Lecture notes on Scattering Amplitudes and Gravitational Waves

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1 Introduction

In the current realm of theoretical physics there is a constant tension between theory and experiment. Often times experiment predates a theoretical discovery, as an 'anomaly" in our understanding, something to be explained by a new model. Think of black body radiation as explained by Quantum Mechanics. But other times, it is the theory that drives the search for a phenomena, a prediction rather than an explanation. It has happened with the Higgs Boson, and it has happened with the topic of these lectures, Gravitational Waves.

Historically particle physics has been a predictive subject, generating high-precision calculations to be tested at particle colliders. Indeed, the new connection between particle physics and black hole physics stems from the need of generating astrophysical predictions for the latter. However, at the same time, it has the potential to resolve 'anomalies" in our very own understanding of black holes and nature.

These notes explore how modern scattering amplitude techniques illuminate gravitational-wave physics, ultimately aimed to understanding the waveform of a binary black hole. However, in doing so, we will discover that even a stationary black hole such as the Schwarzschild or the Kerr solution require a new perspective.

By recasting classical gravitational observables in terms of quantum field theory amplitudes, one can derive gravitational potentials, scattering angles and radiation in a unified framework. We will rederive certain classical amplitudes that match black hole observables. We will cover the role of spin and its relation with rotating black holes. Finally we will discuss radiative observables and non-local in time effects.

1. Asymptotic Observables.
2. Kerr BH: The Teukolsky equation and Compton amplitude

3. Radiation

2 Lecture I: Asymptotic observables

A dynamical system is defined by its equations of motion. Classically, we learned that given initial conditions of any system we should be able to determine its state at a certain time.

However, Quantum Mechanics told us we are not so powerful. Indeed, we can only know certain states by doing measurements. If we are interested in an operator O all we can ask for is to understand its expectation $\langle\psi(t)|O|\psi(t)\rangle = \langle\psi|O(t)|\psi\rangle$ which amounts to preparing the state but also the operator. However, such expectations can get arbitrarily complicated at early times for systems of more than one component, showing chaotic and unpredictable behaviour.[1] To sort this out one is left to understand the expectations at late times, $t \to \infty$, and so we are naturally led to the notion of scattering.

Let us be more precise. We define the S matrix elements via

$$\mathcal{M} = \langle\psi_{out}|\psi_{in}\rangle = \langle\psi'_{\text{in}}|S|\,\psi_{\text{in}}\rangle = \lim_{t\to+\infty,\, t'\to-\infty}\langle\psi'_{in}|U(t,t')|\psi_{in}\rangle \tag{1}$$

This does not look like an expectation value. But we are ultimately insterested in the cross-section

$$\sum_{\psi'\in\Delta}\mathcal{M}^*\mathcal{M} = \langle\psi_{\text{in}}|S^\dagger\mathcal{N}_\Delta S|\,\psi_{\text{in}}\rangle = \langle\psi_{out}|\mathcal{N}_\Delta|\psi_{out}\rangle \tag{2}$$

where $\mathcal{N}$ is a projector, the particle number operator in a region Δ of our multiparticle phase space. The cross-section itself is not an amplitude, but an observable determined by an initial preparation of a beam at a particle collider, or of a thermal state in a quantum simulation. In both cases we have direct control only over $\psi_{in}\rangle$. This is the modern imprint of determinism.

We have learned that in GW generation we are interested in conservative (binding energy) and radiative (fluxes) observables. In fact, conservative and radiative observables in scattering can be put in terms of these expectation values. The basic pattern is always

$$\langle O\rangle_{\text{in-in}} = \langle\Psi_{\text{in}}|S^\dagger O S|\Psi_{\text{in}}\rangle, \tag{3}$$

with different choices of "measurement operator" O:

- impulse: $O = P_i^\mu$ (mechanical momentum of particle i);
- radiated momentum or waveform: $O = K^\mu$ or $O = h_{\mu\nu}(x)$ (operators built from the radiation field);
- cross sections: $O = \widehat{\mathcal{N}}$, a number / flux operator selecting a region of phase space.

[1] The question of chaos being a intrinsic property of QM at early/intermediate times is an active current area of research, see for instance The Semiclassical Way. The same goes for our ability to measure arbitrary precision bulk processes.

2.1 From classical to quantum

In order to understand the prescriptions let us revisit for instance, the treatment of momentum deflection. Recall the Newtonian expression for the momentum deflection of a particle m_1 moving in a potential from another particle, namely

$$\Delta \mathbf{p}_1 = \int_{-\infty}^{+\infty} dt \, \frac{d\mathbf{p}_1}{dt}(t) = - \int_{-\infty}^{+\infty} dt \, \nabla_{\mathbf{r}} V\big(\mathbf{r}(t)\big) \, . \tag{4}$$

On the other hand, Quantum Mechanics tells us, via the Born approximation, that such potential is related to the classical limit of a scattering amplitude,

$$V(\mathbf{r}) = - \frac{1}{4m_1 m_2} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \, e^{i\mathbf{q}\cdot\mathbf{r}} \, \langle \mathcal{A}_4(\mathbf{q}) \rangle \, . \tag{5}$$

Homework 1. *As we saw in previous lectures, for a weak scattering process, the trajectory can be approximated by a straight line with constant relative velocity* $\mathbf{v}$ *and impact parameter* $\mathbf{b}$. *Show that deflection can be written directly in terms of the four-point amplitude as*

$$\Delta \mathbf{p}_1(\mathbf{b}) = \frac{i}{4m_1 m_2} \int \frac{d^3\mathbf{q}}{(2\pi)^2} \, \mathbf{q} \, e^{i\mathbf{q}\cdot\mathbf{b}} \, \delta(\mathbf{q}\cdot\mathbf{v}) \, \langle \mathcal{A}_4(\mathbf{q}) \rangle \tag{6}$$

This does not look at all like an in-in observable. There are two key ingredients that will allow us to interpret it as such: The Fourier transform to impact parameter space $\mathbf{b}$ (which is an initial condition!) and the classical limit procedure.

Let us first discuss the classical limit. We consider the scattering of two stable scalar particles with masses m_1 and m_2, representing (for example) two non-spinning black holes. The incoming state is taken to be a product of wavepackets, separated by a transverse impact parameter b^μ,

$$|\psi\rangle_{\text{in}} = \int d\Phi(p_1) \, d\Phi(p_2) \; \phi_1(p_1) \, \phi_2(p_2) \, e^{ib\cdot p_1/\hbar} \, |p_1 p_2\rangle_{\text{in}} \, , \tag{7}$$

where $d\Phi(p)$ denotes the Lorentz-invariant on-shell phase-space measure and $\phi_i(p)$ are sharply peaked around classical momenta $\bar{p}_i^\mu$. The impact parameter satisfies $p_i \cdot b = 0$ for $i = 1, 2$.

The classical limit is obtained by taking $\hbar \to 0$ while keeping the momenta p_i^μ fixed and requiring a hierarchy of scales: the Compton wavelengths $\ell_c^{(i)} = \hbar/m_i$ are much smaller than both the wavepacket widths ℓ_w and the impact parameter,

$$\ell_c^{(i)} \ll \ell_w \ll \sqrt{-b^2}, . \tag{8}$$

In this regime the intrinsic quantum length scale is the shortest one in the problem, while the impact parameter is the largest, so that many de Broglie oscillations fit inside the wavepacket and many wavepacket widths fit across the scattering geometry. This is an explicit realization of Bohr's correspondence principle: when all macroscopic scales are large compared to $\hbar$–controlled microscopic scales, the expectation values of observables in the state (7) reproduce the classical trajectories and impulses of the scattering process.

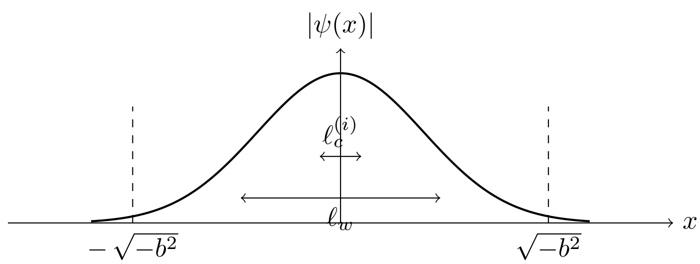

Figure 1: Schematic wavefunction illustrating the Goldilocks regime $\ell_c^{(i)} \ll \ell_w \ll \sqrt{-b^2}$.

In this "Goldilocks" regime the expectation values of suitably chosen observables in the state (7) reproduce classical dynamics.

2.2 Impulse from the S-matrix: KMOC

The Kosower–Maybee–O'Connell (KMOC) formalism provides a systematic way to compute classical observables, such as $\Delta\mathbf{p}_1(\mathbf{b})$, directly from on-shell scattering amplitudes. Instead of first constructing an effective classical theory and then solving its equations of motion, one works with the S-matrix of the underlying quantum field theory and extracts observables as expectation values in suitable wavepacket states.

Let P_i^μ be the momentum operator associated with particle i. The *impulse* on particle 1 during the scattering event is defined as the difference between the expectation values of its outgoing and incoming momenta,

$$\langle \Delta p_1^\mu \rangle = \langle \psi | S^\dagger P_1^\mu S | \psi \rangle - \langle \psi | P_1^\mu | \psi \rangle \,, \tag{9}$$

and analogously for particle 2. Here S is the full scattering matrix, $S = U(+\infty, -\infty)$.

Writing $S = 1 + iT$ in terms of the transition operator T, and using the unitarity relation $S^\dagger S = 1$, the impulse can be expressed purely in terms of T,

$$\langle \Delta p_1^\mu \rangle = \langle \psi | i[P_1^\mu, T] | \psi \rangle + \langle \psi | T^\dagger [P_1^\mu, T] | \psi \rangle \equiv I_{(1)}^\mu + I_{(2)}^\mu \,. \tag{10}$$

The first term $I^\mu_{(1)}$ is linear in the scattering amplitude, while the second term $I^\mu_{(2)}$ is quadratic and encodes the effect of intermediate states.

After expanding the wavefunctions and inserting complete sets of intermediate states, the linear term can be written schematically as

$$I^\mu_{(1)} = \int d\Phi(p_1)\, d\Phi(p_2) d^4q\; e^{-ib\cdot q/\hbar}\, q^\mu\, \Phi(p_1, p_2; q)\, \mathcal{A}_4(p_1, p_2, q)\,. \tag{11}$$

Here q^μ is a "momentum mismatch" between the momenta appearing in the wavefunction and in its conjugate and Φ is a positive wavepacket weight. The amplitude $\mathcal{A}_{2\to 2}$ is evaluated with incoming momenta p_1, p_2 and outgoing momenta $p_1 + q, p_2 - q$.

The quadratic term $I^\mu_{(2)}$ is historically called Born iteration and corrects the relation (5) to subleading order. It involves a sum over states X of additional quanta that may be produced during the scattering,

$$\begin{aligned} I^\mu_{(2)} = \sum_X \int d\Phi(p_i)\; d^4w_i\; d^4q\; e^{-ibq/\hbar}\, w_1^\mu\, &\mathcal{A}_{2\to 2+X}(p_1, p_2 \to p_1 + w_1, p_2 + w_2, r_X) \\ &\times \mathcal{A}^*_{2\to 2+X}(p_1 + q, p_2 - q \to p_1 + w_1, p_2 + w_2, r_X)\,, \end{aligned} \tag{12}$$

with $w_i = r_i - p_i$ the momentum transfer along each massive line and r_X the total momentum carried by the state X. All phase-space measures are on-shell and accompanied by energy–momentum conserving delta functions.

These two terms are smeared amplitudes. The smearing amounts to the classical limit.

Comment on PM expansion. In gravitational scattering one usually organises the result in a post-Minkowskian (PM) expansion in powers of Newton's constant G. At n-th PM order ($\mathcal{O}(G^n)$) the relevant information for the impulse is contained in $(n-1)$-loop amplitudes. For example, the conservative and radiative contributions to the impulse at $\mathcal{O}(G^3)$ (third PM order) are determined by two-loop four-point amplitudes and their cuts.

2.3 The in-in formalism

As we learn in quantum field theory, scattering amplitudes as above are computed from the Feynman time–ordered prescription:

$$\langle T\{\phi(x)\phi(y)\}\rangle = \theta(x^0 - y^0)\, \langle\phi(x)\phi(y)\rangle + \theta(y^0 - x^0)\, \langle\phi(y)\phi(x)\rangle. \tag{13}$$

In classical physics however, we learn to respect causality by imposing *retarded* boundary conditions:

$$G_{\text{ret}}(x - y) = \theta(x^0 - y^0)\, \big\langle [\, \phi(x), \phi(y)\,]\big\rangle. \tag{14}$$

How can these two give the same answer? These two perspectives are again unified once we agree that we are interested in classical in-in observables constructed from amplitudes.

We introduce the Schwinger–Keldysh (in–in) contour C, which runs from $t = -\infty$ to $t = +\infty$ and back to $-\infty$. Fields on the two branches are denoted $\phi_+(x)$ and $\phi_-(x)$, and we define the contour-ordered correlator

$$\big\langle C\, \phi_a(x)\phi_b(y)\big\rangle, \qquad a, b \in \{+, -\}, \tag{15}$$

The usual Feynman propagator is obtained by putting both insertions on the forward branch,

$$G_F(x-y) = \big\langle C\, \phi_+(x)\phi_+(y)\big\rangle = \big\langle T\, \phi(x)\phi(y)\big\rangle, \tag{16}$$

while the retarded propagator is the causal combination

$$G_{\rm ret}(x-y) = \big\langle C\, \phi_+(x)\big[\phi_+(y) - \phi_-(y)\big]\big\rangle = \theta(x^0 - y^0)\, \big\langle [\, \phi(x), \phi(y)\,]\big\rangle. \tag{17}$$

In this way the same Schwinger–Keldysh contour simultaneously encodes both the in–out (Feynman) and causal (retarded) Green's functions.

The same can be applied to the S matrix. It is useful to reinterpret (9) in the language of the in–in formalism. Writing the S–matrix in terms of the interaction Hamiltonian H_I,

$$S = T \exp\Big[-i \int_{-\infty}^{+\infty} dt\, H_I(t)\Big], \qquad S^\dagger = \bar{T} \exp\Big[+i \int_{-\infty}^{+\infty} dt\, H_I(t)\Big], \tag{18}$$

Note that our momentum operator cab inserted via

$$P_1^\mu(t)_C = U_C(+\infty, t)\, P_1^\mu\, U_C(t, -\infty) \tag{19}$$

Then impulse can be written as

$$\begin{aligned} \langle \Delta p_1^\mu \rangle &= \langle \psi_{\rm in} | \bar{T} \exp\Big[i \int_{-\infty}^{+\infty} dt\, H_I(t)\Big]\, P_1^\mu\, T \exp\Big[-i \int_{-\infty}^{+\infty} dt\, H_I(t)\Big] |\psi_{\rm in}\rangle - \langle \psi_{\rm in} | P_1^\mu | \psi_{\rm in}\rangle \\ &\equiv \langle \psi_{\rm in} | P_1^\mu(+\infty)_C - P_1^\mu(-\infty)_C | \psi_{\rm in}\rangle, \end{aligned} \tag{20}$$

which shows that the impulse is an in–in expectation value evaluated on a closed time contour C, with the operator P_1^μ inserted on the forward branch!

Consider elastic $2 \to 2$ scattering of massive particles,

$$p_1, p_2 \to p_1', p_2', \qquad q = p_1' - p_1 = -(p_2' - p_2)\,. \tag{21}$$

The usual in–out 4-point amplitude is

$$\mathcal{A}_4(p_i) \sim \prod_{i=1}^{4} \int d^4 x_i\, e^{ip_i \cdot x_i} \langle 0 | T\{\Phi_1(x_1)\Phi_2(x_2)\Phi_1(x_3)\Phi_2(x_4)\} | 0\rangle, \tag{22}$$

which we regard as a function $\mathcal{A}_4(s, t)$ of Mandelstam invariants.

In the KMOC framework the impulse of particle 1 is

$$\Delta p_1^\mu = \langle \Psi_{\rm in} | S^\dagger P_1^\mu S | \Psi_{\rm in}\rangle - \langle \Psi_{\rm in} | P_1^\mu | \Psi_{\rm in}\rangle. \tag{23}$$

Using $S = T\exp(-i\int dt\, H_I)$ and $S^\dagger = \bar{T}\exp(+i\int dt\, H_I)$, this becomes an in–in expectation value on a closed Schwinger–Keldysh contour C,

$$\Delta p_1^\mu = \langle \Psi_{\rm in}| P_1^\mu(+\infty)_C - P_1^\mu(-\infty)_C |\Psi_{\rm in}\rangle, \tag{24}$$

where the subscript C indicates evolution along C.[2] Expanding the exponentials, and then LSZ-amputating the external massive legs, the leading (linear) contribution to the impulse is obtained from an *amputated retarded* 4*-point SK correlator*. In momentum space, after inserting the wavepackets and going to impact-parameter space, this gives schematically

$$I^\mu_{(1)}(b) \sim \int \hat{d}^4 q\, e^{-ib\cdot q/\hbar}\, q^\mu\, \Phi(p_1,p_2;q)\, \mathcal{R}_4(p_1,p_2 \to p_1+q, p_2-q), \tag{25}$$

where $\mathcal{R}_4$ is the retarded 4-point function (the SK version of $\mathcal{A}_4$).

Comment on crossing Both the amplitude $\mathcal{A}_4$ and the cross section $\sigma[\Delta]$ arise as different boundary values of a single analytic function of complexified energies.

- Feynman $i\epsilon$ prescription $\Rightarrow$ usual in–out amplitude $\mathcal{A}_4$;
- retarded / cut prescription $\Rightarrow$ inclusive observable such as the cross section or impulse.

Ordinary 4-point crossing (e.g. $s \leftrightarrow u$) is then analytic continuation of this common analytic function; what changes between $\mathcal{A}_4$ and generalized observables such as the impulse and cross section is not the function, but which side of its branch cuts one approaches.

2.4 Radiated momentum

Another observable naturally captured by the KMOC framework is the total four-momentum carried away by radiation during the scattering. Let K^μ be the momentum operator for the massless field. If the incoming state contains no messenger quanta, $K^\mu|\psi\rangle_{\rm in} = 0$, the expectation value of the radiated momentum is

$$R^\mu \equiv \langle k^\mu \rangle = \langle \psi | S^\dagger K^\mu S |\psi\rangle = \langle \psi | T^\dagger K^\mu T |\psi\rangle\,. \tag{26}$$

The second equality follows because the unity piece of S annihilates the incoming state.

Inserting a complete set of final states containing at least one messenger of momentum k^μ, one obtains an expression of the form

$$R^\mu = \sum_X \int d\Phi(k)\, d\Phi(r_1)\, d\Phi(r_2)\; k_X^\mu \left|\langle k\, r_1 r_2 X | T | \psi\rangle\right|^2, \tag{27}$$

where k_X^μ is the total messenger momentum in the state $|k\, r_1 r_2 X\rangle$ and the phase-space integrals include the massive particles as well as any additional radiation contained in X.

[2]This is the content of eq. (14) in the notes.

Homework 2 *After expanding the initial wavepacket and trading the duplicated initial momenta for a momentum mismatch q, the radiated momentum takes a form closely analogous to* (12). *Furthermore, show energy–momentum conservation at the observable level,*

$$\langle \Delta p_1^\mu \rangle + \langle \Delta p_2^\mu \rangle + R^\mu = 0\,, \tag{28}$$

which is automatically satisfied when both impulse and radiated momentum are computed from the same set of amplitudes and cuts.

In gravitational applications, the radiated momentum is directly related to the emitted gravitational-wave energy and the recoil of the binary system. Within KMOC, the first non-vanishing contribution to R^μ in black-hole scattering appears at $\mathcal{O}(G^3)$, corresponding to two-loop cuts with real graviton emission. These same cuts also enter the computation of the impulse with radiation reaction, so that the formalism naturally incorporates dissipative effects in a unified on-shell language.

2.4.1 Waveforms and fluxes

Radiative observables first appear at 5 points, when there is at least one real massless quantum with momentum k^μ in the final state. This is the particle flux, or the probability to emitting one state. As can be read off from the radiated momentum Kernel, it is given by

$$\int d\Phi(k)\, d\Phi(r_1)\, d\Phi(r_2) \big| \langle k\, r_1 r_2 X | T | \psi \rangle \big|^2 \tag{29}$$

This is itself an in-in observable

$$\langle \psi_{in} | S^\dagger \mathcal{N}_k S | \psi_{in} \rangle \tag{30}$$

where $\mathcal{N}_k = a_k^\dagger a_k$ is the number operator of state of momentum k. Note that this is a time ordered two-point function in the state $S|\psi_{in}\rangle$.

On the other hand, the standard in–out 5-point amplitude is

$$\mathcal{A}_5(p_i; k) \sim \langle p_1', p_2', k | S | p_1, p_2 \rangle, \tag{31}$$

LSZ-reduced from a time-ordered 5-point correlator.

In KMOC the radiated four-momentum is written as

$$R^\mu = \langle K^\mu \rangle_{\text{in-in}} = \langle \Psi_{\text{in}} | S^\dagger K^\mu S | \Psi_{\text{in}} \rangle, \tag{32}$$

with K^μ the total momentum operator of the messenger field (photon or graviton). Inserting a complete set of final states with at least one messenger of momentum k^μ gives the weighted-cut formula (cf. eq. (17) in the notes),

$$R^\mu \sim \sum_X \int d\Phi(k)\, d\Phi(p_i)\, \hat{d}^4 w_i\, \hat{d}^4 q\, e^{-ib\cdot q/\hbar}\, k_X^\mu\, \mathcal{A}_{2\to 2+k+X}(p_1, p_2 \to p_1{+}w_1, p_2{+}w_2, k, r_X) \mathcal{A}^*_{2\to 2+k+X}(p_1{+}q, p_2{-} \tag{33}$$

where k_X^μ is the total messenger momentum. This is again of the form (3), now with $O = K^\mu$.

In the Schwinger–Keldysh language, R^μ is obtained from an amputated *retarded* 5*-point response function*:

1. work with doubled fields on a closed time contour;
2. represent K^μ (or the stress tensor $T^{\mu\nu}$ at null infinity) as a single "r-type" insertion;
3. amputate the external massive and massless legs and put the messenger on shell.

The resulting object, often denoted $\mathcal{R}_5$, is a generalized amplitude: it has the same on-shell external states as a usual 5-point amplitude, but is computed with a retarded SK prescription (and, at loop level, with cut propagators) instead of the Feynman one.[1, 2]

We introduce precisely such objects in the guise of generalized amplitudes like

$$\mathrm{Exp}_3(k) \equiv {}_{\mathrm{in}}\langle\psi|\, b_3(k)\, |\psi\rangle_{\mathrm{in}}, \tag{34}$$

with $b_3(k)$ the annihilation operator for a radiation quantum in the far future and $|\psi\rangle_{\mathrm{in}}$ an incoming two-body state.[1] Using $b_3 = S^\dagger a_3 S$ and inserting complete sets of states, one finds an expression structurally identical to the KMOC formula for the waveform/radiated momentum—a sum over products of amplitudes with a radiation leg on both sides of the cut. LSZ reduction of the SK correlator reproduces this generalized amplitude.

Cross sections with radiation. For inclusive bremsstrahlung cross sections (e.g. $2 \to 2 + k + X$ with any number of additional soft quanta in X), the same pattern appears. Let $\widehat{\mathcal{N}}_{\Delta,k}$ count quanta (of given polarization, say) in a region Δ of the $(p_1', p_2', k, \dots)$ phase space; then

$$\sigma_{\mathrm{rad}}[\Delta] = \frac{1}{\mathsf{flux}}\,\langle\Psi_{\mathrm{in}}|S^\dagger \widehat{\mathcal{N}}_{\Delta,k} S|\Psi_{\mathrm{in}}\rangle \tag{35}$$

is again a generalized amplitude in the sense of (3), computable either as a SK retarded correlator or as a weighted cut of multi-loop amplitudes. In the framework of [1, 2], bremsstrahlung cross sections, waveforms, and radiated momenta all belong to the same class of in–in asymptotic observables, differing only in the choice of measurement operator O and the weight inserted under the phase-space integrals.

References

[1]

[2]