Classical observables from scattering amplitudes

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Motivation & Introduction

- The detection of binary black hole and neutron star mergers by the LIGO-VIRGO-KAGRA (LVK) collaboration has sparked a renaissance in solving the gravitational two-body problem and computing theoretical waveforms to high precision.
- One of the most useful and fascinating quantities is the radiative (waveform) field[1], $\frac{1}{R}h_{\mu\nu}(t,\hat{n})$ measured by a detector at a distance R from the source, emitted during the inspiral/scattering of astrophysical objects.

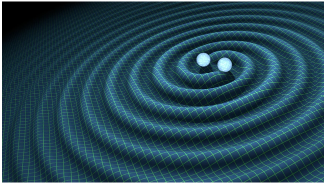


Figure: Ripples in spacetime discovered by LIGO from a binary black hole merger, *Caltech*

[1] LIGO measures the "gravitational wave strain" $h(t) = F_+(\theta,\phi,\psi)h_+(t) + F_\times(\theta,\phi,\psi)h_\times(t)$, where $h_{+,\times} = h^{TT}_{ij}e^{ij}_{+,\times}$ and $h^{TT}_{\mu\nu} = \Lambda_{\mu\nu\alpha\beta}h^{\alpha\beta}$.

- Alongside direct numerical integration of the Einstein field equations,
  - the post-Newtonian (PN) approximation: $Gm/rc^2 \sim v^2/c^2 \ll 1$.

$$h_{\mu\nu}(\omega, \hat{n}) = \sum_{n=0}^{\infty} \frac{1}{c^n} h_{\mu\nu}^{PN}(\omega, \hat{n})$$

[Blanchet, Levi, Porto, Sturani, Foffa, ...]

  - the post-Minkowskian (PM) approximation: $\frac{Gm}{rc^2} \ll 1$, but $v^2/c^2 \lesssim 1$.

$$h_{\mu\nu}(\omega, \hat{n}) = \sum_{n=1}^{\infty} G^n h_{\mu\nu}^{PM}(\omega, \hat{n})$$

(Same as perturbative expansion in QFT)

[Westpfahl, Bjerrum-Bohr, Bern+, Parra-Martinez+, Cheung, Bini, Damour, Porto+, Guevara+, Steinhoff+,Heissenberg, Aoude+, Mogull+, Kosower+,Huang,...]

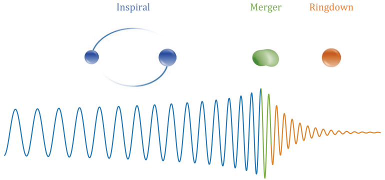


Figure: Temporal evolution of a binary system, *LIGO*

Why Amplitudes?

- Computing the waveform field is intrinsically difficult due to the existence of numerous physical scales that are nonlinearly coupled via general relativity.
- We focus on the inspiral phase, where compact objects can be modeled as point particles within an effective field theory (EFT). [Goldberger, Rothstein]
- Weak field perturbation theory is a crucial input, and Amplitudes methods is demonstrating its potential to surpass traditional classical tools.
- Amplitudes are gauge-invariant objects that compactly encode the perturbative scattering dynamics of point particles in quantum field theory (QFT) and can be computed efficiently in analytic form.
- The motivation is to apply the modern techniques like unitarity cut methods, integration-by-parts (IBP) reduction, double copy relations, soft factorization theorems, etc. developed for amplitude calculations in gauge theory and gravity to problems involving gravitational waves from binary black holes.

Outline

- The application of on-shell scattering amplitudes for deriving classical gravitational observables has seen a remarkable development in recent years.
- One of the most potent formalisms to generate such observables was proposed in a seminal paper by Kosower, Maybee, and O'Connell (KMOC) which uses scattering amplitudes to compute a set of asymptotic quantities, whose classical limits are the observables of interest.
- In this talk, I will focus on computing observables in KMOC formalism, such as the scattering angle/linear impulse or the radiation flux in the hyperbolic scattering of the two bodies, where PM expansion is applicable[2].
- We also discuss how soft factorization theorems in scattering amplitudes reveal universal features of gravitational radiation, such as the memory effects.

[2] There exist dictionaries that relate gravitational scattering data to observables for bound states in generic configurations, utilizing Firsov's formula, the Impetus formula, and a sequence of analytic continuations. [Cho, Kälin,Porto]

Overview

Introduction to KMOC Formalism

- Prior to KMOC, treating gravity as an EFT had already emphasized that scattering amplitudes, including loop amplitudes, encode the classical gravitational potential between two masses. [Donoghue, Holstein]
- KMOC formalism is a framework to compute classical observables such as the scattering angle, radiative flux carried by electromagnetic or gravitational field using perturbative amplitudes in large impact parameter (b) or large J = mb regime.
- The basic idea is to take wave packets for incoming (classical) particles, evolve them using quantum S-matrix operator and then compute expectation value of an observable in the final state.

$$\begin{aligned}\langle \Delta\mathcal{O}\rangle &= \lim_{\hbar\to 0}\hbar^{\kappa_{\mathcal{O}}}\left[{}_{in}\langle\Psi|S^{\dagger}\hat{\mathcal{O}}S|\Psi\rangle_{in} - {}_{in}\langle\Psi|\hat{\mathcal{O}}|\Psi\rangle_{in}\right]\\ &= \lim_{\hbar\to 0}\hbar^{\kappa_{\mathcal{O}}}\left[{}_{in}\langle\Psi|i[\hat{\mathcal{O}},T]|\Psi\rangle_{in} + {}_{in}\langle\Psi|T^{\dagger}[\hat{\mathcal{O}},T]|\Psi\rangle_{in}\right],\end{aligned}$$

  where the factor of $\hbar^{\kappa_{\mathcal{O}}}$ ensures $\langle\Delta\mathcal{O}\rangle \sim \hbar^0$ i.e. classical scaling.
- The S-matrix-governed evolution sums over all possible out states and thus characterizing this framework as an "in-in" formalism.

KMOC Formalism

- $|\Psi\rangle_{in}$ is the incoming two particle state in which the particles are separated by impact parameter b and their momenta localised around p_1, p_2. $\phi_i(p_i)$s are the minimum uncertainty wave packets (in momentum space)

$$|\Psi\rangle_{in} = \int \prod_i \left[\hat{d}^4 p_i \hat{\delta}^{(+)}(p_i^2 - m_i^2)\phi_i(p_i)e^{ib\cdot p_i/\hbar}\right]|p_1 p_2\rangle$$

- During a small angle (large impact parameter) scattering, the semi-classical in-state evolves to a semi-classical out-state where the momenta are peaked around $p_i + \mathcal{O}(\frac{1}{b})$ as $b \to \infty$.
- This is naturally incorporated in KMOC by re-scaling all the massless momenta and replacing them with their wave numbers i.e the exchange momenta ($q_i = \bar{q}_i \hbar$). The $\hbar$ is also restored in the couplings as $e/\sqrt{\hbar}$ (QED), $\kappa/\sqrt{\hbar}$ (gravity), where $\kappa = \sqrt{32\pi G}$.
- Quantum-mechanical expectation values reproduce their classical counterparts only when the wave-packet spread falls within this 'Goldilocks' regime:

$$l_c \ll l_w \ll \sqrt{-b^2}\,.$$

Example: 2-2 scattering in scalar QED

- Let us compute the total change in the momentum (linear impulse) of one of the particles — say particle 1 — during the scattering, where $\mathcal{O} = \mathbb{P}^\mu$
- Using the on-shell completeness relation, unitarity of the S-matrix and plugging in the expression for the initial state,

$$\langle \Delta p_1^\mu \rangle = i \int \hat{d}^4 q\ \hat{\delta}(2p_1 \cdot q + q^2)\hat{\delta}(2p_2 \cdot q - q^2)\ e^{iq\cdot b/\hbar} q^\mu \mathcal{A}_4(p_1, p_2 \to p_1 + q, p_2 - q)$$

$$+ \int \hat{d}^4 q\ \hat{\delta}(2p_1 \cdot q + q^2)\hat{\delta}(2p_2 \cdot q - q^2)\ e^{iq\cdot b/\hbar} \sum_X \int \prod_{i=1,2} \hat{d}^4 w_i\ \hat{\delta}(2p_i \cdot w_i + w_i^2)$$

$$\times w_1^\mu \hat{\delta}^{(4)}(w_1 + w_2 + r_X)$$

$$\times \mathcal{A}(p_1, p_2 \to p_1 + w_1, p_2 + w_2, r_X)\mathcal{A}^*(p_1 + q, p_2 - q \to p_1 + w_1, p_2 + w_2, r_X)\,.$$

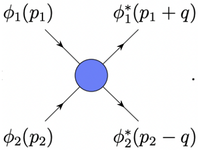


Figure: (1)

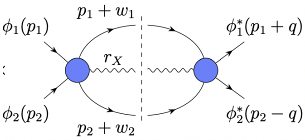


Figure: (2)

Classical observables

- The classical observables are then expressed in terms of perturbative amplitudes. The LO classical linear impulse is given by

$$\Delta p^\mu = \frac{\hbar^3}{4} \langle\langle \int \hat{d}^4\bar{q}\hat{\delta}(p_1 \cdot \bar{q})\hat{\delta}(p_2 \cdot \bar{q}) e^{i\bar{q}\cdot b} i\bar{q}^\mu \; A_4(p_1, p_2 \to p_1 + \hbar\bar{q}, p_2 - \hbar\bar{q})\rangle\rangle \,|_{\hbar\to 0} \,,$$

  where $\langle\langle f(p_1, p_2, q \ldots)\rangle\rangle$ denotes the integration over the minimum uncertainty wave packets which localizes the momenta onto their classical values.
- Similarly, to leading order in the coupling, the radiative gauge field is given by

$$\mathcal{R}^{(0)}_{cl}(\bar{k}) = \lim_{\hbar\to 0} \frac{\hbar^{3/2}}{4} \prod_{i=1,2} \int \hat{d}^4\bar{q}_i \hat{\delta}(p_i \cdot \bar{q}_i)\hat{\delta}^{(4)}(\bar{q_1} + \bar{q_2} - \bar{k}) e^{ib\cdot\bar{q_2}}$$
$$\times \bar{\mathcal{A}}^{(0)}_5(p_1 + \hbar\bar{q_1}, p_2 + \hbar\bar{q_2} \to p_1, p_2, \hbar\bar{k}) \,.$$

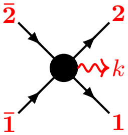


Figure: The five-point amplitude appearing in the radiation kernel at leading order.

Classical physics from "Loops"

- For finite classical contributions from loops, the massless momenta q scaling also forces the same scaling for the loop momenta automatically, i.e. $(l_i = \hbar \bar{l}_i)$ such that $\bar{l}$ is fixed, as the $l \gg q$ regime leads to contact/quantum contributions.
- But why do loops contribute to classical physics?:
  Some regions of loop momentum where internal propagators go on shell remove the naive $\hbar$ suppression. Those on-shell conditions encode long-distance propagation and radiation, which are classical.
- Tree-level classical: exchange of a long-range field (Coulomb, Newton)
  Loop-level classical: corresponds to iterated classical propagation, like the field going out and then rescattering.
- KMOC formalism systematically identifies these so-called "classical regions" prior to performing the loop integration.

Advantages of KMOC Formalism

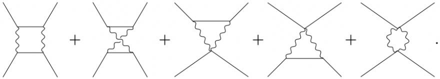


Figure: Relevant 1-loop diagrams in scalar QED

- For a diagram to contribute classically, it must contain *at least one internal matter line* in the loop.
- The power of the formalism is that the classical limit is taken at the level of loop integrands, before evaluating the full amplitude. This drastically simplifies the quantum computation as only a subset of Feynman diagrams contributes in this limit.
- Additionally, the radiation-reaction effects are naturally inbuilt within the framework unlike the traditional formulations of classical physics, where one must include the ALD force (MiSaTaQuWa self-force in gravity) by hand in order to enforce momentum conservation.

$$\begin{aligned}\langle \Delta p_1^\mu \rangle + \langle \Delta p_2^\mu \rangle &= -{}_{\text{in}}\langle \Psi | T^\dagger [\mathbb{K}^\mu, T] | \Psi \rangle_{\text{in}} = -{}_{\text{in}}\langle \Psi | T^\dagger \mathbb{K}^\mu T | \Psi \rangle_{\text{in}} \\ &= -\langle k^\mu \rangle = -P^\mu_{\text{rad}}\,.\end{aligned}$$

Some interesting objects

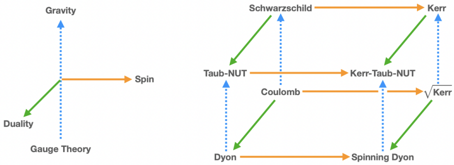


Figure: Classical solutions related by EM duality, double copy and the NJ shift

- Electromagnetic duality is a symmetry in the classical equations of electromagnetism (Maxwell's equations) that relates the electric field ($\mathbf{E}$) and the magnetic field ($\mathbf{B}$).
- Gravity amplitudes can be realized as a "double copy" of gauge theory amplitudes.

$$\mathcal{M}_{\text{gauge}} \times \mathcal{M}_{\text{gauge}} \sim \mathcal{M}_{\text{gravity}}$$

  Classical double copy: Relations between classical solutions of Yang-Mills theory and gravity.
- The Newman-Janis (NJ) shift is a complex coordinate transformation that maps the Schwarzschild solution to the Kerr solution.

$$\phi_{Schw}(r + ia\cos\theta) = \phi_{Kerr}(r)$$

  Here a is the ring radius of the Kerr black hole.
- This mapping has now been generalized to an algorithm that is used to generate rotating solutions from static solutions [Erbin].

The NJ exponentiated amplitudes

- The NJ algorithm on the space of classical solutions in GR and EM could be used in the space of scattering amplitudes to map an amplitude with external scalar particles to an amplitude associated with the infinite spin limit of certain massive spin S amplitudes [Arkani-Hamed, Huang, Huang].
- The minimal coupling of these particles to the gravitational or Maxwell field is equivalent to the classical coupling of the Kerr black hole with linearized gravity or the $\sqrt{\text{Kerr}}$ charged object[3] with the electromagnetic field [Guevara, Ochirov, Vines].
- $$\mathcal{A}^{\pm}_{3,\sqrt{\text{Kerr}}} = Q\mathcal{A}^{\pm}_{3,scalar}\, e^{\pm\bar{q}\cdot a_2}$$

  as $S \to \infty$ and $\hbar \to 0$ with $S\hbar$ being fixed.
- For Kerr BH,

  $$\mathcal{A}^{\pm}_{3,Kerr} = \kappa(\mathcal{A}^{\pm}_{3,scalar})^2\, e^{\pm\bar{q}\cdot a_2}\ .$$
- This exponentiation was identified as the NJ algorithm at the level of 3-point on-shell amplitudes [Arkani-Hamed, Huang, O'Connell].

[3] It is a solution of the free Maxwell's equations with infinite multipole moments expressed solely in terms of the charge, mass and angular momentum of the classical object (same as the no-hair theorem for BHs)

Results

- Using the 3-point minimally-coupled exponentiated-spin amplitudes, the linear impulse, Δp_1^μ for the scalar particle in the background of a $\sqrt{\text{Kerr}}$ object was obtained from a scalar-scalar scattering by complexifying the impact parameter with the ring radius a_2 [Arkani-Hamed, Huang, O'Connell].

$$\Delta p^\mu_{1,scalar}(b) \xrightarrow{NJ:b\to b+ia_2} \Delta p^\mu_{1,\sqrt{\text{Kerr}}}(b,a_2) = \text{Re}\left[\Delta p^\mu_{1,scalar}(b+ia_2)\right]$$
$$= \frac{Q_1Q_2}{2\pi\gamma\beta}\ \text{Re}\left[\frac{\gamma(b+ia_2)^\mu - i\epsilon^\mu(b+ia_2,u_1,u_2)}{(b+ia_2)^2}\right].$$

- The results for the Kerr black hole can then be obtained via double-copy methods:

$$\Delta p^\mu_{1,Kerr} = -\frac{2m_1m_2G}{\gamma\beta}\ \text{Re}\left[\frac{(2\gamma^2-1)(b+ia_2)^\mu - 2i\gamma\,\epsilon^\mu(b+ia_2,u_1,u_2)}{(b+ia_2)^2}\right].$$

- We used the NJ algorithm on the space of scalar QED amplitudes and computed classical observables such as the radiative gauge field emitted by the scalar particle and the net angular momentum impulse for scalar-$\sqrt{\text{Kerr}}$ scattering, to all order in spin at 1PL via the KMOC formalism. [SA, Manna, Manu]

Soft radiation

- In both PN and PM expansion, we compute radiation emitted from a source which is constrained by low velocities or large separations.
- But there is a different perturbative expansion which leads to different insights about gravitational radiation emitted in a scattering process.
- Consider a detector placed at infinity whose frequency band $(\omega, \omega + d\omega)$ is in the infrared: $\omega \ll \omega_c$. The radiation measured by such a detector is called soft radiation.
- Soft expansion is remarkably constrained in terms of momenta and the spin of the initial and final objects.
- At any given order in soft expansion, the radiative field is exact to all orders in PM and PN expansion. It is hence a non-perturbative probe and offers remarkable insights into universal modes of gravitational radiation in classical scattering.

Classical soft theorems in $D \geq 4$ dimensions

- For a generic gravitational scattering, the leading behavior of the radiative field long before and long after the main burst of radiation reaches the detector—admits universal expressions and is given by the classical soft theorems:

$$h_{\mu\nu}(\omega, r, \hat{n}) = \frac{1}{r^{\frac{D-2}{2}}} e^{i\omega r} \left[\sum_{N=-1}^{\infty} \omega^N h^N_{\mu\nu}(\hat{n}) + \sum_{m=0}^{\infty} \omega^m (\log \omega)^{m+1} h^{\log\, m}_{\mu\nu}(\hat{n}) \right.$$
$$\left. + \sum_{N,M \,|\, M-N>-1} \omega^M (\log \omega)^N h^{\log(N,M)}_{\mu\nu}(\hat{n}) \right].$$

[Laddha,Saha,Sahoo,Sen]

- **Universality**: If a term in the soft expansion is independent of the details of the scattering and only depends on the linear momenta, angular momenta, and spins of the scattering objects, we call it universal.
- The leading term, $h^{-1}_{\mu\nu}(\hat{n})$ is the displacement memory and is universal in all dimensions. It is also independent of the spin of the objects and is an observable. It measures (after passage of gravitational wave) a permanent physical change in the asymptotic metric at future null infinity $\mathcal{I}^+$ [Zeldovich, Thorne, Christodolou, Garfinkle,...].

$$h^{-1}_{\mu\nu}(\hat{n}) = \sum_{i\in(in+out)} \frac{p_{i,\mu} p_{i,\nu}}{p_i \cdot \hat{n}} + \int_{S^2} d^2\hat{n}' \frac{A_\mu(\hat{n}') A_\nu(\hat{n}')}{A(\hat{n}') \cdot \hat{n}}.$$

Logarithmic soft theorems

- Blue terms exist only in $D = 4$ and are shown to be universal. The first term in this series is known as tail to the memory.
- Universality of these coefficients is proved in a series of papers. These are known as the Log soft theorems [Laddha, Sahoo, Sen]. They depend only on the incoming and outgoing momenta and angular momenta of the scattering objects.
- It has been conjectured that the universality extends to all m [Sahoo,Sen] and a specific formula for the coefficient of $\omega^m (\log \omega)^{m+1}$ terms have been put forward in the case of $2-2$ scattering [Alessio, Di Vecchia, Heissenberg].
- Tail contributions to the memory may become observable in future gravitational-wave detectors. They could also help distinguish binary black-hole mergers from other types of mergers, since tail terms are absent in the former.

Spin-memories

- The sub-leading term, $h^0_{\mu\nu}(\hat{n})$ is universal in $D > 4$. In $D = 4$, this term is completely fixed by the incoming and outgoing momenta and angular momenta of the initial and final objects respectively. It's called the spin-memory.

$$h^0_{\mu\nu}(\hat{n}) = \sum_{i\in(in+out)} \frac{p_{i,(\mu}J_{i,\nu)\rho}\hat{n}^\rho}{p_i \cdot \hat{n}} + \int_{S^{D-2}} d^{D-2}\hat{n}' \, \frac{A_\mu(\hat{n}')B_{\nu\rho}(\hat{n}')\hat{n}^\rho}{A(\hat{n}')\cdot\hat{n}} \,,$$

  where J_i is the classical angular momentum of the i-th object.
- We then have an infinite tower of higher-spin memories, $h^N_{\mu\nu}(\hat{n})$ for $N \geq 1$ that are non-universal.
- Finally, the third family (red terms) encapsulates the logarithmic terms which (for a given N) are sub-leading relative to the corresponding leading log soft factors.
- In the (retarded) time domain, the leading log terms decay as $\frac{1}{u}$, $\frac{\log u}{u^2}$, $\cdots \frac{(\log u)^m}{u^{m+1}}$ as $u \to \infty$.
- The physical origin of all the logarithmic soft modes is the long-range $(\frac{1}{r})$ interaction between the scattering states.

From S-matrix to classical soft theorems

- Soft universality of gravitational S-matrix:

$$\mathcal{M}_{N+1}(p_1, \cdots, p_n, \omega\hat{n}) = \sum_{i\in(in+out)} \Big[\frac{1}{\omega}\frac{p_{i,\mu}p_{i,\nu}}{p_i\cdot\hat{n}} + \frac{p_{i,(\mu}J_{i,\nu)\rho}\hat{n}^\rho}{p_i\cdot\hat{n}}\Big]\mathcal{M}_N(p_1, \cdots, p_n) + \mathcal{O}(\omega)\,.$$

[Weinberg], [Cachazo, Strominger], [Laddha, Sen]

- As KMOC proved, taking $\hbar \to 0$ limit as early as possible in the computation of expectation value, one can use the tools of scattering amplitudes to compute classical observables efficiently (for perturbative scattering).
- Classical radiative field from KMOC:

$$h_{\mu\nu}(\omega, \hat{n}) = \lim_{\hbar\to 0}\langle in|S^\dagger \hat{h}_{\mu\nu}(\omega\hat{n})S|out\rangle \approx \lim_{\hbar\to 0}\mathcal{M}_{N+1}(p_1, \cdots, p_n, \omega\hat{n})$$

- From $D > 4$ to $D = 4$: The logarithmic-drag is specific to $D = 4$ and is intrinsically tied to the fact that S-matrix is IR divergent in $D = 4$. The asymptotic trajectory:

$$x^\mu(t)|_{|t|\to\infty} = b^\mu + \frac{p^\mu}{m}|t| + C^\mu \log(|t|)\,.$$

So the sub-leading soft radiation contains a $\log(|t|) \to \log(\omega^{-1})$ term in $D = 4$. Later this was rigorously proved using asymptotic analysis of EOMs, valid for both large impact-parameter scattering as well as mergers [Saha, Sahoo, Sen].

Factorization theorems

- The most fascinating part of the story is the soft theorems for gravitational amplitudes. In $D = 4$,

$$\frac{\mathcal{M}_{N+1}(p_1, \cdots, p_n, k)}{\mathcal{M}_N(p_1, \cdots, p_n)} = \sum_{N=-1}^{1} \omega^N (\log \omega)^{N+1} S^{\log (N)}(p_1, \cdots, p_n, \hat{k})$$ [Sahoo,Sen]

- $n = -1$ is the Weinberg soft graviton theorem. $n = 0, 1$ are loop exact soft factor and have been computed in a series of papers [Sahoo,Sen].
- Classical limit of the Weinberg soft theorem is the displacement memory effect [Laddha,Sen] [Bautista, Guevara].
- Classical limit of the first quantum log soft theorem is tail to the memory to $\mathcal{O}(\kappa^5)$ [Athira, Ghosh, Laddha, Manu] [Alessio, Di Vecchia, Heissenberg].

(Sub)^n-leading soft theorems

- These theorems reveal the extent to which a gravitational amplitude factorizes when one of the gravitons becomes soft compared to other external momenta.
- For a generic theory of gravity with arbitrary matter coupling, the (sub)^n-leading soft graviton theorem can be written as [Hamada,Shiu]

$$\lim_{\omega\to 0}\partial_\omega^n[\omega\mathcal{M}_5(\tilde{p}_1,\tilde{p}_2\to p_1,p_2,k)] = \hat{S}^n\mathcal{M}_4+\mathcal{B}_n(p_1,p_2,\tilde{p}_1,\tilde{p}_2,\hat{k})\,\mathcal{M}_4 + \mathcal{R}_n(p_1,p_2,\tilde{p}_1,\tilde{p}_2,\hat{k})\,,$$

- $\mathcal{B}_n$ is the non-universal part of the factorization formula that depends on the irrelevant three-point couplings in the theory [Elvang,Jones,Naculich].
- $\mathcal{R}^n\neq 0\,\forall\, n\geq 3$ is the so-called remainder term that spoils factorization.
- How do the higher-order tree-level (sub)^n-leading soft graviton theorems constrain the gravitational scattering?
  We prove that despite their limitations, the (sub)^n-leading soft theorems can capture all the logarithmic soft factors in the limit of vanishing deflection [SA].

Gravitational scattering of massive spinless particles

- We consider massive scalars minimally coupled to gravity and analyze five-point scattering amplitude in which two scalar particles with momenta $\tilde{p}_1, \tilde{p}_2$ scatter into p_1, p_2 and a graviton of momentum k with the on-shell conditions $p_i^2 = \tilde{p}_i^2 = m_i^2$.

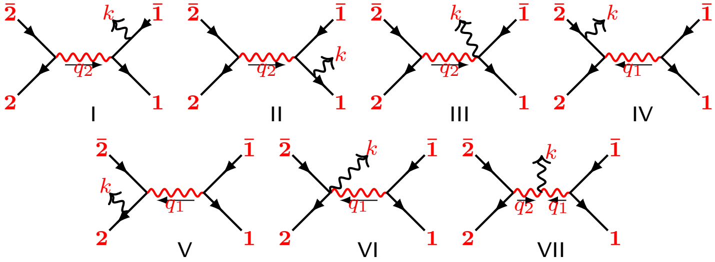


Figure: Tree-level five-point amplitudes for gravitational scattering of massive particles

- The remainder terms do not contribute to the logarithmic contributions that arise in the classical limit.
- In the deflection less limit ($|b| \to \infty$) such that ωb is fixed, all the log terms of the form $(\omega b)^m \log(\omega b) | m \geq 1$ survive and can be completely determined by the (sub)^n-soft "factorization" theorems for tree-level gravitational amplitudes.

Introduction to the IR-triangle

- Each term in the soft expansion of the radiative field is a field on the sphere at null-infinity.
- The fact that a specification of the asymptotic data completely fixes these terms independent of the gravitational dynamics is striking.
- Can we derive the universality of the terms in the soft expansion by simply studying the property of gravitational physics at the boundary?
- Asymptotically flat space-times (AFS): Class of spacetimes in which gravitational radiation *peels off* as a polynomial in $1/R$ so that the spacetime asymptotes to a flat space-time.
- In flat spacetime, the metric on S^2 at null infinity, $q_{AB} \approx d\theta^2 + \sin^2\theta \; d\phi^2$. When gravitational waves arrive, the metric gets disturbed:

$$q_{AB} \to q_{AB} + \frac{1}{r} C_{AB}(u, \theta, \phi)\,,$$

where C_{AB} is the shear field that characterizes the free radiative data at null infinity. The displacement memory effect is precisely this permanent change in the shear at early and late retarded times u.

IR-triangle

- As the spacetime is flat at null-infinity, spacetime translations are a symmetry. Bondi, Matzner, and Sachs (BMS) realized that even if we consider more general translations of u, then the geometry asymptotes to a flat metric at null infinity.
- Super-translations:

$$u \to u + f(\theta, \phi)$$
$$C_{AB} \to C_{AB} + f\partial_u C_{AB} + \partial_A \partial_B f \,.$$

- The displacement memory effect can be re-written as the conservation law of the supertranslation charges [Strominger]: $Q_{\mathcal{I}^+}[f] = Q_{\mathcal{I}^-}[f]$.
- We thus see that the super-translation symmetry, displacement memory effect and the Weinberg soft graviton theorem are three facets of the same universal abstract object (the first IR-triangle)

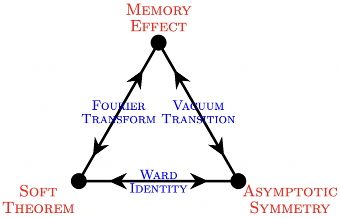


Figure: The IR-triangle

Conclusions

- We saw that unlike the usual path of computing the classical observables by solving the EOMs iteratively, one can define these observables in the QFT and compute them efficiently in terms of perturbative amplitudes.
- The power of the KMOC formalism is that the classical limit is taken as early as possible in the full amplitude. This drastically simplifies the quantum computation as only a subset of Feynman diagrams contributes in this limit.
- Additionally, the radiation-reaction effects are naturally inbuilt within the framework.
- We saw that the infinite tower of the low-frequency radiative modes is an interesting class of non-perturbative radiative observables whose analytic form can be used to probe infrared aspects of classical and quantum gravitational scattering.
- These modes can be efficiently computed from the classical limit of the quantum soft theorems.
- We also saw that the asymptotic symmetries, memory effects, and the quantum soft theorems are three facets of the same IR-triangle.

Outlook

- Since the relationship between classical and quantum soft theorems is non-perturbative and is valid regardless of the nature of hard scattering, one may compute the radiative field at null infinity to incorporate large deflection scattering processes in a different perspective in KMOC approach [SA, Laddha, Manna, Manu].
- A natural generalization of the program is to compute classical gravitational observables for the Kerr black hole at higher PM orders, including radiation-reaction effects. These involve the Kerr-Compton amplitudes [Cangemi, Chiodaroli, Johansson, Ochirov, Skvortsov] and using double-copy methods and tools from BHPT.
- There is an infinite dimensional enhancement of the $SL(2, C)$ on sphere at infinity, "super-rotations". Together with supertranslations and diff(S^2), they form a Generalized BMS group [Campiglia, Laddha].
  Tail to the memory $\leftrightarrow$ Log soft theorem $\leftrightarrow$ conservation of super-rotation charges

Outlook

- The existence of higher-order tree-level soft theorems has been recently linked to the discovery of the $w_{1+\infty}$ asymptotic algebra [Freidel, Pranzetti,Raclariu] [Guevara, Himwich,Pate,Strominger] [Geiller]. Our analysis can potentially reveal the link between higher spin asymptotic symmetries and a subset of logarithmic terms in the soft expansion of gravitational radiation.
- Like Love numbers of black holes, the vanishing of tail to the memory in a black hole merger, perhaps implies some deeper structure of gravitational scattering.
- Observables for bound states from amplitudes?
- Soft theorems for bound orbits?

Current and Future directions

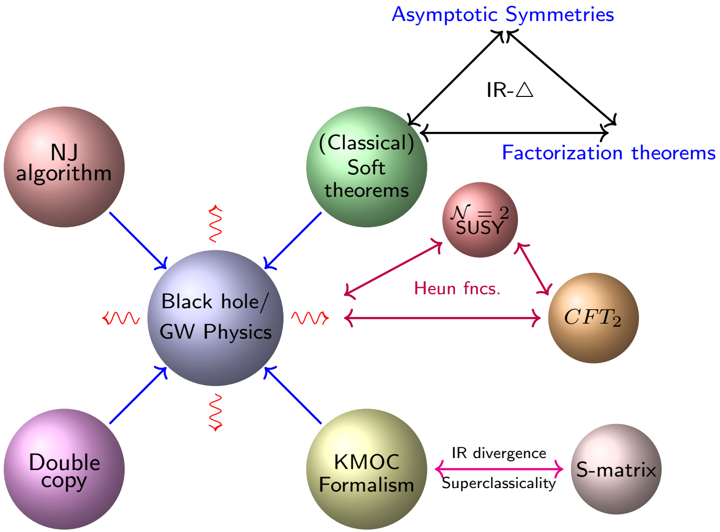


Figure: Current and future directions of my research work.

Thank You!

Backup slides

Effective action that reproduces the leading interactions of a Kerr Black hole [Levi,Steinhoff],[Vines]:

$$S_{int} = m \int d\tau \Big[\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} C_{ES^{2n}} (a \cdot \nabla)^{2n-2} R_{\alpha\beta\mu\nu} u^\alpha a^\beta u^\mu a^\nu$$

$$+ \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} C_{BS^{2n}} (a \cdot \nabla)^{2n-1} \, {}^*R_{\alpha\beta\mu\nu} u^\alpha a^\beta u^\mu a^\nu \Big]_{x=r(\tau)} .$$

with $C_{ES^{2n}} = 1$ and $C_{BS^{2n}} = -1$.

Radiation kernel

- The scaling of massless momenta makes the KMOC formalism particularly effective for calculating radiative observables at null infinity $\mathcal{I}^+$. One of such observables is the radiative gauge field.
- To obtain radiative gauge field, consider expectation value of photon momentum operator

$$\langle \hat{K}^\mu \rangle = \langle \Psi | S^\dagger \hat{K}^\mu S | \Psi \rangle = \langle \Psi | T^\dagger \hat{K}^\mu T | \Psi \rangle \ , \hat{K}^\mu = \sum_{h=\pm} \int d\phi(k)\, k^\mu \, a_h^\dagger(k)\, a_h(k)$$

- In the classical limit,

$$\langle \hat{K}^\mu \rangle_{cl} = \sum_X \int d\phi(\bar{X}) \bar{k}_X^\mu |\mathcal{R}_{cl}(\bar{X})|^2 \, .$$

- Here $\mathcal{R}_{cl}(X)$ defines the radiative gauge field. To leading order in the coupling with single photon emission,

$$\mathcal{R}_{cl}^{(0)}(\bar{k}) = \frac{\hbar^{3/2}}{4} \prod_{i=1,2} \int \hat{d}^4 \bar{q}_i \hat{\delta}(p_i \cdot \bar{q}_i) \hat{\delta}^{(4)}(\bar{q_1} + \bar{q_2} - \bar{k}) e^{ib\cdot \bar{q_2}}$$
$$\times \bar{\mathcal{A}}_5^{(0)}(p_1 + \hbar \bar{q_1}, p_2 + \hbar \bar{q_2} \to p_1, p_2, \hbar \bar{k}) \, .$$

- The starting point is formula for the linear impulse, computed using the KMOC formalism,

$$\Delta p_1^\mu = \frac{\hbar^3}{4}\int \hat{d}^4\bar{q}\hat{\delta}(2p_1\cdot\bar{q})\hat{\delta}(2p_2\cdot\bar{q})e^{i\bar{q}\cdot b}i\bar{q}^\mu\ A_4(p_1,p_2\to p_1+\hbar\bar{q},p_2-\hbar\bar{q})|_{\hbar\to 0}\,, \tag{1}$$

  where p_1 and p_1+q are the momenta of the scalar and p_2 and p_2-q are the momenta of the massive higher spin particle.

- To construct the 4-pt amplitude, we will glue the corresponding 3-pt amplitudes for the scalar as well as the higher spin particles. The 3-pt amplitudes for a spinning particle are

$$h=+1:\ i\sqrt{2}Q_2\ x\frac{\langle\mathbf{22'}\rangle^{2S}}{m_2^{2S-1}};\quad h=-1:\ i\sqrt{2}Q_2\frac{1}{x}\frac{[\mathbf{22'}]^{2S}}{m_2^{2S-1}}\,, \tag{2}$$

  where $x=\frac{\langle\zeta|p_2|q]}{m_2\langle\zeta q\rangle}$and S is the spin of the particle, with $S=0$ gives the 3-pt amplitudes of a scalar minimally coupled to the photon.

- Since q is small, the spinor $|\mathbf{2}'\rangle$ is only a small boost of the spinor $|\mathbf{2}\rangle$. Taking account of the on-shell relation $2p_2 \cdot q = q^2 \simeq 0$,

$$\frac{1}{m_2}\langle \mathbf{2}\mathbf{2}'\rangle = \mathbb{I} + \frac{1}{2Sm_2}\bar{q} \cdot s_2 \,, \tag{3}$$

where s_2^μ is the Pauli-Lubanski pseudovector associated with a spin S particle: $s^\mu = \frac{S\hbar}{m_2}\langle \mathbf{2}|\sigma^\mu|\mathbf{2}]$.
- We now take the limit $S \to \infty$ and $\hbar \to 0$ with $S\hbar$ fixed. The amplitudes become

$$h = \pm 1 : iQ_2\sqrt{2}m_2 \; x^{\pm 1} e^{\pm q \cdot a_2} \,, \tag{4}$$

where $a_2 = s_2/m_2$.
- We now use the expression for x and $x\bar{x} = 1$ to get the classical limit of the amplitude

$$\begin{aligned} A_4(p_1, p_2 \to p_1 + \hbar q, p_2 - \hbar q)|\hbar \to 0 &= \frac{Q_1 Q_2 m_1 m_2}{\hbar^3 q^2} p_1^\mu \left(\epsilon_\mu^+ \epsilon_\nu^- e^{q \cdot a_2} + \epsilon_\mu^- \epsilon_\nu^+ e^{-q \cdot a_2}\right) p_2^\nu \\ &= \frac{Q_1 Q_2 m_1 m_2}{\hbar^3 q^2} \left(e^{-w} e^{q \cdot a_2} + e^{w} e^{-q \cdot a_2}\right) , \end{aligned} \tag{5}$$

where in going from the first equality to the second we have used

$$\frac{x_{11'}}{x_{22'}} = e^w \text{ with } \cosh w = u_1 \cdot u_2 \tag{6}$$

with w being the rapidity.

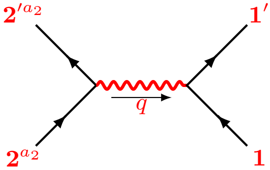


Figure: The four-point scalar - $\sqrt{\text{Kerr}}$ amplitude with photon exchange. Here 'a_2' denotes the rescaled spin of the $\sqrt{\text{Kerr}}$ particle.

- Plugging this amplitude into the linear impulse formula and using the identity,

$$q_\mu \sinh w = i\epsilon_{\mu\nu\rho\sigma} u_1^\nu u_2^\rho q^\sigma \, , \tag{7}$$

the linear impulse can be written as

$$\Delta p_1^\mu = Q_1 Q_2 \text{Re} \int \hat{d}^4 q \hat{\delta}(u_1 \cdot q) \hat{\delta}(u_2 \cdot q) \frac{i}{q^2} \left[(q^\mu \cosh w - i\epsilon^{\mu\nu\rho\sigma} u_{1\nu} u_{2\rho} q_\sigma) e^{-iq\cdot(b+ia_2)} \right] . \tag{8}$$

Hence, we see that the NJ shift manifests itself in the simple shift of the impact parameter, $b \to b + ia_2$. This tells us that the linear impulse for the scalar particle in the background of a $\sqrt{\text{Kerr}}$ object can be obtained from a scalar-scalar scattering by simply complexifying the impact parameter.

LO Angular Impulse

- The leading order orbital angular impulse in the KMOC formalism is given by

$$\Delta L_i^{\mu\nu} = \frac{\hbar^2}{4}\int \hat{d}^4\bar{q}_1 \hat{d}^4\bar{q}_2 \hat{\delta}(p_1\cdot\bar{q}_1)\hat{\delta}(p_2\cdot\bar{q}_2) e^{-i(b\cdot\bar{q}_2)}\left[\left(\tilde{p}_i \wedge \frac{\partial}{\partial\tilde{p}_i}\right)^{\mu\nu} + \left(p_i \wedge \frac{\partial}{\partial p_i}\right)^{\mu\nu}\right] \hat{\delta}^{(4)}(\bar{q}_1+\bar{q}_2)\, \mathcal{A}_4(p_1,p_2\to\tilde{p}_1,\tilde{p}_2)\,, \tag{9}$$

where p_i's are initial momenta and we denote the final momenta as $\tilde{p}_i = p_i + \hbar\bar{q}_i$. Here $\mathcal{A}_4(p_1,p_2\to\tilde{p}_1,\tilde{p}_2)$ is the deformed four-point scalar$-\sqrt{\text{Kerr}}$ scattering amplitude.
- After a linear transformation $(p_i,\tilde{p}_i)\to(p_i,q_i)$, the orbital angular impulse is given by

$$\Delta L_i^{\mu\nu} = \frac{\hbar^2}{4}\int \hat{d}^4\bar{q}_1 \hat{d}^4\bar{q}_2 \hat{\delta}(p_1\cdot\bar{q}_1)\hat{\delta}(p_2\cdot\bar{q}_2) e^{-ib\cdot\bar{q}_2}\left[(p_i\wedge\partial_{p_i})^{\mu\nu} + (\bar{q}_i\wedge\partial_{\bar{q}_i})^{\mu\nu}\right] \left\{\hat{\delta}^{(4)}(\bar{q}_1+\bar{q}_2)\mathcal{A}_4(p_1,p_2\to p_1+\hbar\bar{q}_1, p_2+\hbar\bar{q}_2)\right\}\,. \tag{10}$$

For the scalar particle, we do integration by parts on the second term in the first line in eq.(10) and integrate over $\bar{q}_2$ to obtain

$$\Delta L_1^{\mu\nu} = \Delta L_{1,I}^{\mu\nu} + \Delta L_{1,II}^{\mu\nu}\,, \tag{11}$$

with

$$\Delta L_{1,I}^{\mu\nu} = \frac{\hbar^2}{4}\int \hat{d}^4\bar{q} e^{i\bar{q}\cdot b}\hat{\delta}(p_1\cdot\bar{q})\hat{\delta}(p_2\cdot\bar{q})\left(p_1\wedge\frac{\partial}{\partial p_1}\right)^{\mu\nu}\mathcal{A}_4(p_1,p_2\to p_1+\hbar\bar{q}, p_2-\hbar\bar{q})$$
$$\Delta L_{1,II}^{\mu\nu} = -\frac{\hbar^2}{4}\int \hat{d}^4\bar{q} e^{i\bar{q}\cdot b}\ \hat{\delta}'(p_1\cdot\bar{q})\ \hat{\delta}(p_2\cdot\bar{q})(\bar{q}\wedge p_1)^{\mu\nu}\ \mathcal{A}_4(p_1,p_2\to p_1+\hbar\bar{q}, p_2-\hbar\bar{q})\,. \tag{12}$$

- We use[4] $\frac{\partial}{\partial p_j^\mu} p_i^\alpha = \delta_i^j \, \delta_\mu^\alpha, \qquad \partial_{p_1^\mu} a_2^\alpha = 0.$
- Summing the two expressions $\Delta L_{1,I}^{\mu\nu}$ and $\Delta L_{1,II}^{\mu\nu}$, we obtain the orbital angular impulse of the scalar particle

$$\Delta L_1^{\mu\nu} = Q_1 Q_2 \int \hat{d}^4\bar{q}\hat{\delta}(p_1\cdot\bar{q})\hat{\delta}(p_2\cdot\bar{q})\frac{e^{i\bar{q}\cdot b}}{\bar{q}^2}\Big[(p_1\wedge p_2)^{\mu\nu}\cosh(a_2\cdot\bar{q})\left(1-\frac{(p_1\cdot p_2)^2}{\mathcal{D}}\right)$$
$$+ i\frac{\sinh(a_2\cdot\bar{q})}{(a_2\cdot\bar{q})}\left(p_1^{[\mu}\epsilon^{\nu]}(p_2,a_2,\bar{q}) + \frac{p_1\cdot p_2}{\mathcal{D}}\epsilon(p_1,p_2,a_2,\bar{q})(p_2\wedge p_1)^{\mu\nu}\right)$$
$$+ \frac{m_2^2}{\mathcal{D}}(a_2\cdot p_1)(p_1\wedge\bar{q})^{\mu\nu}\left\{(p_1\cdot p_2)\sinh(a_2\cdot\bar{q}) + i\mathcal{Y}\epsilon(p_1,p_2,a_2,\bar{q})\right\}\Big], \tag{13}$$

where $\mathcal{Y} = \left[\frac{\cosh(a_2\cdot\bar{q}_\perp)}{(a_2\cdot\bar{q}_\perp)} - \frac{\sinh(a_2\cdot\bar{q}_\perp)}{(a_2\cdot\bar{q}_\perp)^2}\right]$. This expression matches with the answer obtained through solving classical EOMs perturbatively.

[4] The 4-point amplitude depends on the external momenta of the scattering particles as well as the (classical) spin vector a_2^μ. It is related to the spin tensor $S_2^{\mu\nu}$ via $a_2^\mu = \frac{1}{2m_2^2}\epsilon^{\mu\nu\rho\sigma}p_{2\nu}S_{2\rho\sigma}$. It is rather natural to interpret $S_2^{\mu\nu}$ as the independent spin tensor which can be thought of as an "intrinsic" spin angular momentum of a classical particle.

- The orbital angular impulse for the scalar particle to linear order in spin written in terms of $S_2^{\mu\nu}$ is

$$\Delta L_1^{\mu\nu} = \frac{Q_1 Q_2}{2\pi\sqrt{\mathcal{D}}}\Big[\frac{1}{\beta^2\gamma^2}(p_2\wedge p_1)^{\mu\nu}\log|\mu_1 b| + \frac{1}{b^2}\Big(p_1^{[\mu}S_2^{\nu]\rho}b_\rho + (p_2\wedge p_1)^{\mu\nu}\frac{(p_1\cdot p_2)}{\mathcal{D}}S_2^{\rho\sigma}p_{1\rho}b_\sigma\Big)\Big] + \mathcal{O}(S_2^2)\,, \tag{14}$$

where μ_1 is the IR cutoff.

- For the $\sqrt{\text{Kerr}}$ particle, the integral of (10) is rewritten as follows

$$\Delta L_2^{\mu\nu} = -(b\wedge\Delta p_2)^{\mu\nu} + \Delta L_{2,I}^{\mu\nu} + \Delta L_{2,II}^{\mu\nu}\,, \tag{15}$$

where

$$\Delta L_{2,I}^{\mu\nu} = \frac{\hbar^2}{4}\int \hat{d}^4\bar{q}e^{i\bar{q}\cdot b}\hat{\delta}(p_1\cdot\bar{q})\hat{\delta}(p_2\cdot\bar{q})\left(p_2\wedge\frac{\partial}{\partial p_2}\right)^{\mu\nu}\mathcal{A}_4(p_1,p_2\to p_1+\hbar\bar{q},p_2-\hbar\bar{q})\,,$$

$$\Delta L_{2,II}^{\mu\nu} = -\frac{\hbar^2}{4}\int \hat{d}^4\bar{q}e^{i\bar{q}\cdot b}\ \hat{\delta}(p_1\cdot\bar{q})\ \hat{\delta}'(p_2\cdot\bar{q})\ (\bar{q}\wedge p_2)^{\mu\nu}\ \mathcal{A}_4(p_1,p_2\to p_1+\hbar\bar{q},p_2-\hbar\bar{q})\,.$$

$$\Delta p_2^{\mu} = \frac{\hbar^2}{4}\int \hat{d}^4\bar{q}\hat{\delta}(p_1\cdot\bar{q})\hat{\delta}(p_2\cdot\bar{q})\ e^{i\bar{q}\cdot b}\ (-i\bar{q}^\mu)\ \mathcal{A}_4(p_1,p_2\to p_1+\hbar\bar{q},p_2-\hbar\bar{q})\,. \tag{16}$$

- The evaluation of $\Delta L_{2,I}^{\mu\nu}$ is rather subtle as for any function $f(a_2, \bar{q})$ we obtain terms involving $\frac{\partial}{\partial p_2^\mu} f(a_2, \bar{q}) = -\frac{1}{2m_2^2} \frac{\partial f(a_2, \bar{q})}{\partial a_2^\alpha} \epsilon_\mu{}^{\alpha\rho\sigma} S_{2\rho\sigma}$, where we have used $\frac{\partial a_2^\alpha}{\partial p_2^\mu} = \frac{1}{2m_2^2} \epsilon^{\alpha\beta\rho\sigma} S_{2\rho\sigma} \delta_{\mu\beta}$.
- We simply evaluate the impulse to linear order in $S_2^{\mu\nu}$.

$$
\begin{aligned}
\Delta L_2^{\mu\nu} &= Q_1 Q_2 \int \hat{d}^4 \bar{q} \hat{\delta}(\bar{q} \cdot p_1) \hat{\delta}(\bar{q} \cdot p_2) e^{i\bar{q}\cdot b} \frac{1}{\bar{q}^2} \Big[\frac{1}{\beta^2 \gamma^2} (p_1 \wedge p_2)^{\mu\nu} - (b \wedge \bar{q})^{\mu\nu} \, (S_2^{\rho\sigma} p_{1\rho} \bar{q}_\sigma) \\
&\qquad + i (p_1 \wedge p_2)^{\mu\nu} \frac{(p_1 \cdot p_2)}{\mathcal{D}} S_2^{\rho\sigma} p_{1\rho} \bar{q}_\sigma \Big] \\
&= \frac{Q_1 Q_2}{2\pi \sqrt{\mathcal{D}}} \Big[\frac{1}{\beta^2 \gamma^2} (p_1 \wedge p_2)^{\mu\nu} \log |\mu_2 b| - \frac{1}{b^2} \Big(b^{[\mu} S_2^{\nu]\rho} p_{1\rho} - (p_1 \wedge p_2)^{\mu\nu} \frac{(p_1 \cdot p_2)}{\mathcal{D}} S_2^{\rho\sigma} p_{1\rho} b_\sigma \Big) \Big]
\end{aligned}
\tag{17}
$$

where μ_2 is the IR cutoff.

Conservation of Angular Momentum

- The spin kick Δa_2^μ imparted on the $\sqrt{\text{Kerr}}$ particle with mass m_2 and ring radius a_2 in a scalar-$\sqrt{\text{Kerr}}$ scattering to leading order in coupling using the KMOC formalism is given by:

$$\Delta a_2^\mu = \frac{i\hbar^2}{4}\int \hat{d}^4\bar{q}\hat{\delta}(p_1\cdot\bar{q})\hat{\delta}(p_2\cdot\bar{q})e^{i\bar{q}\cdot b}\Big\{\left[a_2^\mu(p_2),\mathcal{A}_4\right] + \frac{\hbar}{m_2}(a_2\cdot\bar{q})u_2^\mu\mathcal{A}_4\Big\}. \tag{18}$$

Using the NJ deformed 4-point amplitude, we have

$$\Delta a_2^\mu = \frac{e^2}{m_2}\,\mathrm{Re}\int \hat{d}^4\bar{q}\hat{\delta}(\bar{q}\cdot u_1)\hat{\delta}(\bar{q}\cdot u_2)\frac{i}{\bar{q}^2}e^{-iq\cdot(b+ia_2)}\left[u_1^\mu(a_2\cdot\bar{q}) - \bar{q}^\mu(a_2\cdot u_1) - i\epsilon^\mu(u_1,a_2,\bar{q})\right]. \tag{19}$$

This matches with the spin kick obtained using classical equations of motion.

- Using the dual relation, we have $\Delta S_2^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\Delta p_{2\rho}a_{2\sigma} + \epsilon^{\mu\nu\rho\sigma}p_{2\rho}\Delta a_{2\sigma}$. Therefore, we obtain the following expression for spin angular impulse

$$\begin{aligned}\Delta S_2^{\mu\nu} = -Q_1Q_2\int \hat{d}^4\bar{q}\hat{\delta}(\bar{q}\cdot u_1)\hat{\delta}(\bar{q}\cdot u_2)\frac{e^{i\bar{q}\cdot b}}{\bar{q}^2}\Big[\cosh(a_2\cdot\bar{q})\Big\{iu_1^{[\mu}\epsilon^{\nu]}(\bar{q},u_2,a_2)\\ - i\bar{q}^{[\mu}\epsilon^{\nu]}(u_1,u_2,a_2)\Big\} + \sinh(a_2\cdot\bar{q})\Big\{u_2^{[\mu}u_1^{\nu]}(a_2\cdot\bar{q}) - u_2^{[\mu}\bar{q}^{\nu]}(a_2\cdot u_1) + \gamma a_2^{[\mu}\bar{q}^{\nu]}\Big\}\Big].\end{aligned} \tag{20}$$

- At linear order in $S_2^{\mu\nu}$, the spin angular impulse can be evaluated.

$$\Delta S_2^{\mu\nu} = -iQ_1Q_2 \int \hat{d}^4\bar{q}\hat{\delta}(p_1\cdot\bar{q})\hat{\delta}(p_2\cdot\bar{q})e^{i\bar{q}\cdot b}\frac{1}{\bar{q}^2}\left[p_1^{[\mu}S_2^{\nu]\sigma}\bar{q}_\sigma - \bar{q}^{[\mu}S_2^{\nu]\sigma}p_{1\sigma}\right] + \mathcal{O}(S_2^2) \tag{21}$$

$$\Delta S_2^{\mu\nu} = \frac{Q_1Q_2}{2\pi\sqrt{\mathcal{D}}\,b^2}(b^{[\mu}S^{\nu]\alpha}p_{1\alpha} - p_1^{[\mu}S^{\nu]\alpha}b_\alpha). \tag{22}$$

- We now have all the expressions to compute the total angular impulse for the the scalar-$\sqrt{\text{Kerr}}$ scattering to linear order in spin. This is given by the sum of eqs.(14),(17) and (21). We obtain the following result

$$\Delta J^{\mu\nu} = \frac{Q_1Q_2}{2\pi\sqrt{\mathcal{D}}}\frac{1}{\beta^2\gamma^2}(p_1\wedge p_2)^{\mu\nu}\log\left|\frac{\mu_2}{\mu_1}\right| + \delta^{\mu\nu}_{\text{scalar-scoot}} = 0\,, \tag{23}$$

where

$$\delta^{\mu\nu}_{\text{scalar-scoot}} = -\frac{Q_1Q_2}{2\pi\sqrt{\mathcal{D}}}\frac{1}{\beta^2\gamma^2}(p_1\wedge p_2)^{\mu\nu}\log\left|\frac{\tau_1}{\tau_2}\right|. \tag{24}$$

The IR cutoffs are related to the proper times of the two particles via $\frac{\mu_2}{\mu_1} = \frac{\tau_1}{\tau_2}$ *Veneziano, et al."22, Bhardwaj, Lippstreu"22*. Hence, the total angular momentum for scalar - $\sqrt{\text{Kerr}}$ scattering is conserved, to linear order in spin.

NJ exponentiation [Arkani-Hamed, Huang, O'Connell]

When dressed with the external polarization tensors, the three-point, minimally-coupled amplitude is given as [13]

$$h=+1: \quad \sqrt{2} i e_2 x \frac{\langle \mathbf{2 2}'\rangle^{2S}}{m^{2S-1}}, \quad h=-1: \quad \sqrt{2} i e_2 \frac{1}{x} \frac{[\mathbf{2 2}']^{2S}}{m^{2S-1}}. \tag{18}$$

Since q is small, the spinor $|\mathbf{2}'\rangle$ is only a small boost of the spinor $|\mathbf{2}\rangle$. We may therefore write

$$|\mathbf{2}'\rangle = |\mathbf{2}\rangle + \frac{1}{8}\omega_{\mu\nu}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)|\mathbf{2}\rangle, \tag{19}$$

where the boost parameters $\omega_{\mu\nu}$ are small. It is easy to compute these boost parameters because

$$p_2'^{\mu} = (\delta^\mu_\nu + \omega^\mu{}_\nu)p_2^\nu \implies \omega_{\mu\nu} = -\frac{1}{m_2^2}(p_{2\mu}q_\nu - p_{2\nu}q_\mu), \tag{20}$$

taking account of the on-shell relation $2p_2\cdot q = q^2 \simeq 0$. We therefore learn that

$$|\mathbf{2}'\rangle = |\mathbf{2}\rangle + \frac{1}{2m_2}\not{q}|\mathbf{2}]. \tag{21}$$

Thus, we have,

$$\frac{1}{m_2}\langle \mathbf{2 2}'\rangle = \mathbb{I} + \frac{1}{2m_2^2}\hbar\langle \mathbf{2}|\not{\bar{q}}|\mathbf{2}] = \mathbb{I} + \frac{1}{2Sm_2}\bar{q}\cdot s, \tag{22}$$

where s^μ is the Pauli-Lubanski pseudovector associated with a spin S particle:

$$s^\mu = \frac{1}{m_2}S\hbar\langle \mathbf{2}|\sigma^\mu|\mathbf{2}]. \tag{23}$$

The operators $\mathbb{I}$ and s^μ are now operators acting on the little group space of particle 2. In the end, all little group

Figure: NJ exponentiation

indices will be contracted with appropriate wave functions. We now take the limit $S\to\infty$ and $\hbar\to 0$ with $S\hbar$ fixed. The amplitudes become

$$h=\pm 1: \quad \lim_{S\to\infty} i e_2\sqrt{2}m\, x^{\pm 1}\left(\mathbb{I}\pm\frac{\bar{q}\cdot s}{2Sm}\right)^{2S} = i e_2\sqrt{2}m x^{\pm 1}e^{\pm\bar{q}\cdot a} \tag{24}$$

where the quantity $a = \frac{s}{m}$ parameterises the spin, but has dimensions of length.

Now let's consider the classical limit of the four point amplitude between a charged particle of spin S, with $S\to\infty$, and a scalar particle:

$$M_4(1,2\to 1',2')\,|_{q^2\to 0} = 2\frac{e_1e_2m_1m_2}{2\bar{q}^2}\left(\frac{x_{11'}}{x_{22'}}e^{-\bar{q}\cdot a} + \frac{x_{22'}}{x_{11'}}e^{\bar{q}\cdot a}\right) \tag{25}$$

Note that it is given by two terms with different helicity configurations. We will see that these terms have a crucial role, allowing us to understand the emergence of the real-part operation in the impulse. The x ratios are little group invariant, and can be shown to be given by:

$$\frac{x_{11'}}{x_{22'}} = e^{w}, \quad \frac{x_{22'}}{x_{11'}} = e^{-w}, \tag{26}$$

where w is the rapidity. Thus we have,

$$M_4(1,2\to 1',2')\,|_{q^2\to 0} = 2\frac{e_1e_2m_1m_2}{\hbar^3\bar{q}^2}\left(e^{w}e^{-\bar{q}\cdot a} + e^{-w}e^{\bar{q}\cdot a}\right) \tag{27}$$

Figure: NJ exponentiated amplitude

Gravitational scattering of massive spinless particles

- $$\bar{\mathcal{M}}_5^{\mu\nu}[p_1 + \hbar\bar{q}_1, p_2 + \hbar\bar{q}_2 \to p_1, p_2,, \hbar\bar{k}]$$
$$= \hat{\delta}^{(4)}(\bar{q}_1 + \bar{q}_2 - \bar{k})\bar{\mathcal{A}}_5^{\mu\nu}[p_1 + \hbar\bar{q}_1, p_2 + \hbar\bar{q}_2 \to p_1, p_2, \hbar\bar{k}]\,.$$
- The stripped amplitude is given by

$$\bar{\mathcal{A}}_5^{\mu\nu}[p_1 + \hbar\bar{q}_1, p_2 + \hbar\bar{q}_2 \to p_1, p_2,, \hbar\bar{k}] = -\frac{\kappa^3 m_1^2 m_2^2}{\hbar^2}\Big[\frac{4P^\mu P^\nu}{\bar{q}_1^2\bar{q}_2^2} + \frac{2\gamma}{\bar{q}_1^2\bar{q}_2^2}\Big(Q^\mu P^\nu + Q^\nu P^\mu\Big)$$
$$+\Big(\gamma^2 - \frac{1}{2}\Big)\Big(\frac{Q^\mu Q^\nu}{\bar{q}_1^2\bar{q}_2^2} - \frac{P^\mu P^\nu}{\omega_1^2\omega_2^2}\Big)\Big], \tag{25}$$

where $\kappa = \sqrt{32\pi G}$ and

$$P^\mu = -\omega_1 u_2^\mu + \omega_2 u_1^\mu$$
$$Q^\mu = (\bar{q}_1 - \bar{q}_2)^\mu + \frac{\bar{q}_1^2}{\omega_1}u_1^\mu - \frac{\bar{q}_2^2}{\omega_2}u_2^\mu\,,\ \omega_1 = -\bar{k}\cdot u_1,\ \omega_2 = -\bar{k}\cdot u_2\,.$$

- The soft graviton factors are given by

$$S^{(0),\mu\nu} = \kappa \sum_{i=1,2} \left[\frac{1}{p_i \cdot k} p_i^{(\mu} p_i^{\nu)} - \frac{1}{\tilde{p}_i \cdot k} \tilde{p}_i^{(\mu} \tilde{p}_i^{\nu)} \right]$$

$$S^{(1),\mu\nu} = i \frac{\kappa}{2} \sum_{i=1,2} \left[\frac{1}{p_i \cdot k} p_i^{(\mu} \hat{J}_i^{\nu)\rho} k_\rho - \frac{1}{\tilde{p}_i \cdot k} \tilde{p}_i^{(\mu} \hat{\tilde{J}}_i^{\nu)\rho} k_\rho \right]$$

$$S^{(n),\mu\nu} = \frac{\kappa}{2} \sum_{i=1,2} \left[\frac{\hat{J}_i^{\mu\rho} k_\rho \hat{J}_i^{\nu\sigma} k_\sigma}{p_i \cdot k} \left(k \cdot \frac{\partial}{\partial p_i} \right)^{n-2} + \frac{\hat{\tilde{J}}_i^{\mu\rho} k_\rho \hat{\tilde{J}}_i^{\nu\sigma} k_\sigma}{\tilde{p}_i \cdot k} \left(k \cdot \frac{\partial}{\partial \tilde{p}_i} \right)^{n-2} \right] \quad , n \geq 2 \, .$$

- The remainder term for the amplitude is given by

$$\mathcal{X}^{\mu\nu} = \frac{\kappa^3 m_1^2 m_2^2}{4} \sum_{r=3}^{n} \frac{(-1)^{n-r}}{(n-r)!} (k \cdot \partial)^{n-r} (\hat{\delta}^{(4)}(q_1 + q_2)) \Lambda_{r-1}^{\mu\nu} + (1 \leftrightarrow 2) \, ,$$

[SA]

where the polynomial $\Lambda_n^{\mu\nu}$ is defined as

$$\Lambda_{n\geq 2}^{\mu\nu} = H_2^{\mu\nu} \frac{2^{n-2} (\bar{q} \cdot \bar{k})^{n-2}}{(\bar{q}^2)^{n-2}} \, , \quad H_2^{\mu\nu} = -\frac{4}{(\bar{q}^2)^2} \left(\omega_2^2 u_1^\mu u_1^\nu - \frac{\omega_1 \omega_2}{2} (u_2^\mu u_1^\nu + u_2^\nu u_1^\mu) \right) . \tag{26}$$

LO gravitational radiation

- Using the expansions, the $(\text{sub})^n|_{n\geq 2}$-leading soft radiation is given by

$$\mathcal{R}^{(n),\mu\nu}(\bar{k}) = \frac{\kappa^3 m_1 m_2}{4}\int \hat{d}^4\bar{q}\Big[\sum_{r=0}^{n}\frac{1}{(n-r)!}e^{-ib\cdot\bar{q}}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})(ib\cdot\bar{k})^{n-r}K^{\mu\nu}_{r-1}$$
$$+\sum_{r=0}^{n-1}\frac{(-1)^{n-r}}{(n-r)!}\Big\{e^{-ib\cdot\bar{q}}\hat{\delta}^{(n-r)}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})(u_1\cdot\bar{k})^{n-r}\Big(K^{\mu\nu}_{r-1}\Big)+e^{ib\cdot\bar{q}}\Big(1\leftrightarrow 2\Big)\Big\}$$
$$+\sum_{r=0}^{n-2}\sum_{\substack{t,s\geq 1\\ \ni(t+s)=n-r}}\frac{(-1)^s}{t!s!}e^{-ib\cdot\bar{q}}(ib\cdot\bar{k})^t(u_1\cdot\bar{k})^s\hat{\delta}^{(s)}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})K^{\mu\nu}_{r-1}\Big],$$

where the polynomial $K^{\mu\nu}_n$ is defined as

$$K^{\mu\nu}_{n\geq 1} = H^{\mu\nu}_1\frac{2^{n-1}(\bar{q}\cdot\bar{k})^{n-1}}{(\bar{q}^2)^{n-1}} + H^{\mu\nu}_0\frac{2^n(\bar{q}\cdot\bar{k})^n}{(\bar{q}^2)^n},$$

where

$$H^{\mu\nu}_1 = \frac{4\gamma}{\bar{q}^2}\frac{\omega_2}{\bar{q}^2}\bar{q}^{(\mu}u_1^{\nu)} \quad \text{and} \quad H^{\mu\nu}_0 = -\frac{2}{\bar{q}^2}\Big(\gamma^2-\frac{1}{2}\Big)\frac{\bar{q}^\mu\bar{q}^\nu}{\bar{q}^2}.$$

Remainder terms

- Collecting the logarithmic terms and upon simplifying, it matches with the log terms of (sub)^n-leading order soft radiation in eq. (**??**).
- Some terms in the (unstripped) five-point amplitude do not factorize as soft factors times the four-point amplitude. These are known as the remainder terms. We have identified such terms at (sub)^n-leading order for $n \geq 3$ in the soft radiation given by

$$\mathcal{X}^{\mu\nu}_{\mathcal{R},\omega^{(n-1)}} = \frac{\kappa^3 m_1 m_2}{4} \int \hat{d}^4\bar{q} \Big[\sum_{r=3}^{n} \frac{1}{(n-r)!} e^{-ib\cdot\bar{q}} \hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})(ib\cdot\bar{k})^{n-r}\Lambda^{\mu\nu}_{r-1}$$
$$+ \sum_{r=3}^{n-1} \frac{(-1)^{n-r}}{(n-r)!} \Big\{ e^{-ib\cdot\bar{q}} \hat{\delta}^{(n-r)}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})(u_1\cdot\bar{k})^{n-r} \Big(\Lambda^{\mu\nu}_{r-1}\Big) + e^{ib\cdot\bar{q}}\Big(1 \leftrightarrow 2\Big)\Big\}$$
$$+ \sum_{r=3}^{n-2} \sum_{\substack{t,s\geq 1 \\ \ni (t+s)=n-r}} \frac{(-1)^s}{t!s!} e^{-ib\cdot\bar{q}} (ib\cdot\bar{k})^t (u_1\cdot\bar{k})^s \hat{\delta}^{(s)}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})\Lambda^{\mu\nu}_{r-1}\Big],$$

  where the polynomial $\Lambda^{\mu\nu}_n$ is defined in eq. (26).
- One can check that these remainder terms do not contribute to the logarithmic contributions that arise in the classical limit. Hence, the low-frequency radiative field can simply be obtained from the (sub)^n soft graviton "factorization" theorems.

- However, the tensorial structure of $\mathcal{R}_n$ [Hamada,Shiu] [Li, Lin, Zhang]

$$\mathcal{R}_n = \epsilon_{\mu\nu}\hat{k}_{\alpha_1}\hat{k}_{\alpha_2}\cdots\hat{k}_{\alpha_{n-1}}A^{\mu\nu\alpha_1\alpha_2\cdots\alpha_{n-1}}$$

leads to the following "factorization formula" for tree-level amplitude at all orders in the soft expansion

$$\lim_{\omega\to 0}\partial_\omega^n[\omega\Pi_{\hat{n}}^{-}\mathcal{M}_5(\tilde{p}_1,\tilde{p}_2\to p_1,p_2,k)] = \Pi_{\hat{n}}^{-}\hat{S}^n\mathcal{M}_4(\tilde{p}_1,\tilde{p}_2\to p_1,p_2)\,,$$

For minimally-coupled gravity theories, $\mathcal{B}_n = 0\,\forall\, n$. $\Pi_{\hat{n}}^{-} := D_z^{n+1}(1+|z|^2)^{-1}$ is the projection operator, where $\hat{k} = (1,\hat{n}(z,\bar{z}))$ and $z,\bar{z}$ are the stereographic coordinates [Campiglia,Laddha].

Classical soft theorems

For a generic gravitational scattering in $D=4$, the radiative field has the following form under soft expansion,

$$h_{\mu\nu}(\omega, r, \hat{n}) = \frac{1}{r} e^{i\omega r} \sum_{N=-1}^{\infty} \omega^N \, h_{\mu\nu}^{N}(\hat{n}) + \sum_{m=0}^{\infty} \omega^m \, (\log\omega)^{m+1} \, h_{\mu\nu}^{\log\ m}(\hat{n})$$

$$+ \sum_{N,M \mid M-N>-1} \omega^M (\log\omega)^N \, h_{\mu\nu}^{\log(N,M)}(\hat{n}) + \mathcal{O}\left(\frac{1}{r^2}\right) \quad (27)$$

where the leading term (memory), the leading log term (tail to the memory), and the $\omega \log\omega$ term (spin-dependent tail memory) for $2 \to 2$ scattering are given by

$$h_{\mu\nu}^{-1}(\hat{n}) = \frac{\kappa}{4}\left(S_{\mu\nu}^{(0)}(\{p_a\}, \hat{n}) - S_{\mu\nu}^{(0)}(\{p'_a\}, \hat{n})\right) ,$$

$$h_{\mu\nu}^{\log}(\hat{n}) = \frac{\kappa^3}{16\pi}\Bigg(\sum_{a,b=1|b\neq a}^{2} S_{\mu\nu}^{(1)}(\{p_a\}, \hat{n}) + \sum_{a,b=1|b\neq a}^{2} S_{\mu\nu}^{(1)}(\{p'_a\}, \hat{n})$$

$$+ \sum_{b=1}^{2} (p'_b \cdot \hat{n}) S_{\mu\nu}^{(0)}(\{p'_a\}, \hat{n}) - \sum_{b=1}^{2} (p_b \cdot \hat{n}) S_{\mu\nu}^{(0)}(\{p_a\}, \hat{n}) \Bigg) ,$$

$$h_{\mu\nu}^{\log(1,1)}(\hat{n}) = \frac{\kappa^3}{32\pi}\Bigg(\sum_{a,b=1|b\neq a}^{2} S_{\mu\nu}^{(2)}(\{p_a\}, \{\mathcal{S}_a\}, \{r_a\}, \hat{n}) + \sum_{a,b=1|b\neq a}^{2} S_{\mu\nu}^{(2)}(\{p'_a\}, \{\mathcal{S}'_a\}, \{r'_a\}, \hat{n})$$

$$+ 2\sum_{b=1}^{2} (p'_b \cdot \hat{n}) \Big(\sum_{a=1}^{2} \frac{p'_{a,(\mu} \hat{n}^{\rho}}{p'_a \cdot \hat{n}} [(r'_a \wedge p'_a)_{\nu)\rho} + \mathcal{S}'_{a,\nu)\rho}]$$

$$- \sum_{a=1}^{2} \frac{p_{a,(\mu} \hat{n}^{\rho}}{p_a \cdot \hat{n}} [(r_a \wedge p_a)_{\nu)\rho} + \mathcal{S}_{a,\nu)\rho}] \Big) \Bigg) . \quad (28)$$

Here

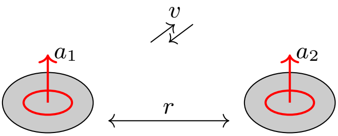
$$S^{(0),\mu\nu}(\{p_a\},\hat{n}) = \sum_{a=1}^{2} \frac{p_a^\mu p_a^\nu}{p_a \cdot \hat{n}} \,,$$

$$S^{(1),\mu\nu}(\{p_a\},\hat{n}) = (p_1 \cdot p_2) \frac{(2(p_1 \cdot p_2)^2 - 3m_1^2 m_2^2)}{[(p_1 \cdot p_2)^2 - m_1^2 m_2^2]^{3/2}} \frac{\hat{n}_\rho}{p_a \cdot \hat{n}} p_a^{(\mu} (p_a \wedge p_b)^{\nu)\rho} \,,$$

$$S^{(2),\mu\nu}(\{p_a\},\{\mathcal{S}_a\},\{r_a\},\hat{n}) = (p_1 \cdot p_2) \frac{(2(p_1 \cdot p_2)^2 - 3m_1^2 m_2^2)}{[(p_1 \cdot p_2)^2 - m_1^2 m_2^2]^{3/2}} \frac{\hat{n}_\rho \hat{n}_\sigma}{p_a \cdot \hat{n}} \Big((p_a \wedge p_b)^{\mu\rho} (r_a \wedge p_a + \mathcal{S}_a)^{\nu\sigma} + (p_a \wedge p_b)^{\nu\sigma} (r_a \wedge p_a + \mathcal{S}_a)^{\mu\rho} \Big) \,. \quad (29)$$

Here, $\{p_a, \mathcal{S}_a, r_a\}$ and $\{p'_a, \mathcal{S}'_a, r'_a\}$ denote the initial and final {momenta, spin tensors and the unperturbed trajectories} of the particles. $\kappa = \sqrt{32\pi G}$ and $\hat{n}$ is the unit vector on the celestial sphere.

Solving the binary spinning black hole problem



masses m_1, m_2
spins $S_1 = m_1 a_1$
$S_2 = m_2 a_2$
$(c = 1)$

Figure: The binary spinning black hole

- The no hair theorem states that black holes are characterized by only their mass, charge and angular momentum, implying that externally the black hole behaves as a point particle.
- Effective field theory (EFT), Effective One-Body (EOB) formalism to solve for the dynamics of binary systems in gravity via PM expansion.
  Scattering angle/Impulse, spin kick,...(perturbative in spin!)

Is there a way one can compute these observables exact to all orders in spin?

Kerr solution in Effective field theory

- Metric in Kerr-Schild form:

$$g_{\mu\nu}^{KS} = \eta_{\mu\nu} + \phi\, l_\mu l_\nu$$
$$= \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(G^2)\,,$$

where $h \sim \mathcal{O}(G)$ is the linear metric perturbation and l_μ is null w.r.t. to both $g^{\mu\nu}$ and $\eta^{\mu\nu}$. Here $\phi_{Schw} = \frac{2GM}{r}$ and $\phi_{Kerr} = \frac{2GMr}{r^2+a^2\cos^2\theta}$.

- Trace-reversed metric perturbation ($\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}h_\rho{}^\rho \eta^{\mu\nu}$) for Schwarzschild:

$$\bar{h}^{\mu\nu}_{Schw} = \frac{4GM}{r} u^\mu u^\nu\,.$$

- $h^{\mu\nu}_{Schw}$ solves the harmonic-gauge linearized field equation ($\Box h_{\mu\nu} = -16\pi G\, T_{\mu\nu}$) with the source

$$T^{\mu\nu}_{Schw} = m \int d\tau\, u^\mu u^\nu \hat{\delta}^4(x - r(\tau))\,.$$

Kerr solution in Effective field theory

- Similarly for linearized Kerr [Vines]:

$$\bar{h}^{\mu\nu}_{Kerr} = u^\rho u^{(\mu} \exp\left(a * \partial\right)^{\nu)}_{\rho} \frac{4GM}{r} \text{ , where } (a*\partial)^\mu_\nu = \epsilon^\mu_{\nu\rho\sigma} a^\rho \partial^\sigma \text{ ,}$$

$$\text{momentum : } p^\mu = m u^\mu (u^2 = 1), \text{spin vector : } S^\mu = m a^\mu$$

- The conserved stress tensor of the linearized Kerr BH is given by

$$\begin{aligned} T^{\mu\nu}_{Kerr}(x) &= \frac{1}{m}\int d\tau \left[p^\mu p^\nu + p^{(\mu} S^{\nu)\rho}\partial_\rho + \sum_{n\geq 2} C^{\mu\nu}_n(p,a,\partial)\right]\hat{\delta}^4(x - r(\tau)) \\ &= \frac{1}{m}\int d\tau p^{(\mu} p^\rho \exp\left(a*\partial\right)^{\nu)}{}_\rho \hat{\delta}^4(x - r(\tau)) \,, \end{aligned}$$

  where $S^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} p_\alpha a_\beta$ is the spin tensor.
- One can write an effective action that reproduces the leading interactions of a Kerr Black hole [Levi,Steinhoff],[Vines].

Kerr as the double copy of $\sqrt{\text{Kerr}}$

- Kerr = a spinning BH in GR
- $$\bar{h}^{\mu\nu} = u^\rho u^{(\mu} \exp{(a * \partial)}_\rho^{\nu)} \frac{4GM}{r}$$

- $\sqrt{\text{Kerr}}$ = the EM field of a certain rotating charge distribution
- $$A_\mu = u_\nu \exp{(a * \partial)}^\nu{}_\mu \frac{Q}{r}$$

- Classical double copy [Luna, Monteiro, O'Connell, White ++]
- The object which sources the gauge field A_μ is called the $\sqrt{\text{Kerr}}$ [Arkani-Hamed, Huang O'Connell]. It is a solution of the free Maxwell's equations with infinite multipole moments expressed solely in terms of the charge, mass and angular momentum of the classical object (same as the no-hair theorem for BHs)
- The conserved current for the $\sqrt{\text{Kerr}}$ object:

$$\begin{aligned} J^\mu(x) &= \frac{Q}{m} \int d\tau \left[p^\mu + S^{\mu\rho}\partial_\rho + \sum_{n\geq 2} D_n^\mu(p, a, \partial) \right] \hat{\delta}^4(x - r(\tau)) \\ &= \frac{Q}{m} \int d\tau \, p^\rho(\tau) \exp{(a * \partial)}^\mu{}_\rho \hat{\delta}^4(x - r(\tau)) \,. \end{aligned}$$

Black Hole Amplitudes

- The minimally coupled 3-point amplitudes for two massive spin S particles coupled with a graviton given by [Arkani-Hamed, Huang, Huang][5]

$$\lim_{S\to\infty}\left(\mathcal{A}_3[\mathbf{1}^S,\mathbf{2}^S,3^+] = \kappa\, x^2\left(\frac{\langle\mathbf{12}\rangle}{m}\right)^{2S}\right),\ x=(\varepsilon_3^+\cdot p_1)$$

  matches with the leading interactions of a Kerr Black hole with linearized gravity $(h_{\mu\nu}(k)T^{\mu\nu}_{BH}(-k)|_{k^2\to 0})$, consistent with the no-hair theorem.

- For spin 0 : (universal) monopole coupling
  spin 1/2 : adds (universal) dipole/spin-orbit coupling [Vaidya]
  spin 1: adds Black Hole quadrupole
  spin 2: adds Black Hole octupole.
  [Guevara, Ochirov, Vines, Chung, Huang, Kim, Lee, Bautista, Maybee, O'Connell, Helset, Aoude, Haddad, Damgaard, Bern, Luna, Roiban, Shen, Zeng,...]

---

[5] SL(2,C) representation of momentum 4-vector,

$$p_{\alpha\dot\alpha} = p_\mu \sigma^\mu_{\alpha\dot\alpha} = \sum_{I,J=1}^{2} \epsilon_{IJ}\lambda^I_\alpha \tilde\lambda^J_{\dot\alpha} = \sum_{I,J=1}^{2} \epsilon_{IJ}|p^I\rangle_\alpha [p^J|_{\dot\alpha}\,,$$

where (I,J) are the SU(2) little group indices associated with the massive particle. The variables $\lambda^I_\alpha, \tilde\lambda^J_{\dot\alpha}$ are called massive spinor-helicity variables.

On-shell amplitudes and Newman-Janis

- The minimally coupled 3-point amplitudes for two massive spin S particles coupled with a photon are given by [Arkani-Hamed, Huang, Huang]

$$\mathcal{A}_3[\mathbf{2}^S, \mathbf{2'}^S, q^+] = \frac{Q_2}{m_2}(\varepsilon^+(q)\cdot p_2)\frac{\langle \mathbf{2 2'}\rangle^{2S}}{m_2^{2S-1}}$$
$$\mathcal{A}_3[\mathbf{2}^S, \mathbf{2'}^S, q^-] = \frac{Q_2}{m_2}(\varepsilon^-(q)\cdot p_2)\frac{[\mathbf{2 2'}]^{2S}}{m_2^{2S-1}}$$

- We set $p_2' = p_2 + \hbar\bar{q}$ and using momentum conservation, this amplitude can be written as

$$\mathcal{A}^{\pm}_{3,\sqrt{\text{Kerr}}} = Q\mathcal{A}^{\pm}_{3,scalar}\, e^{\pm\bar{q}\cdot a_2}$$

  as $S\to\infty$ and $\hbar\to 0$ with $S\hbar$ being fixed. Here $a_2^\mu = \frac{S\hbar}{m_2^2}\langle\mathbf{2}|\sigma^\mu|\mathbf{2}]$
- For Kerr BH,

$$\mathcal{A}^{\pm}_{3,Kerr} = \kappa(\mathcal{A}^{\pm}_{3,scalar})^2\, e^{\pm\bar{q}\cdot a_2}\,.$$

- This exponentiation was identified as the NJ algorithm at the level of 3-point on-shell amplitudes [Arkani-Hamed, Huang, O'Connell].

Spin-dressed photon propagator

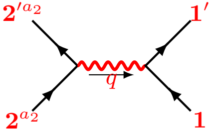


Figure: The four-point scalar - $\sqrt{\text{Kerr}}$ amplitude with photon exchange. Here 'a_2' denotes the rescaled spin of the $\sqrt{\text{Kerr}}$ particle.

- The four-point amplitude involving a pair of particles via photon exchange can be constructed using the three-point amplitudes:

$$\mathcal{A}_4[p_1, p_2 \to p_1' p_2'] = \frac{1}{q^2} \mathcal{A}^{\mu}_{3,\sqrt{\text{Kerr}}}[p_2', p_2, q] \mathcal{P}_{\mu\nu} \ \mathcal{A}^{\nu}_{3,scalar-QED}[p_1', p_1, -q]$$

$$= \mathcal{A}^{\mu}_{3,\text{scalar-QED}} \left[p_2', p_2, \hbar \bar{q}\right] \frac{\widetilde{\mathcal{P}}_{\mu\nu}(\bar{q})}{\hbar^2 \bar{q}^2} \mathcal{A}^{\nu}_{3,\text{scalar-QED}} \left[p_1', p_1, -\hbar \bar{q}\right] \ ,$$

where $\mathcal{P}^{\mu\nu} \to \widetilde{\mathcal{P}}^{\mu\nu}(\bar{q}) := e^{\bar{q}\cdot a_2} \varepsilon^{\mu}_{+}(\bar{q}) \varepsilon^{\nu}_{-}(\bar{q}) + e^{-\bar{q}\cdot a_2} \varepsilon^{\nu}_{+}(\bar{q}) \varepsilon^{\mu}_{-}(\bar{q})$.

LO Gravitational radiation

- The (sub)^n-leading order soft radiation is given by

$$\begin{aligned}\mathcal{R}^{(n),\mu\nu}(\bar{k}) = \frac{\kappa^3 m_1 m_2}{4}\int \hat{d}^4\bar{q}\Big[&\sum_{r=0}^{n}\frac{1}{(n-r)!}e^{-ib\cdot\bar{q}}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})(ib\cdot\bar{k})^{n-r}K^{\mu\nu}_{r-1} \\ &+\sum_{r=0}^{n-1}\frac{(-1)^{n-r}}{(n-r)!}\Big\{e^{-ib\cdot\bar{q}}\hat{\delta}^{(n-r)}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})(u_1\cdot\bar{k})^{n-r}\Big(K^{\mu\nu}_{r-1}\Big)+e^{ib\cdot\bar{q}}\Big(1\leftrightarrow 2\Big) \\ &+\sum_{r=0}^{n-2}\sum_{\substack{t,s\geq 1\\ \ni(t+s)=n-r}}\frac{(-1)^s}{t!s!}e^{-ib\cdot\bar{q}}(ib\cdot\bar{k})^t(u_1\cdot\bar{k})^s\hat{\delta}^{(s)}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})K^{\mu\nu}_{r-1}\Big],\end{aligned}$$

where the polynomial $K^{\mu\nu}_r$ depends on $\bar{q}$, $\bar{k}$, u_1, u_2 and is of the order $\mathcal{O}(\omega^{r-1})$.

LO Gravitational radiation

- Upon simplifying, the logarithmic term in (sub)^n-leading soft radiation at order κ^3 is given by [SA]

$$\mathcal{R}^{(\omega b)^{n-1}\log(\omega b),\mu\nu} = \frac{i^{n-1} m_1 m_2 \kappa^3}{4\pi(n-1)!\gamma^3\beta^3}\gamma(2\gamma^2-3)(\omega b)^{n-1}\log(\omega b) \times \left(u_1^{(\mu}u_2^{\nu)} - \frac{(u_2\cdot\bar{k})}{(u_1\cdot\bar{k})}u_1^{(\mu}u_1^{\nu)}\right) + (1\leftrightarrow 2)\,.$$

- In the deflection less limit ($|b|\to\infty$) such that ωb is fixed, all the log terms of the form $(\omega b)^m\log(\omega b)|m\geq 1$ survive and can be completely determined by the (sub)^n-soft "factorization" theorems for tree-level gravitational amplitudes.
- The source of such a radiative mode then is the asymptotic interaction between the incoming or outgoing states, leading to the emission of gravitational radiation only from $t\to\pm\infty$.

$(\text{Sub})^n$-leading soft theorems

- We consider a $2 \to 2$ scattering process with a large impact parameter. These processes can then be studied within perturbation theory.
- If p'_a is the final momentum of a particle and the initial momentum is p_a, then we have

$$p_a'^{\mu} = p_a^{\mu} + \sum_{n=1}^{\infty} \kappa^{2n} \Delta p_a^{(n)\mu} \,,$$

  where $\Delta p_a^{(1)\mu}$ is the LO linear impulse and κ^{2n}-th term is the N^{n-1}LO impulse.
- Both $\log \omega$ and $\omega \log \omega$ survive even at leading order in the coupling. This then indicates that the terms of the form $\omega^n \log \omega$ can be determined by the so-called $(\text{sub})^n$-leading soft graviton theorems for tree-level gravitational amplitudes.

(Sub)^n-leading soft theorems

For example, in 2-2 scattering the classical log soft graviton factor is given by

$$h^{\mu\nu}(\omega,\hat{n})|_{\log\omega} = \frac{\kappa^3}{16\pi}\log(\omega)\left(\sum_{a,b=1}^{2} S^{(1),\mu\nu}(\{p_a\},k) + \sum_{a,b=1}^{2} S^{(1),\mu\nu}(\{p'_a\},k)\right),$$

where

$$S^{(1),\mu\nu}(\{p_a\},k) = (p_1\cdot p_2)\frac{(2(p_1\cdot p_2)^2 - 3m_1^2m_2^2)}{[(p_1\cdot p_2)^2 - m_1^2m_2^2]^{3/2}}\frac{k_\rho}{p_a\cdot k}p_a^{(\mu}\left(p_a^{\nu)}p_b^{\rho} - p_b^{\nu)}p_a^{\rho}\right).$$

The classical log soft graviton factor at leading order in the coupling takes the following form

$$h^{\mu\nu}(\omega,\hat{n})|_{\log\omega} = -\frac{i\kappa}{4}\log(\omega)\sum_i \frac{1}{p_i\cdot k}p_i^{(\mu}\hat{J}_i^{\nu)\rho}k_\rho\left[\frac{\kappa^2}{2\pi}\frac{(p_1\cdot p_2)^2 - \frac{1}{2}m_1^2m_2^2}{\sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2}}\right],$$

which arises from the action on gravitational tree-level four-point amplitude. Here $\hat{J}_i^{\mu\nu} = i(p_i\wedge\frac{\partial}{\partial p_i})^{\mu\nu}$. Thus one can use the soft graviton theorems for tree-level amplitudes in computing the radiation kernel, which generates the $\omega^n\log\omega$ terms.