}\left(\mathrm{e}^{i\omega_{23}t}+\mathrm{e}^{-i\omega_{23}t}\right)c_3 \tag{183}$$
$$\frac{\mathrm{d}c_3}{\mathrm{d}t} = -i\frac{\Omega_{23}}{2}\left(\mathrm{e}^{i\omega_{23}t}+\mathrm{e}^{-i\omega_{23}t}\right)c_2 - i\omega_{13}^{(0)}c_3 \tag{184}$$

We place ourselves in the rotating frame with the incoming light via

$$c_2' = \mathrm{e}^{i\omega_{12}t}c_2 \tag{185}$$
$$c_3' = \mathrm{e}^{i(\omega_{12}+\omega_{23})t}c_3 \tag{186}$$

When we replace the coherences by the coherences in the rotating frame in the dynamical equations, and we make the RWA $\mathrm{e}^{\pm 2i\omega_{12}t} \simeq \langle \mathrm{e}^{\pm 2i\omega_{12}t}\rangle = 0$, $\mathrm{e}^{\pm 2i(\omega_{12}+\omega_{23})t} \simeq \left\langle \mathrm{e}^{\pm 2i(\omega_{12}+\omega_{23})t}\right\rangle = 0$, we get the set of coupled equations

$$\begin{aligned}
\frac{\mathrm{d}c_1}{\mathrm{d}t} &= -i\frac{\Omega_{12}}{2}c_2' \\
\frac{\mathrm{d}c_2'}{\mathrm{d}t} &= -i\frac{\Omega_{12}}{2}c_1 + i\Delta_{12}\,c_2' - i\frac{\Omega_{23}}{2}c_3' \\
\frac{\mathrm{d}c_3'}{\mathrm{d}t} &= -i\frac{\Omega_{23}}{2}c_2' + i\delta c_3'
\end{aligned} \tag{187}$$

### 5.1.1. Light shifts and two-photon transitions

Let us examine the experimentally relevant case of high intermediate detuning, $|\Delta_{12}| \gg |\delta|, \Omega_{12}, \Omega_{23}$. In this case, the coefficient $c_2'$ will evolve with a frequency much higher than the others, and we can perform an adiabatic elimination of that level, finding the stationary solution for this coefficient as a function of the others, and replacing this stationary solution in the other dynamical equations. This procedure has already been used before to obtain the light forces.

If we set $\frac{\mathrm{d}c_2'}{\mathrm{d}t} = 0$, we obtain

$$c_2' = \frac{\Omega_{12}c_1 + \Omega_{23}c_3'}{2\Delta_{12}}\ . \tag{188}$$

Since $\frac{\Omega_{12}}{\Delta_{12}}, \frac{\Omega_{23}}{\Delta_{12}} \ll 1$, the coefficient $c_2'$ will be always very small. In the regime of intermediate high detuning, the intermediate level is never populated.

The other dynamical equations become

$$\begin{aligned}\frac{\mathrm{d}c_1}{\mathrm{d}t} &= -i\frac{\Omega_{12}^2}{4\Delta_{12}}c_1 - i\frac{\Omega_{12}\Omega_{23}}{4\Delta_{12}}c_3' \\ \frac{\mathrm{d}c_3'}{\mathrm{d}t} &= -i\frac{\Omega_{12}\Omega_{23}}{4\Delta_{12}} + i\left(\delta - \frac{\Omega_{23}^2}{4\Delta_{12}}\right)c_3' \end{aligned} \tag{189}$$

We can cast the equations above just like the equations for the Rabi flipping of the first part of these notes:

$$i\hbar\frac{d}{dt}\begin{pmatrix} c_1 \\ c_3' \end{pmatrix} = \begin{pmatrix} \Delta E_1 & \hbar\frac{\Omega_{\mathrm{eff}}}{2} \\ \hbar\frac{\Omega_{\mathrm{eff}}}{2} & \Delta E_3 - \hbar\delta \end{pmatrix}\begin{pmatrix} c_1 \\ c_3' \end{pmatrix} . \tag{190}$$

The energy shifts and effective Rabi frequency are given by

$$\Delta E_1 = \frac{\hbar\Omega_{12}^2}{4\Delta_{12}} ; \tag{191}$$

$$\Delta E_3 = \frac{\hbar\Omega_{23}^2}{4\Delta_{12}} ; \tag{192}$$

$$\Omega_{\mathrm{eff}} = \frac{\Omega_{12}\Omega_{23}}{2\Delta_{12}} . \tag{193}$$

The energy shifts represent the optical potentials created by the light fields. In this context, we see them as **light shifts** of the resonances.

The effective Rabi frequency describes the effective coupling between the two levels. We see that thetwo light fields can effectively couple the two levels without ever populating the intermediate one. We call it a **two-photon transition**. It depends non-linearly in the intensities of both incoming fields, since the Rabi frequency is proportional to the product of the individual Rabi Frequencies. In order to resonantly excite the upper level, we must fulfill now the resonance condition

$$\delta' = \omega_{12} + \omega_{23} - \left(\omega_{13}^{(0)} + \frac{\Delta E_3}{\hbar} - \frac{\Delta E_1}{\hbar}\right) = 0. \tag{194}$$

## 5.2. Lambda configuration

For lambda configuration, as shown in Fig. 11 most of the calculation follows similar lines. The Hamiltonian is the same as in Eq. (179), but now the frequency $\omega_{13}^0$ is equal to $\omega_{13}^0 = \omega_{12}^{(0)} - \omega_{23}^{(0)}$. Accordingly,t he detuning to reach resonance from state $|1\rangle$ to state $|3\rangle$ is given by $\delta = \Delta_{12} - \Delta_{23}$. Now, to put ourselves in the rotating frame, we do the transformation

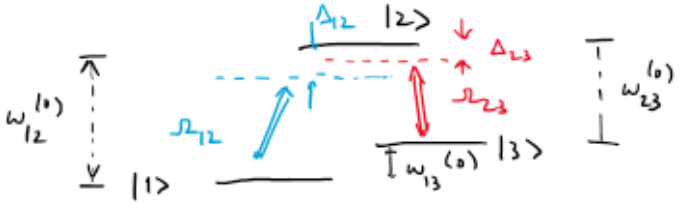



Figure 11: 3 level system in lambda configuration.

$$c_2' = e^{i\omega_{12}t}c_2 \; ; \tag{195}$$
$$c_3' = e^{i(\omega_{12}-\omega_{23})t}c_3 \; , \tag{196}$$

and we end up with the same equations of motion in the RWA as given by Eqs. (187), with the new definitions for $c_3'$ and $\delta$.

### 5.2.1. Coherent Raman Transitions

By imposing $|\Delta_{12}| \gg |\delta|, \Omega_{12}, \Omega_{23}$, we perform the adiabatic approximation just as it has been done before, and we deduce the condition for a two-photon transition between two low-energy levels via an intermediate upper level. In this context, this transition is called a **coherent Raman transition** (which can differ from the definition of Raman transition used by the solid state community).

### 5.2.2. Dark states and STIRAP

Now, let us face the dynamics of the levels without considering that $\Delta_{12}$ is large. First, let us put ourselves in the condition $\delta = 0$, and consider the following linear combination of levels:

$$\begin{cases} |g_+\rangle = \dfrac{\Omega_{12}\,|1\rangle + \Omega_{23}e^{-i\omega_{13}t}\,|3\rangle}{\sqrt{\Omega_{12}^2+\Omega_{23}^2}} \\ \\ |g_-\rangle = \dfrac{\Omega_{23}\,|1\rangle - \Omega_{12}e^{-i\omega_{13}t}\,|3\rangle}{\sqrt{\Omega_{12}^2+\Omega_{23}^2}} \end{cases} \tag{197}$$

The state of the system is written on this new basis as

$$\begin{aligned} |\psi\rangle &= c_1|1\rangle + c_2|2\rangle + c_3|3\rangle \\ &= c_1|1\rangle + c_2'e^{-i\omega_{12}t}|2\rangle + c_3'e^{-i\omega_{13}t}|3\rangle \\ &= c_+|g_+\rangle + c_-|g_-\rangle + c_2'e^{-i\omega_{12}t}|2\rangle. \end{aligned} \tag{198}$$

Replacing the definitions of $|g_+\rangle$ and $|g_-\rangle$ to the equations, we get

$$\begin{cases} c_1 = \dfrac{\Omega_{12}\, c_+ + \Omega_{23}\, c_-}{\sqrt{\Omega_{12}^2 + \Omega_{23}^2}} \\ c_3' = \dfrac{\Omega_{23}\, c_+ - \Omega_{12}\, c_-}{\sqrt{\Omega_{12}^2 + \Omega_{23}^2}} \end{cases} \tag{199}$$

Inverting those, we get

$$\begin{cases} c_+ &= \dfrac{\Omega_{12}\, c_1 + \Omega_{23}\, c_3}{\sqrt{\Omega_{12}^2 + \Omega_{23}^2}}, \\ c_- &= \dfrac{\Omega_{23}\, c_1 - \Omega_{12}\, c_3}{\sqrt{\Omega_{12}^2 + \Omega_{23}^2}} \end{cases} \tag{200}$$

In this new basis, the dynamics equations become

$$i\frac{\lambda}{\Delta t}\begin{bmatrix} c_+ \\ c_2 \\ c_- \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{\Omega_{12}^2 + \Omega_{23}^2} & 0 \\ \sqrt{\Omega_{12}^2 + \Omega_{23}^2} & -\Delta_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_+ \\ c_2 \\ c_- \end{bmatrix} \tag{201}$$

We see that the linear combination $|g_-\rangle$ is a level not coupled to the excited state. Up to now, we have not included decay rates, but if we include it, the excited state can decay to the levels $|g_+\rangle$ and $|g_-\rangle$ - after some cycles, the atomic state ends up in the level $|g_-\rangle$ and it will stay there forever. This level is called a **dark state** of the atom.

There are simple limits in which the dark level assume a simple shape: If $\Omega_{12} \gg \Omega_{23}$,

$$\begin{cases} |g_+\rangle \to |1\rangle \ , \\ |g_-\rangle \to \mathrm{e}^{-i\omega_{31}t}\,|3\rangle \ . \end{cases} \tag{202}$$

In the opposite limit of $\Omega_{23} \gg \Omega_{12}$,

$$\begin{cases} |g_+\rangle \to \mathrm{e}^{-i\omega_{31}t}\,|3\rangle \ , \\ |g_-\rangle \to |1\rangle \ . \end{cases} \tag{203}$$

This allows the effect known as the **Stimulated Raman Adiabatic Passage** (STIRAP). If we first turn on the light resonant with the transition $|2\rangle \to |3\rangle$ (thus increasing $\Omega_{23}$ from zero while keeping $\Omega_{12}$ at zero), and then slowly turning on $\Omega_{12}$ whiel turning off $\Omega_{23}$, we go from $\Omega_{23} \gg \Omega_{12}$ to $\Omega_{12} \gg \Omega_{23}$. We will thus transform slowly the eigenstate $|g_-\rangle$ of the system from $|1\rangle$ to $|3\rangle$ efficiently (probability 1 in the limit of slow passage), without ever populating $|2\rangle$. We can even keep the coherence of a superposition in this way, since there is no spontaneous emission involved.

### 5.2.3. Electromagnetic Induced Transparency (EIT)

For both beams close to resonance, and in the regime $\Omega_{23} \gg \Omega_{12}$, we enter the regime of Electromagnetic Induced Transparency (EIT). We will call $\Omega_{23} = \Omega_c$, the Rabi

frequency of the coupling beam, and $\Omega_{12} = \Omega_p$, the Rabi frequency of the probe beam.

The coupling beam is responsible for making the $|1\rangle$ state to be a dark state, which means that if the resonance condition $\delta = 0$ is satisfied, the probe light cannot couple $|1\rangle$ to the excited state $|2\rangle$, and the light will pass by the atomic sample without being absorbed. This phenomena is called the **Electromagnetic Induced transparency** (EIT): the sample is made transparent by the control beam.

In order to derive the width of the transparency window, one needs to solve the dynamics for the general case of $\delta$ not necessarily equal to zero. Then, one can show that in the regime $\Gamma_{12} \gg \Omega_c \gg \Omega_p$, where $\Gamma_{12}$ is the decay rate from level $|2\rangle$ to level $|1\rangle$, the transparency window is of width $\Omega_c$, as shown in Fig. (12).

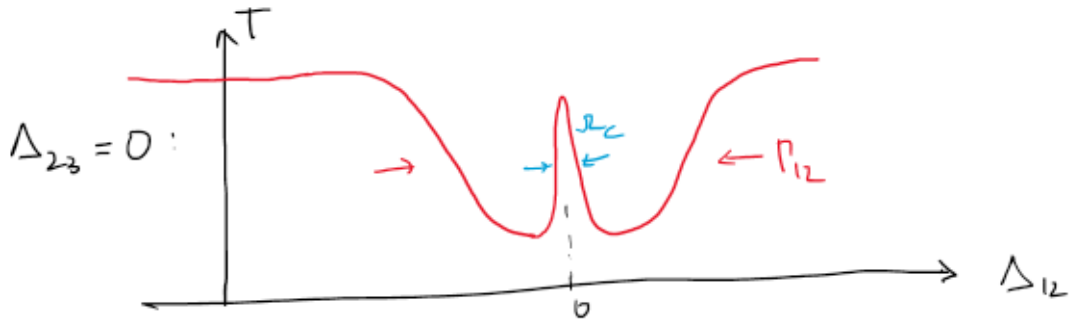



Figure 12: EIT transmission spectrum for $\Gamma_{12} \gg \Omega_c \gg \Omega_p$ and $\Delta_{23} = 0$.

The EIT effect has several interesting applications. First, the transparency window has its width controllable, with applications for metrology and sensing. Also, close to the transparency window, the absorption varies fast. This means that the real part of the electric susceptibility also varies fast, and one can expect very small group velocities for light. The impressive record of 8 m/s was achieved for a light pulse around the transparency window of EIT [8].

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