QCD 2

Giulia Zanderighi (Max Planck Institute für Physik)

2nd Lecture

Joint ICTP-SAIFR school on Particle Physics — July 2026

The beta-function

$$\boxed{\beta(\alpha_s^{\rm ren}) \equiv \mu^2 \frac{d\alpha_s(\mu^2)}{d\mu^2}}$$

The renormalized coupling is

$$\alpha_s(\mu) = \alpha_s^{\rm bare} + b_0 \ln \frac{M_{UV}^2}{\mu^2} \left(\alpha_s^{\rm bare}\right)^2$$

So, one immediately obtains

$$\beta = -b_0 \alpha_s^2(\mu) + \ldots$$

Integrating the differential equation one finds at lowest order

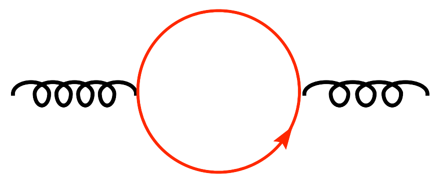
$$\frac{1}{\alpha_s(\mu)} = b_0 \ln \frac{\mu^2}{\mu_0^2} + \frac{1}{\alpha_s(\mu_0)} \qquad \Rightarrow \qquad \alpha_s(\mu) = \frac{1}{b_0 \ln \frac{\mu^2}{\Lambda^2}}$$

Where Λ is the scale at which the coupling would become infinite

More on the beta-function

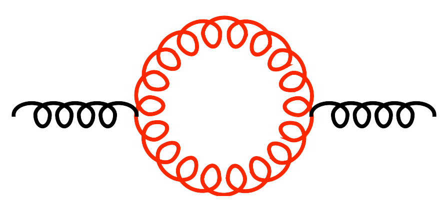
Roughly speaking:

(a) quark loop vacuum polarization diagram gives a negative contribution to $b_0 \sim -2n_f/12\pi$



(a)

(b) gluon loop gives a positive contribution to $b_0 \sim 11N_c/12\pi$



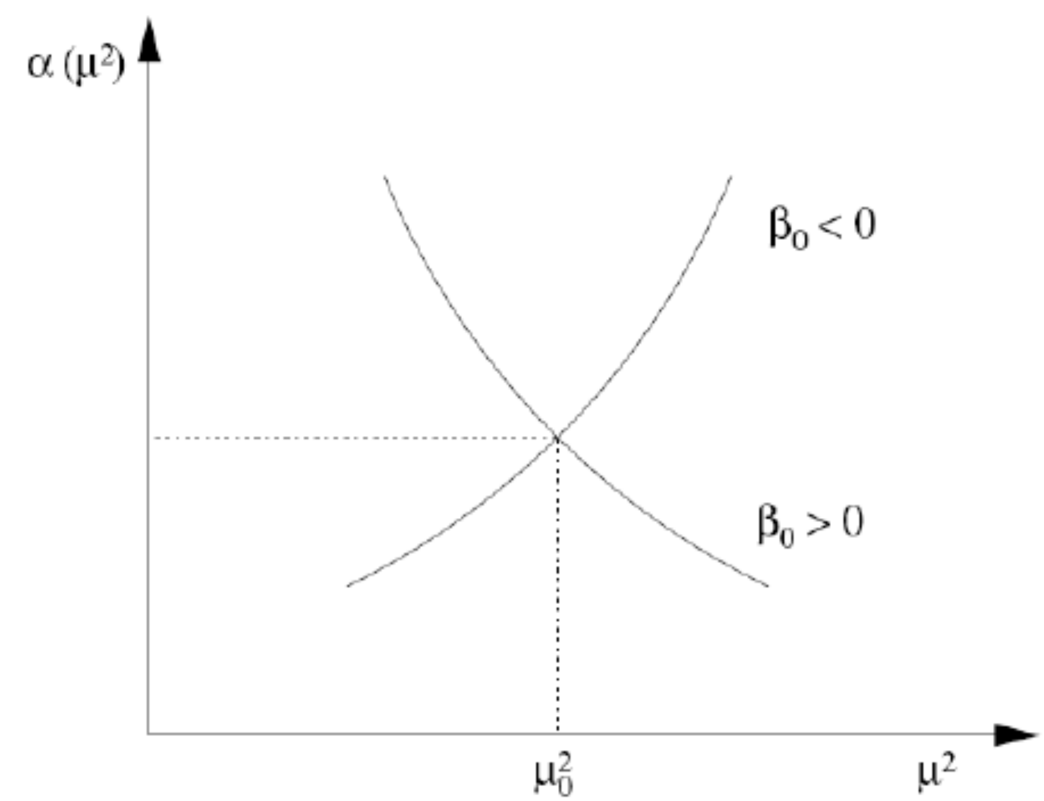
(b)

Since (b) > (a) $\Rightarrow$ $b_{0,QCD} > 0$

$\Rightarrow$ overall negative beta-function in QCD

While in QED (b) = 0 $\Rightarrow$ $b_{0,QED} < 0$

$$\beta_{\mathrm{QED}} = \frac{1}{3\pi}\alpha^2 + \ldots$$



More on the beta-function

- QCD: perturbative picture valid for scales $\mu >> \Lambda_{\text{QCD}}$ (about 200 MeV)
- QED: perturbative picture valid for scales $\mu << \Lambda_{\text{QED}}$

Question: why does nobody talk about Λ_{QED}?

Answer:

$$\Lambda_{\text{QED}} = m_e \exp\left\{-\frac{1}{2b_0\alpha(m_e)}\right\} \sim 10^{90}\text{GeV} >> M_{\text{Planck}}$$

i.e. QED is not a consistent theory up to arbitrary high scales

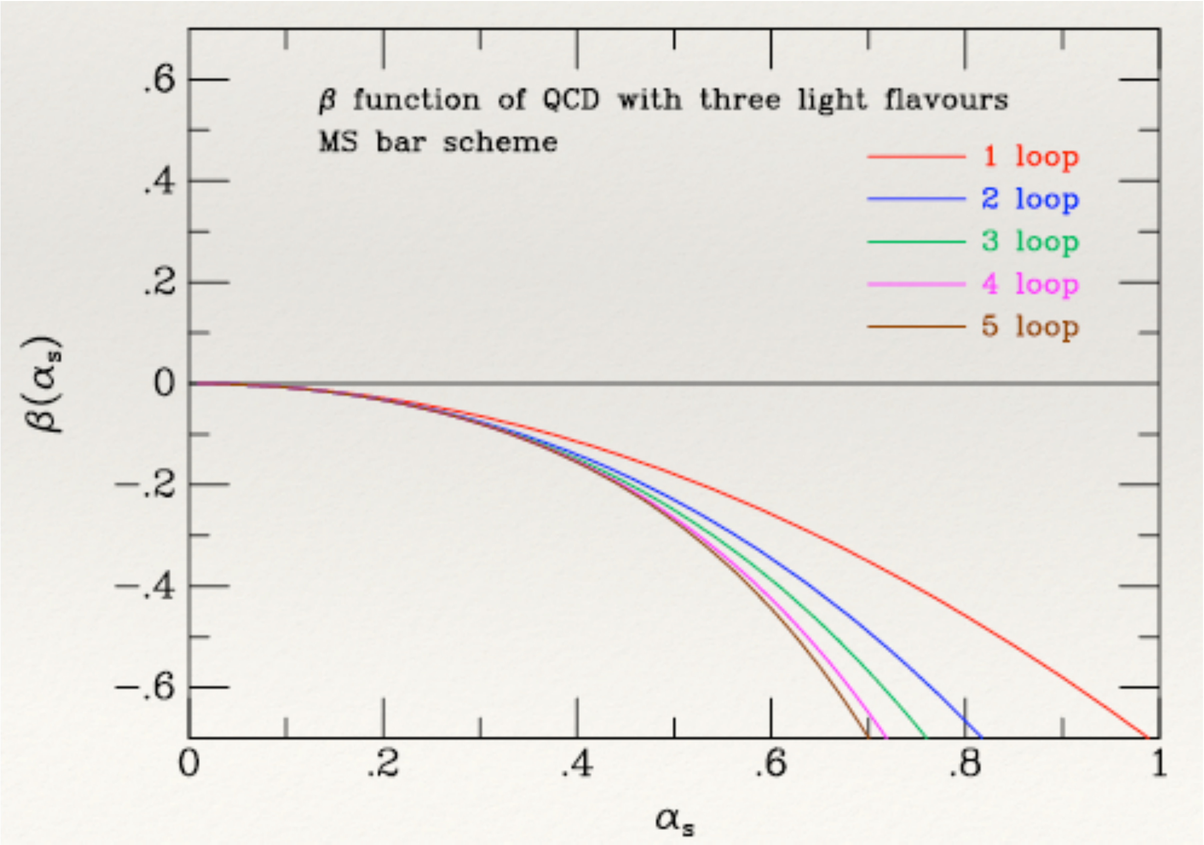
Back to the QCD beta-function

Perturbative expansion of the beta-function:

$$\beta = -\alpha_s^2(\mu) \sum_i b_i \alpha_s^i(\mu)$$

$$b_0 = \frac{11N_c - 4n_f T_R}{12\pi}$$

$$b_1 = \frac{17N_c^2 - 5N_c n_f - 3C_F n_f}{24\pi^2}$$



- n_f is the number of active flavours (depends on the scale)
- today, the beta-function known up to five loops, but only first two coefficients are independent of the renormalization scheme

<u>Exercise:</u> proof the above statement [hint: use the fact that at O(α_s) the coupling in two different schemes is related by a finite change]

Renormalization Group Equation

Consider a dimensionless quantity A, function of a single scale Q. The dimensionless quantity should be independent of Q. However in quantum field theory this is not true, as renormalization introduces a second scale μ

But the renormalization scale is arbitrary. The dependence on it must cancel in physical observables up to the order to which one does the calculation.

So, for any observable A one can write a renormalization group equation

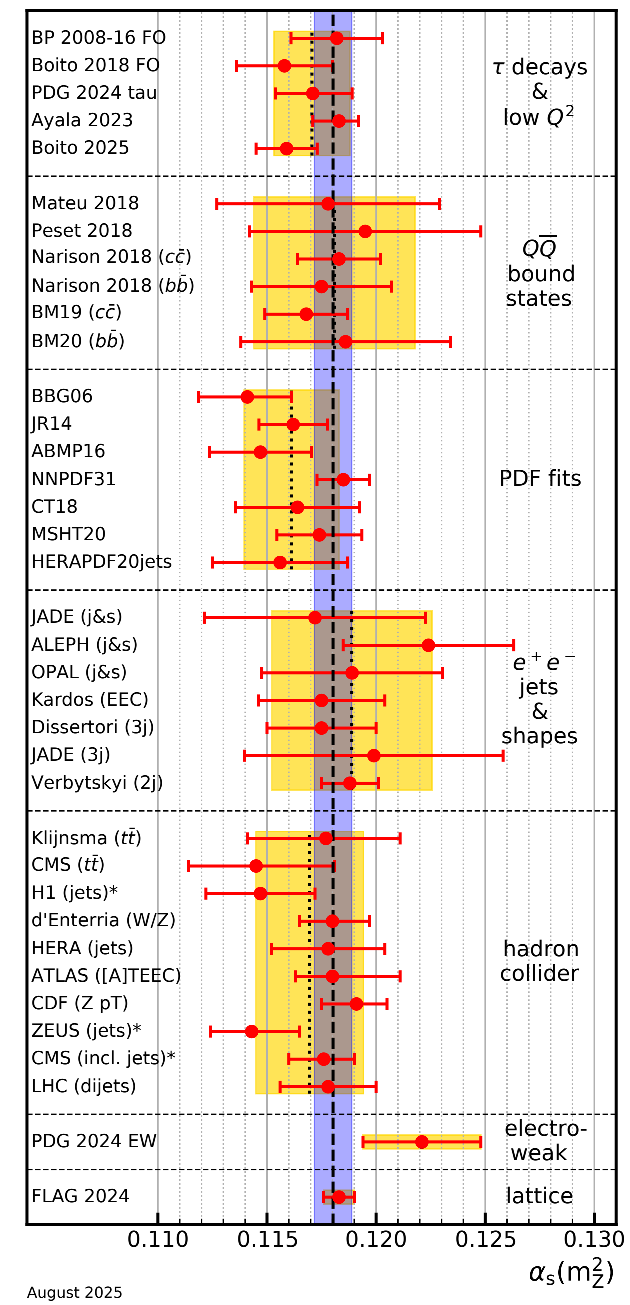
$$\boxed{\left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s}\right] A\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = 0}$$

$$\alpha_s = \alpha_s(\mu^2) \qquad \beta(\alpha_s) = \mu^2 \frac{\partial \alpha_s}{\partial \mu^2}$$

Scale dependence of A enters through the running of the coupling: knowledge of $A(1, \alpha_s(Q^2))$ allows one to compute the variation of A with Q given the beta-function

Measurements of the running coupling

Uncertainty on α_s (and PDF) is in several cases the dominant source of uncertainty to provide precise theory predictions



<u>Procedure to compute world average in PDG</u>:

- subdivide observables in categories
- provide an average for each category
- provide an average of all categories
  ⇒ *the world average of α_s*

$$\boxed{\alpha_s(M_Z) = 0.1180 \pm 0.0009}$$

$$\alpha_s(m_Z^2) = 0.1183 \pm 0.0007 \qquad \text{(FLAG 2024 estimate)}$$

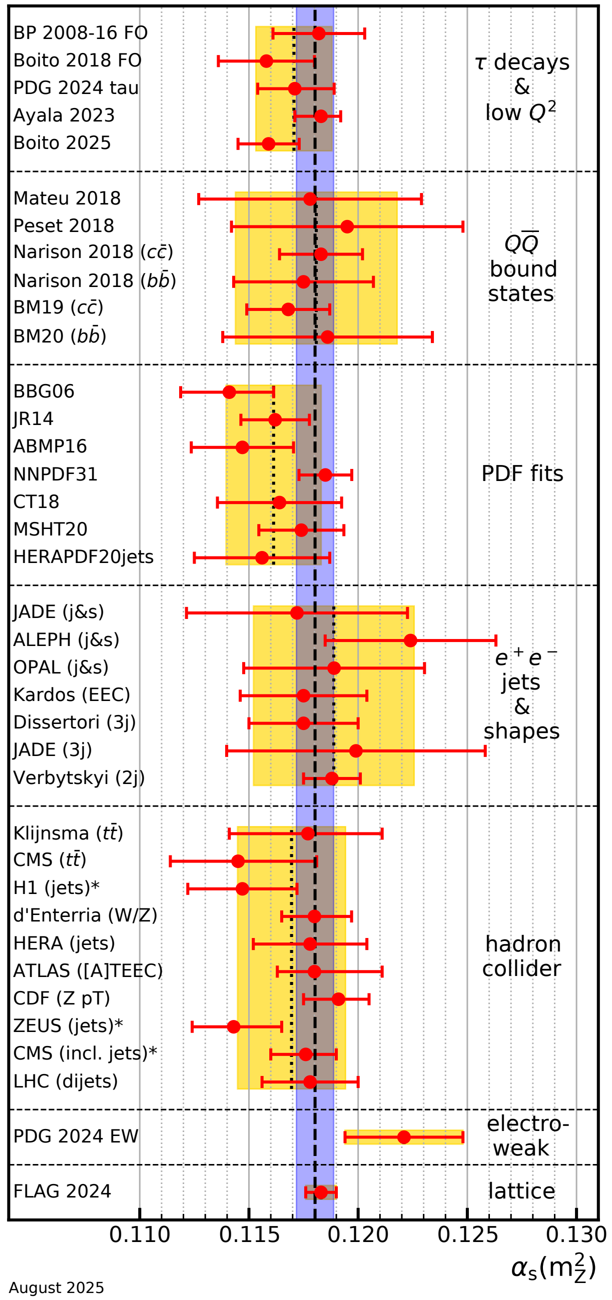
$$\alpha_s(m_Z^2) = 0.1178 \pm 0.0010 \qquad \text{(PDG 2025 without lattice)}$$

Many ambiguities, choices (e.g. treatment of correlations etc.), subtle aspects involved…

Measurements of the running coupling

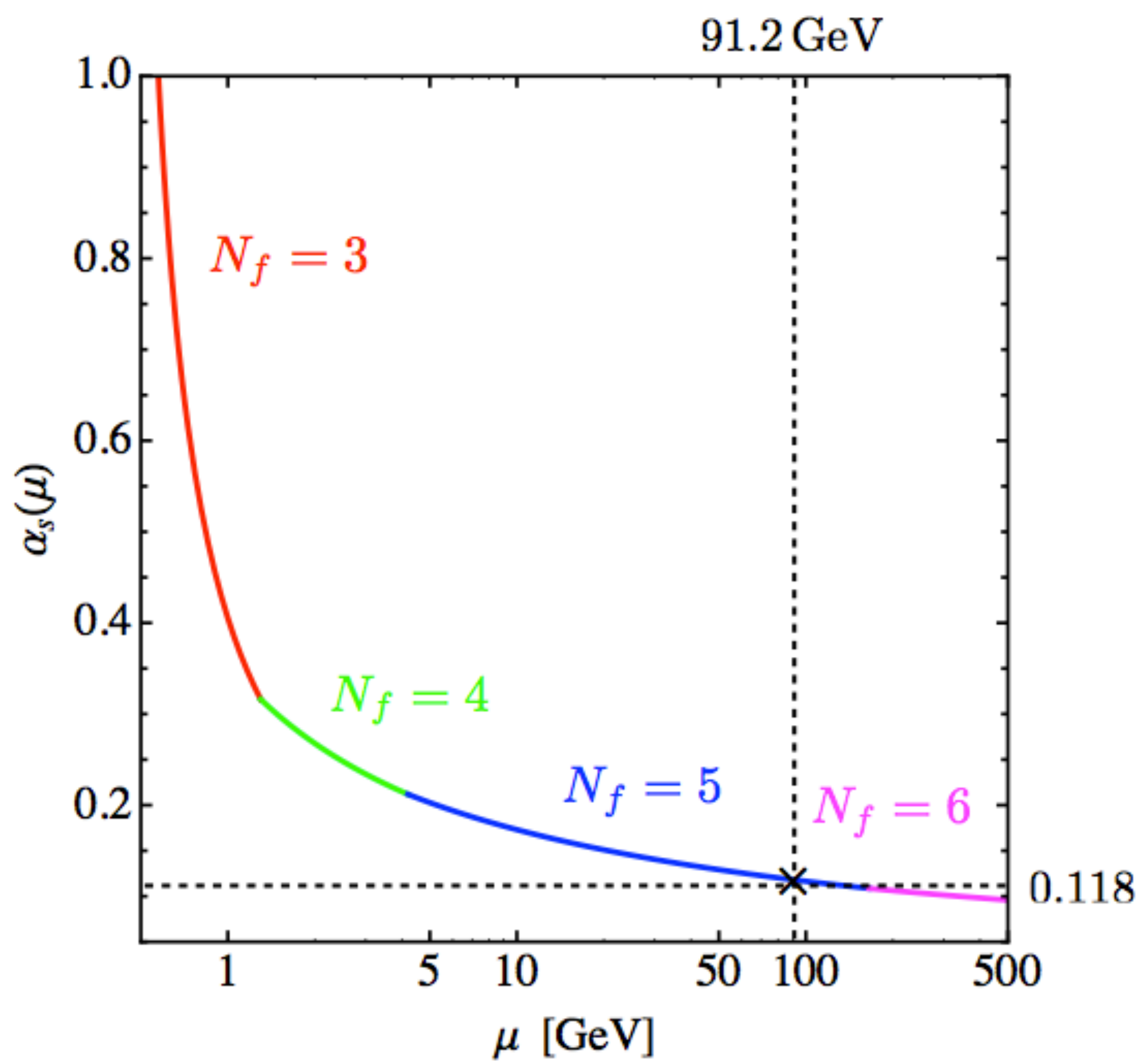
<u>Current Status</u>

- overall consistent picture: α_s from very different observables compatible
- α_s is not so small at current scales
- α_s decreases slowly at higher energies (logarithmic only)
- higher order corrections are and will remain important



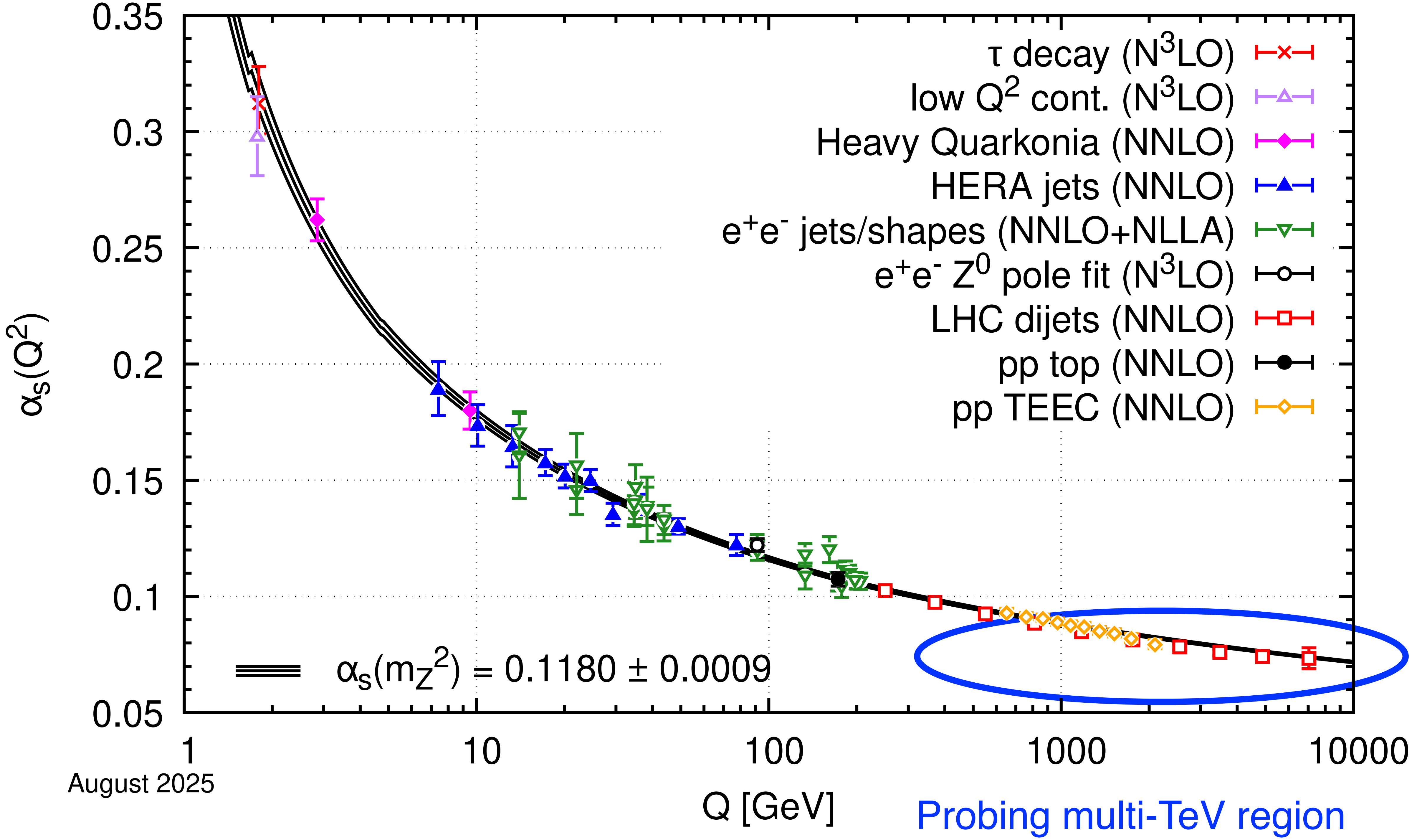
Active flavours & running coupling

The active field content of a theory modifies the running of the couplings



Constrain New Physics by measuring the running at high scales?

Measurements of the running coupling



Recap

We have then discussed the UV behaviour of QCD

- discussed renormalisation of UV divergences
- introduced the running of the coupling constant and the beta-function
- discussed measurements of the coupling constant

Next

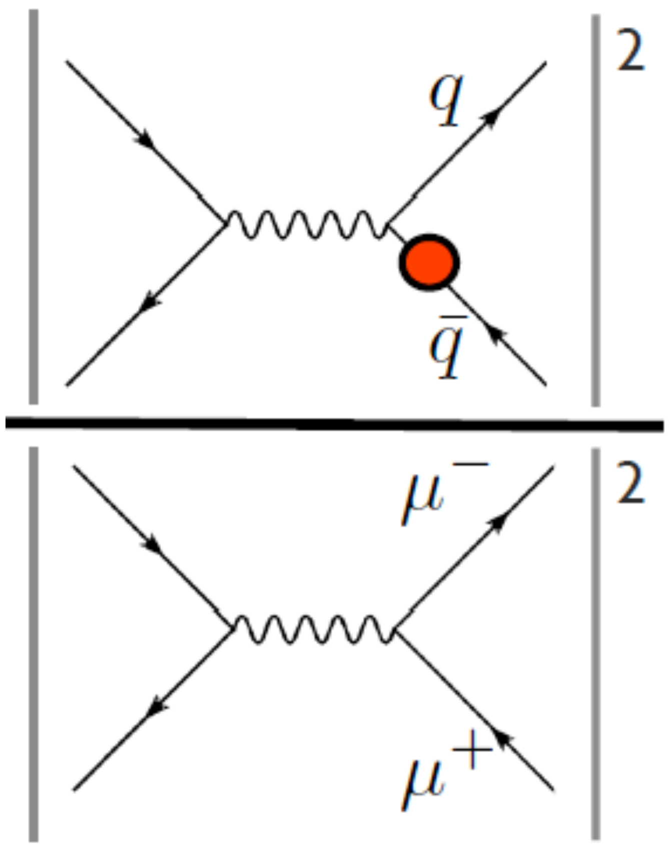
In the following we will concentrate on the perturbative regime of QCD.

In particular, we'll discuss generic properties of QCD amplitudes

- Soft-collinear divergences and how they are dealt with
- Kinoshita-Lee-Nauenberg theorem
- The concept of infrared finiteness
- Sterman Weinberg jets

The soft approximation

Let's consider again the R-ratio

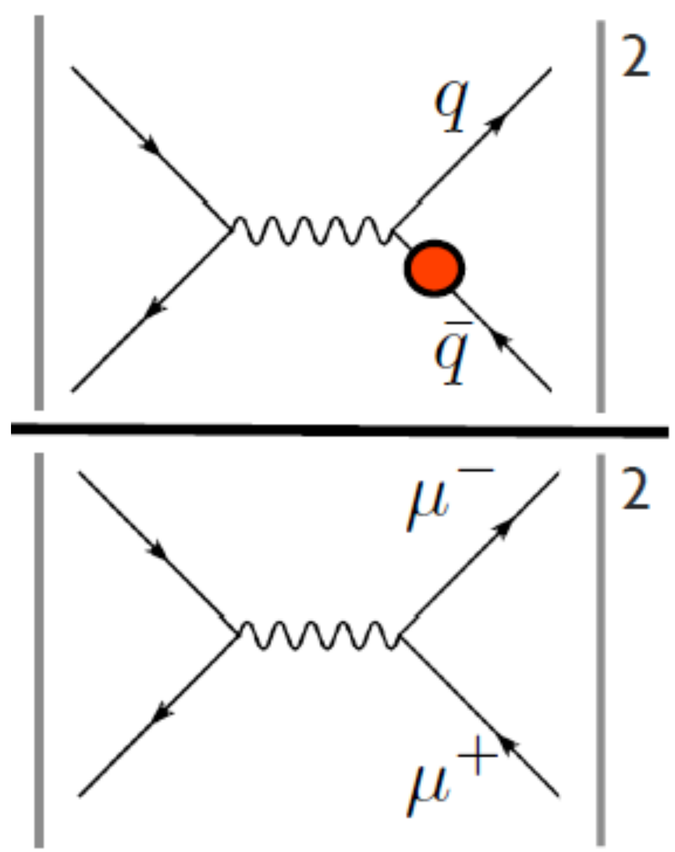


Leading order result

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \approx N_c \sum_q e_q^2$$

The soft approximation

Let's consider again the R-ratio



We have seen a good agreement between the leading order result and data, but there are various unanswered questions

- Since free quarks do not exist, why is the leading order result so good?
- In particular, why can one identify the cross-sections for the production of quarks to that of hadrons?
- Can one probe QCD further by testing more exclusive observables?

Quark-hadron duality

The reliability of parton-level calculations to describe hadron-level observables is known as quark-hadron duality.

This duality relies on the time separation between a hard scattering (partons are produced) and a soft process (quarks hadronize). Since the two processes happen at very different time-scales there is not quantum interference and the soft process does not alter the hard momentum flow “too much”

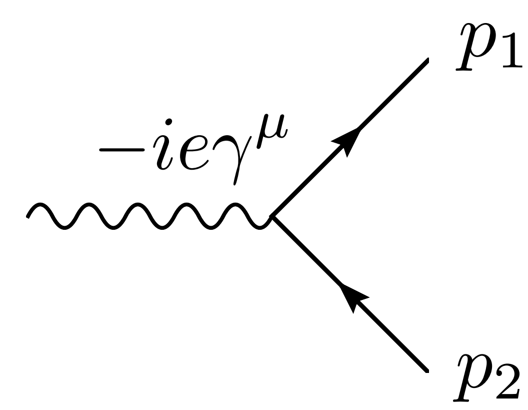
With this in mind, let’s apply the parton description and look for a better approximation of R, i.e. let’s compute QCD corrections, at least in some approximation

The soft approximation

QCD corrections are only in the final state, i.e. corrections to $\gamma^* \to q\bar{q}$

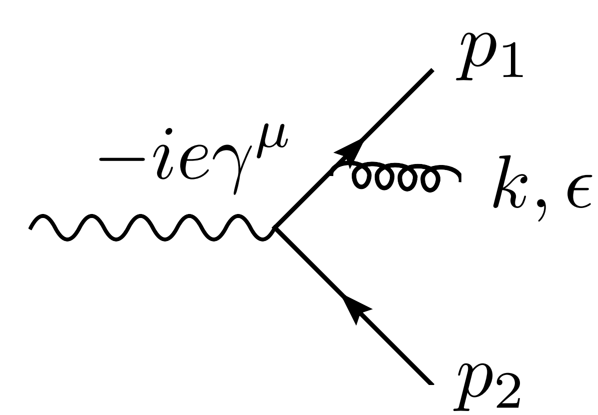
At leading order:

$$M_0^\mu = \bar{u}(p_1)(-ie\gamma^\mu)v(p_2)$$



Emit one gluon:

$$\begin{aligned} M^\mu_{q\bar{q}g} &= \bar{u}(p_1)(-ig_s t^a \not{\epsilon})\frac{i(\not{p}_1+\not{k})}{(p_1+k)^2}(-ie\gamma^\mu)v(p_2) \\ &+ \bar{u}(p_1)(-ie\gamma^\mu)\frac{i(\not{p}_2-\not{k})}{(p_2-k)^2}(-ig_s t^a \not{\epsilon})v(p_2) \end{aligned}$$



Consider the soft approximation: $k \ll p_1, p_2$

$$M^\mu_{q\bar{q}g} = \bar{u}(p_1)\left((-ie\gamma^\mu)(-ig_s t^a)v(p_2)\right)\left(\frac{p_1\epsilon}{p_1 k} - \frac{p_2\epsilon}{p_2 k}\right)$$

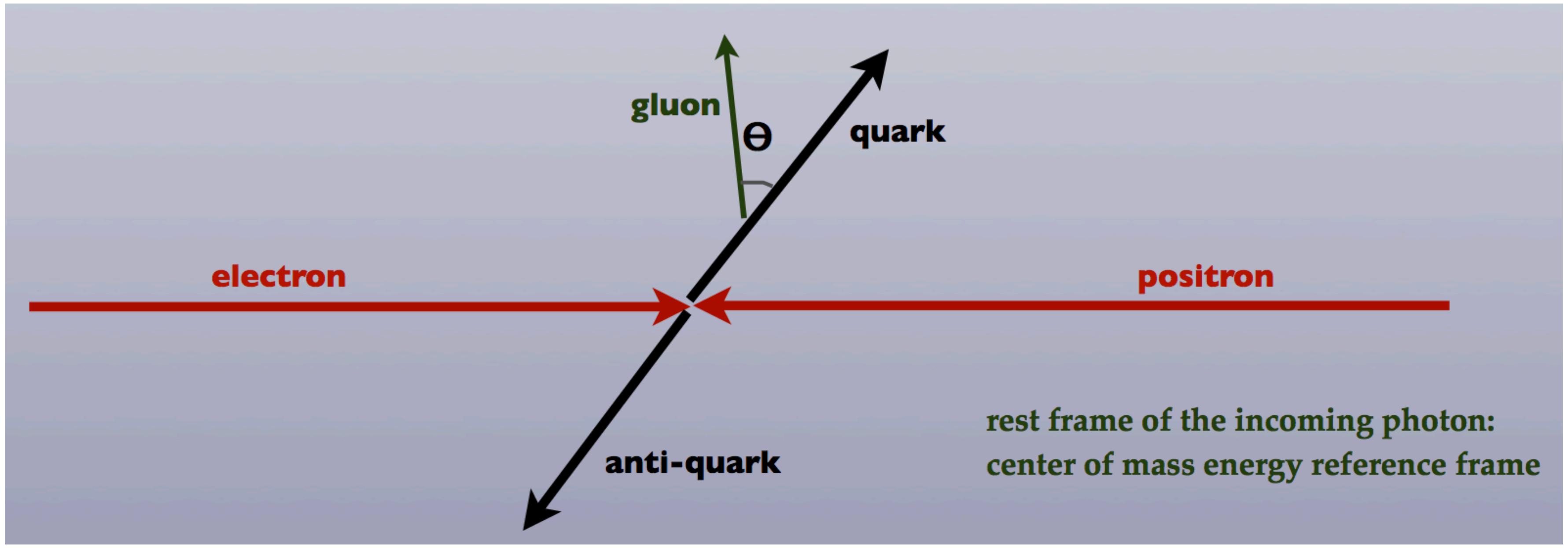
$\Rightarrow$ factorization of soft part (crucial for resummed calculations)

Soft divergences

The squared amplitude becomes

$$|M^{\mu}_{q\bar{q}g}|^2 = \sum_{\rm pol} \left| \bar{u}(p_1)\left((-ie\gamma^{\mu})(-ig_s t^a)v(p_2)\right)\left(\frac{p_1\epsilon}{p_1 k} - \frac{p_2\epsilon}{p_2 k}\right)\right|^2$$
$$= \quad |M_{q\bar{q}}|^2 C_F g_s^2 \frac{2p_1p_2}{(p_1k)(p_2k)}$$

The above is a Lorentz-invariant amplitude. Go to the centre-of-mass frame:



Soft divergences

The squared amplitude becomes

$$|M^{\mu}_{q\bar{q}g}|^2 = \sum_{\text{pol}} \left| \bar{u}(p_1)\left((-ie\gamma^{\mu})(-ig_s t^a)v(p_2)\right)\left(\frac{p_1\epsilon}{p_1 k} - \frac{p_2\epsilon}{p_2 k}\right)\right|^2$$
$$= \quad |M_{q\bar{q}}|^2 C_F g_s^2 \frac{2p_1p_2}{(p_1k)(p_2k)}$$

Including phase space, in this frame, in terms of energy and angle of the gluon one contains

$$d\phi_{q\bar{q}g}|M_{q\bar{q}g}|^2 \quad = \quad d\phi_{q\bar{q}}|M_{q\bar{q}}|^2 \frac{d^3k}{2\omega(2\pi)^3} C_F g_s^2 \frac{2p_1p_2}{(p_1k)(p_2k)}$$
$$= \quad d\phi_{q\bar{q}}|M_{q\bar{q}}|^2 \omega d\omega d\cos\theta \frac{d\phi}{2\pi}\frac{2\alpha_s C_F}{\pi}\frac{1}{\omega^2(1-\cos^2\theta)}$$

The differential cross section becomes

$$d\sigma_{q\bar{q}g} = d\sigma_{q\bar{q}} \frac{2\alpha_s C_F}{\pi}\frac{d\omega}{\omega}\frac{d\theta}{\sin\theta}\frac{d\phi}{2\pi}$$

Soft & collinear divergences

Cross section for producing a q$\bar{q}$-pair and a gluon is infinite (IR divergent)

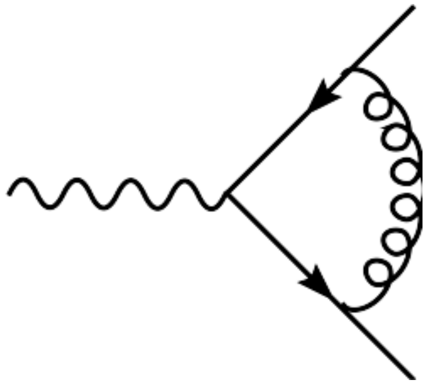
$$d\sigma_{q\bar{q}g} = d\sigma_{q\bar{q}} \frac{2\alpha_s C_F}{\pi} \frac{d\omega}{\omega} \frac{d\theta}{\sin\theta} \frac{d\phi}{2\pi}$$

<u>$\omega \to 0$:</u> soft divergence

<u>$\theta \to 0$:</u> collinear divergence

But the full O(α_s) correction to R is finite, because one must include a virtual correction which cancels the divergence of the real radiation

$$d\sigma_{q\bar{q},v} \sim -d\sigma_{q\bar{q}} \frac{2\alpha_s C_F}{\pi} \frac{d\omega}{\omega} \frac{d\theta}{\sin\theta} \frac{d\phi}{2\pi}$$



NB: here we kept only soft terms, if we do the full calculation one gets a finite correction of α_s/π

Soft & collinear divergences

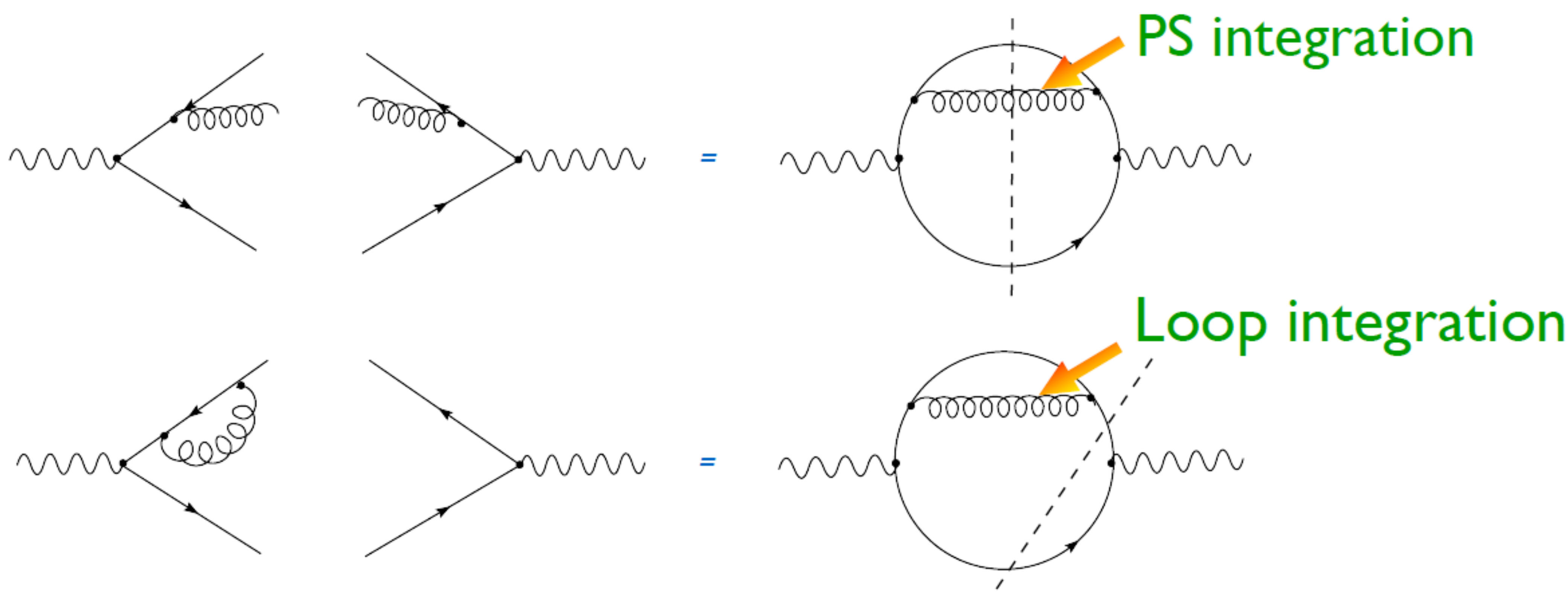
$\underline{\omega \to 0}$ soft divergence: the four-momentum of the emitted particle approaches zero, typical of gauge theories, even if matter (radiating particle) is massive

$\underline{\theta \to 0}$ collinear divergence: particle emitted collinear to emitter. Divergence present only if all particles involved are massless

NB: the appearance of soft and collinear divergences discussed in the specific contect of $e^+e^- \to qq$ are a general property of QCD

Infrared finiteness

Cancellation of IR divergences in R is not a miracle. It follows directly from unitarity provided the measurement is inclusive enough

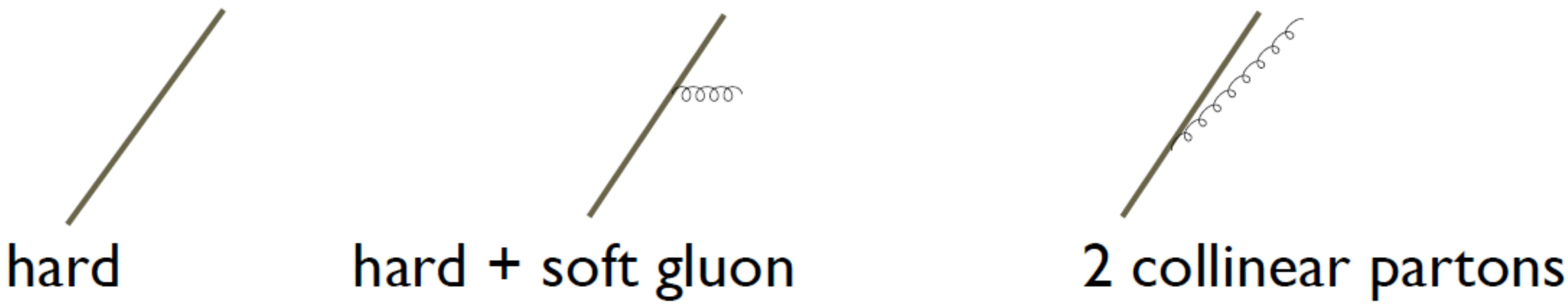


In the infrared region real and virtual are kinematically equivalent but for a (-1) from unitarity

Compute and regulate real and virtual separately, until a cancelation of divergences is achieved

KLN Theorem

Kinoshita-Lee-Nauenberg theorem: Infrared singularities in a massless theory cancel out after summing over degenerate (initial and final) states



Physically a hard parton can not be distinguished from a hard parton plus a soft gluon or from two collinear partons with the same energy. They are degenerate states.
Hence, one needs to add them to get a physically sound observable

Infrared safety (= finiteness)

So, the R-ratio is an infrared safe quantity.

In perturbation theory one can compute only IR-safe quantities, otherwise get infinities, which can not be renormalized away (why not…?)

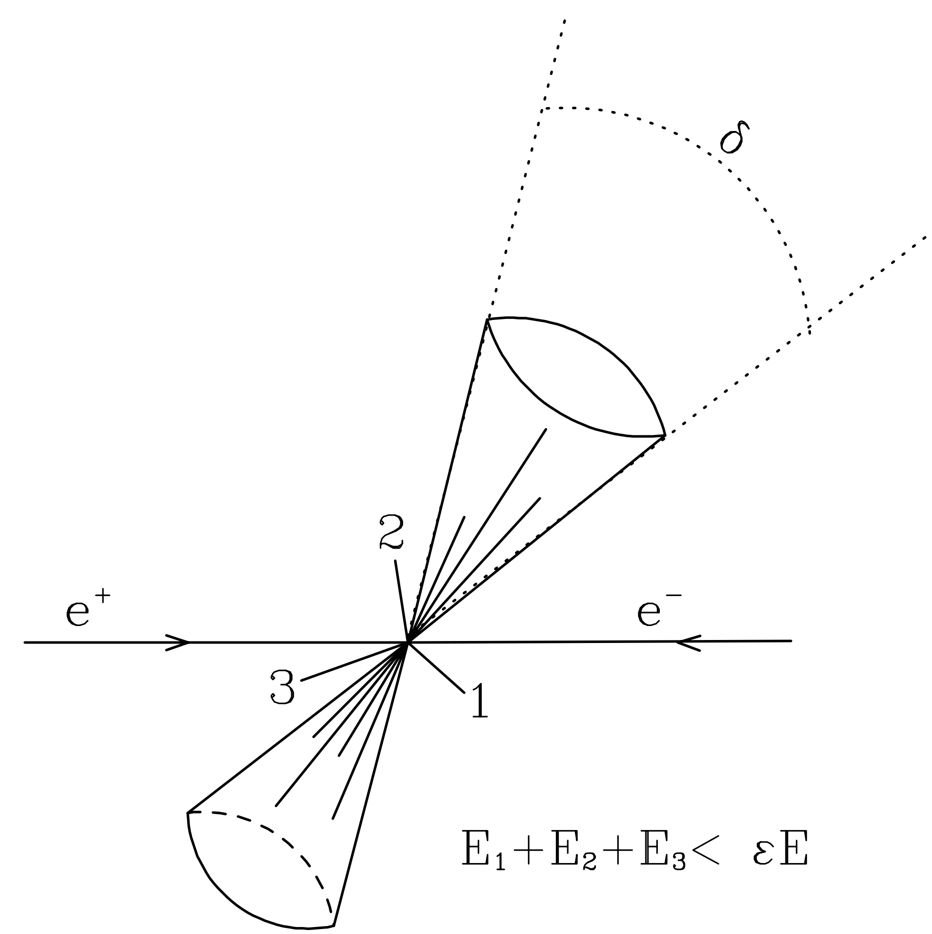
So, the natural questions are:

- are there other IR-safe quantities?
- what property of R guarantees its IR-safety?

Sterman-Weinberg jets

First formulation of cross-sections which are finite in perturbation theory and describe the hadronic final state

Introduce two parameters ε and δ:
a pair of Sterman-Weinberg jets are two cones of opening angle δ that contain all the energy of the event excluding at most a fraction ε

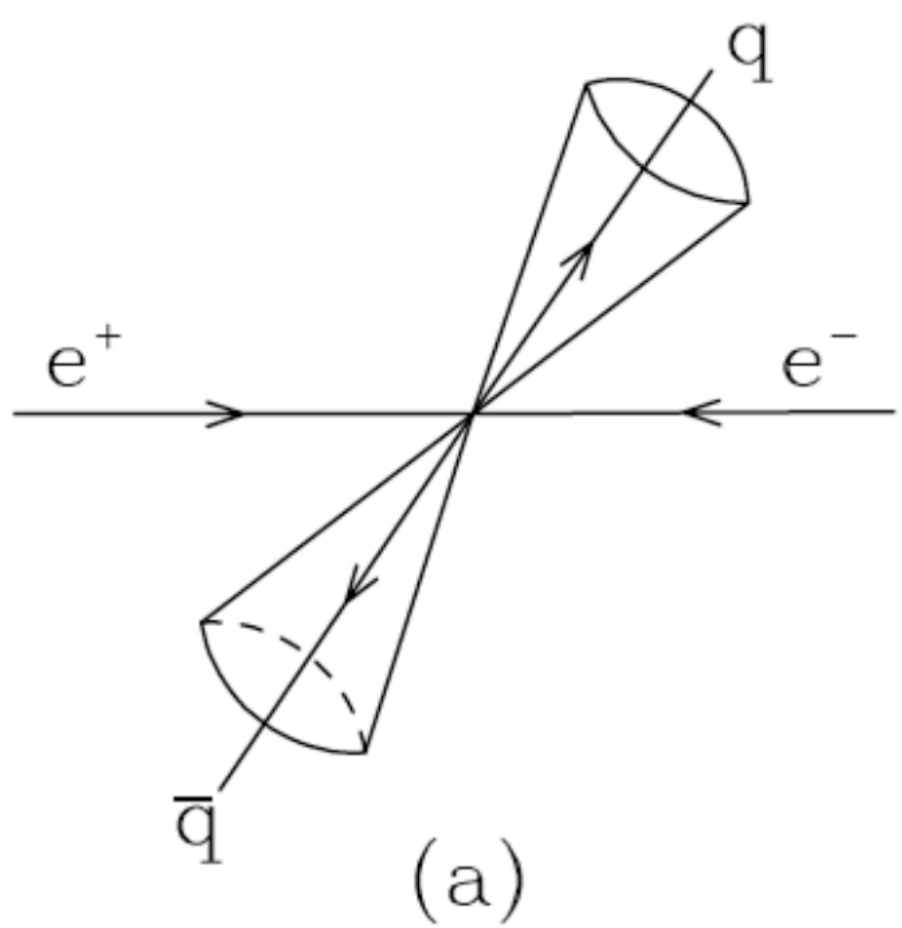


Why finite? the cancelation between real and virtual is not destroyed in the soft/collinear regions

Sterman-Weinberg jets

Let's compute the O(as) correction to the Sterman-Weinberg jet cross-section in the soft-collinear approximation

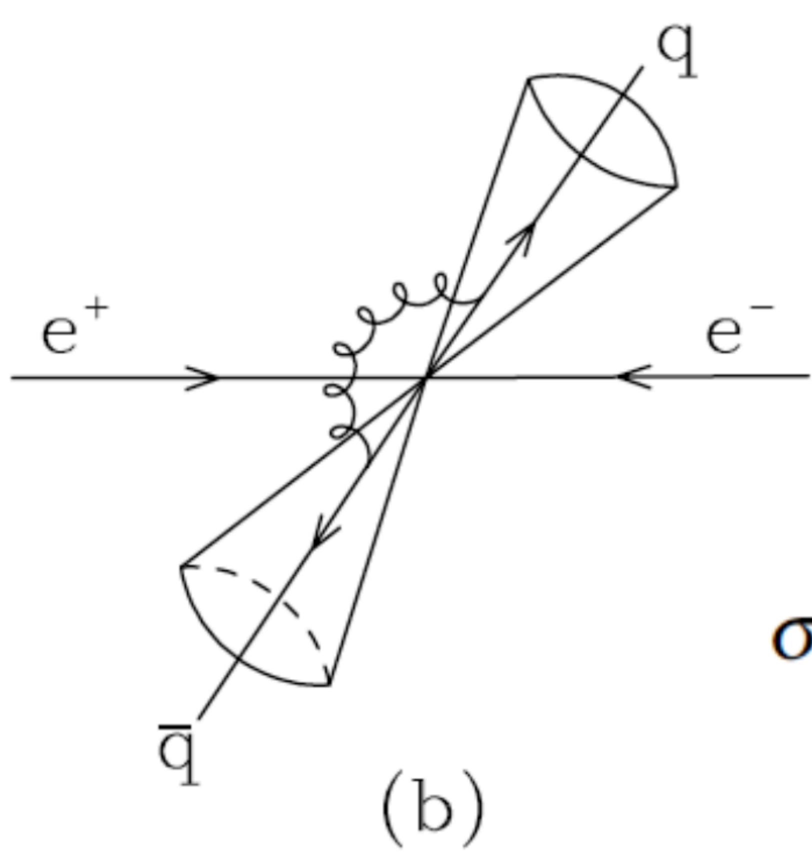
a) We have a Born term σ_B which is completely within the Sterman-Weinberg jet definition: since there are only two quarks they keep all the energy inside the cones



Sterman-Weinberg jets

Let's compute the O(as) correction to the Sterman-Weinberg jet cross-section in the soft-collinear approximation

b) We have a virtual term which is also completely within the Sterman-Weinberg jet definition (only two quarks)

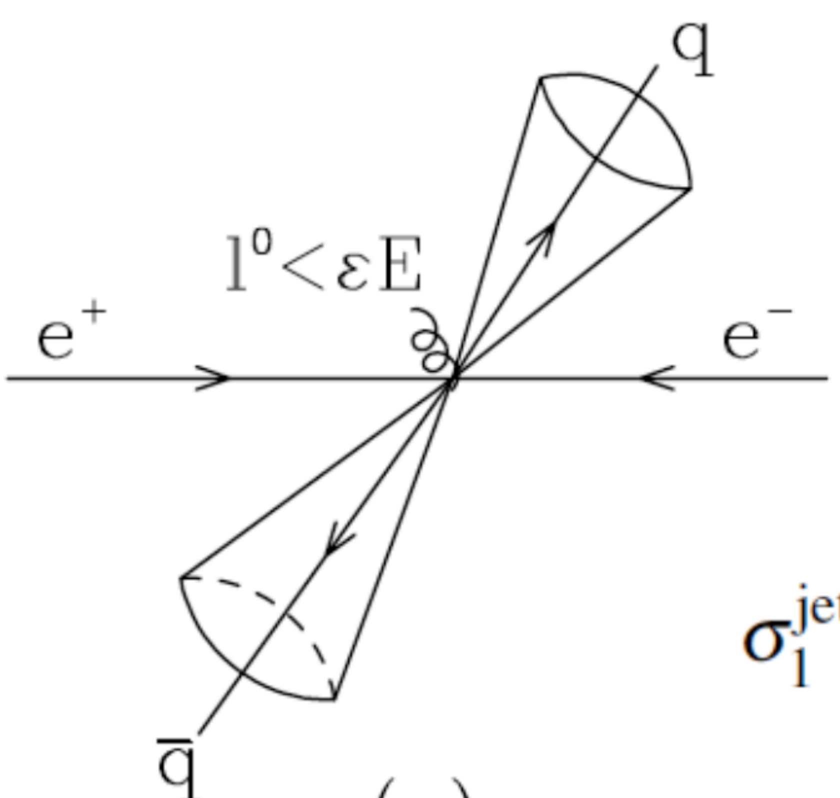


$$\sigma_1^{\text{jet,virt}} = -C_F \frac{g_S^2}{\pi^2} \sigma_{\text{born}} \int_0^E \frac{dl_0}{l_0} \int_0^\pi \frac{d\cos\theta}{1-\cos^2\theta}$$

Sterman-Weinberg jets

Let's compute the O(as) correction to the Sterman-Weinberg jet cross-section in the soft-collinear approximation

c) We have a real term: the emitted gluon can be emitted also outside the jet provided it carries only little energy, or..



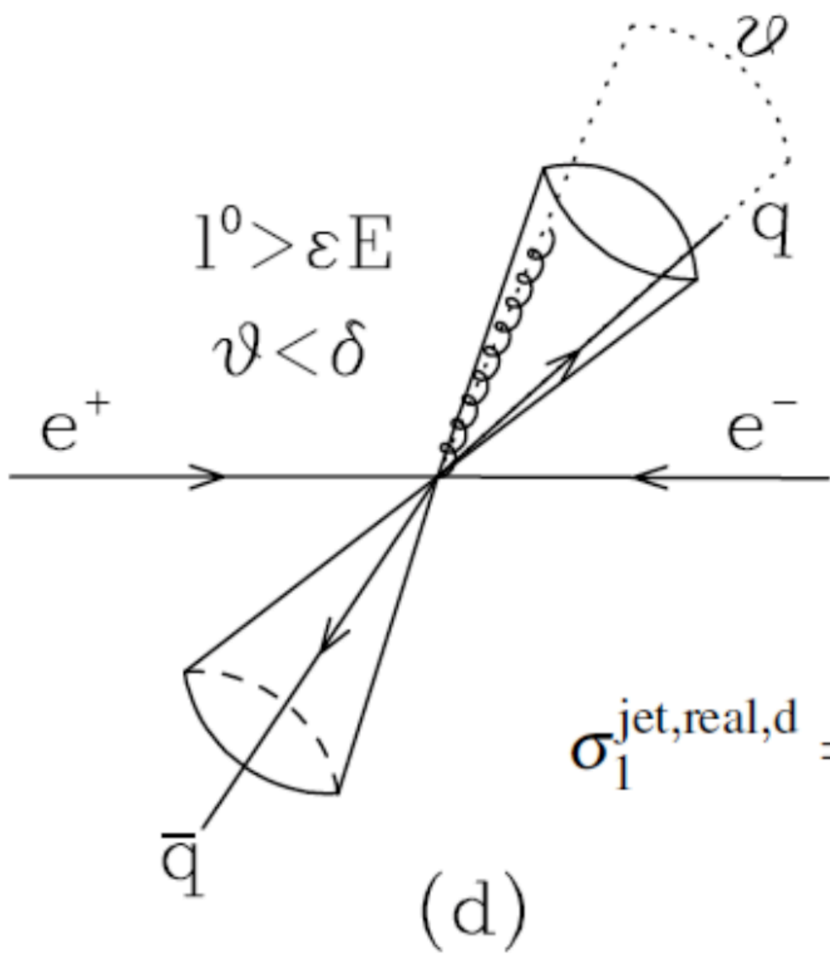
(c)

$$\sigma_1^{\text{jet,real,c}} = C_F \frac{g_S^2}{\pi^2} \sigma_{\text{born}} \int_0^{\varepsilon E} \frac{dl_0}{l_0} \left[\int_0^{\pi} \frac{d\cos\theta}{1-\cos^2\theta} \right]$$

Sterman-Weinberg jets

Let's compute the O(as) correction to the Sterman-Weinberg jet cross-section in the soft-collinear approximation

d) .. or it can carry a considerable fraction of energy provided it is emitted inside the cones



$$\sigma_1^{\text{jet,real,d}} = C_F \frac{g_S^2}{\pi^2} \sigma_{\text{born}} \int_{\varepsilon E}^{E} \frac{dl_0}{l_0} \left[\int_0^{\delta} \frac{d\cos\theta}{1-\cos^2\theta} + \int_{\pi-\delta}^{\pi} \frac{d\cos\theta}{1-\cos^2\theta} \right]$$

Sterman-Weinberg jets

Adding all the contributions, the Sterman-Weinberg jet cross-section up to $O(\alpha_s)$ in the soft-collinear approximation is given by

$$\sigma_1 = \sigma_0 \left(1 + \frac{2\alpha_s C_F}{\pi} \ln \epsilon \ln \delta^2\right)$$

Effective expansion parameter in QCD is often $\alpha_s C_F/\pi$ not α_s

α_s-expansion enhanced by a double log: left-over from real-virtual cancellation

Sterman-Weinberg jets

Adding all the contributions, the Sterman-Weinberg jet cross-section up to $O(\alpha_s)$ in the soft-collinear approximation is given by

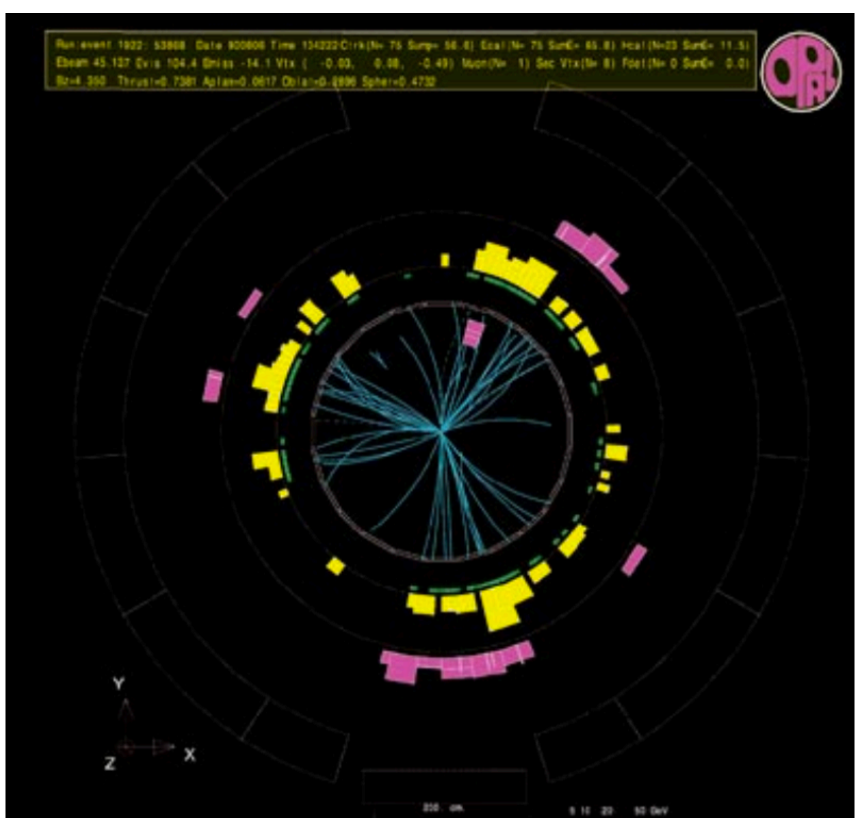
$$\sigma_1 = \sigma_0 \left(1 + \frac{2\alpha_s C_F}{\pi} \ln\epsilon \ln\delta^2 \right)$$

- if more gluons are emitted, one gets for each gluon
  - a power of $\alpha_s C_F/\pi$
  - a soft logarithm $\ln\epsilon$
  - a collinear logarithm $\ln\delta$
- if ε and/or δ become too small the above result diverges
- if the logarithms are large, fixed order become unreliable (diverge). One needs to resum large infrared and collinear logarithms to all orders in the coupling constant

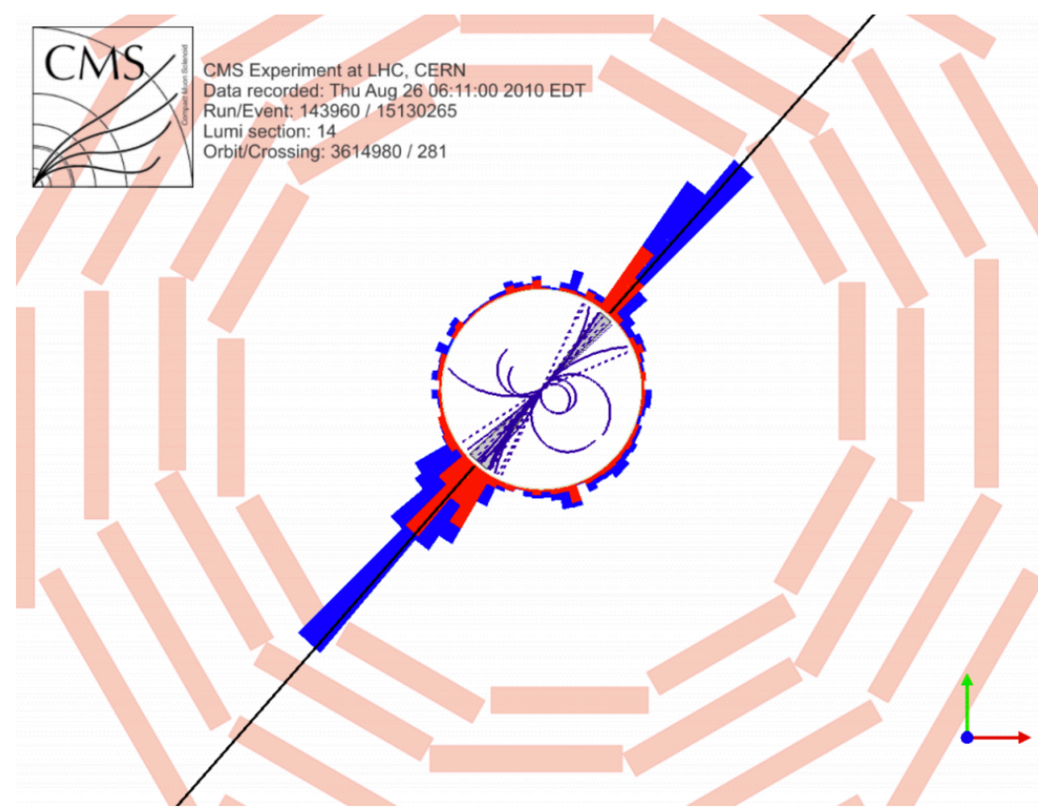
Jets

- Jets were discovered in the late 70s in electron-position collision
- They provided the first direct evidence for the gluon
- In the 80s and 90s jets provided many other stringent tests of QCD at LEP
- Today jets are a powerful tools to look for New Physics at the LHC

Gluon discovery: 3jet event in e^+e^-



High energy di-jet event at CMS



Infrared safety: definition

An observable $\mathcal{O}$ is infrared and collinear safe if

$$\mathcal{O}_{n+1}(k_1, k_2, \ldots, k_i, k_j, \ldots k_n) \to \mathcal{O}_n(k_1, k_2, \ldots k_i + k_j, \ldots k_n)$$

whenever one of the k_i/k_j becomes soft or k_i and k_j are collinear

i.e. the observable is insensitive to emission of soft particles or to collinear splittings

Infrared safety: examples

Infrared safe ?

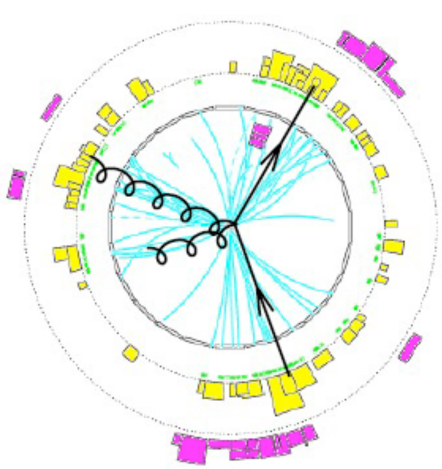
- energy of the hardest particle in the event NO
- multiplicity of gluons NO
- momentum flow into a cone in rapidity and angle YES
- cross-section for producing one gluon with $E > E_{min}$ and $\theta > \theta_{min}$ NO
- jet cross-sections DEPENDS

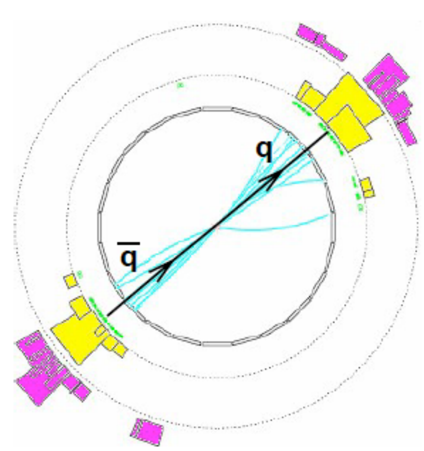
Only for infrared safe quantities is a comparison of data and theory well defined to all orders in perturbation theory

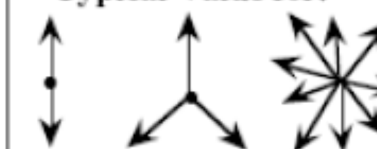
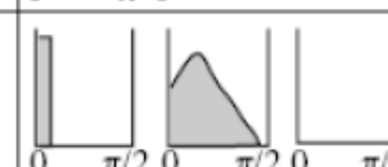
Other IR safe quantities

Event shapes: describe the shape of the event, but are largely insensitive to soft and collinear branching

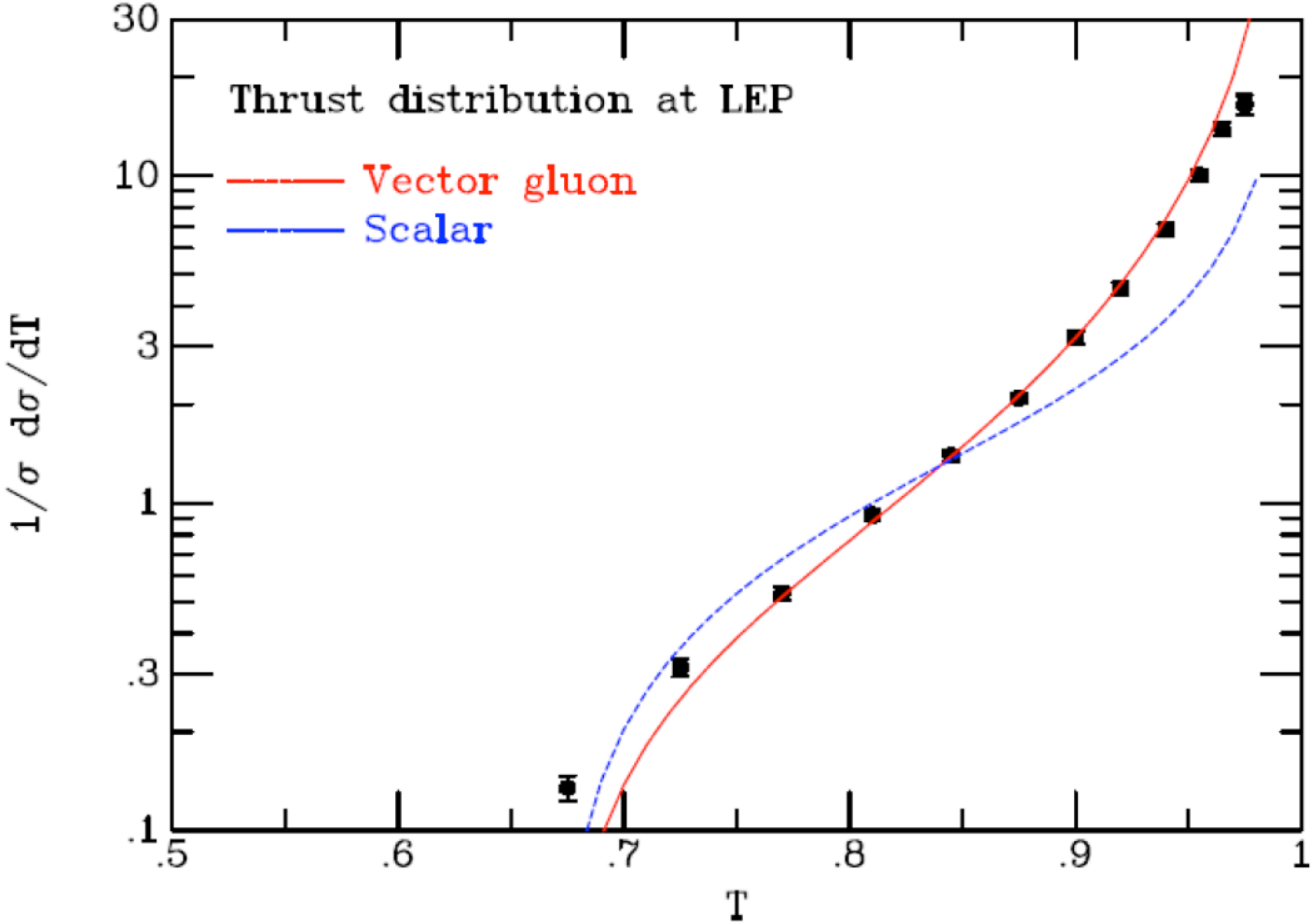
- widely used to measure α_s
- measure color factors
- test QCD
- learn about non-perturbative physics



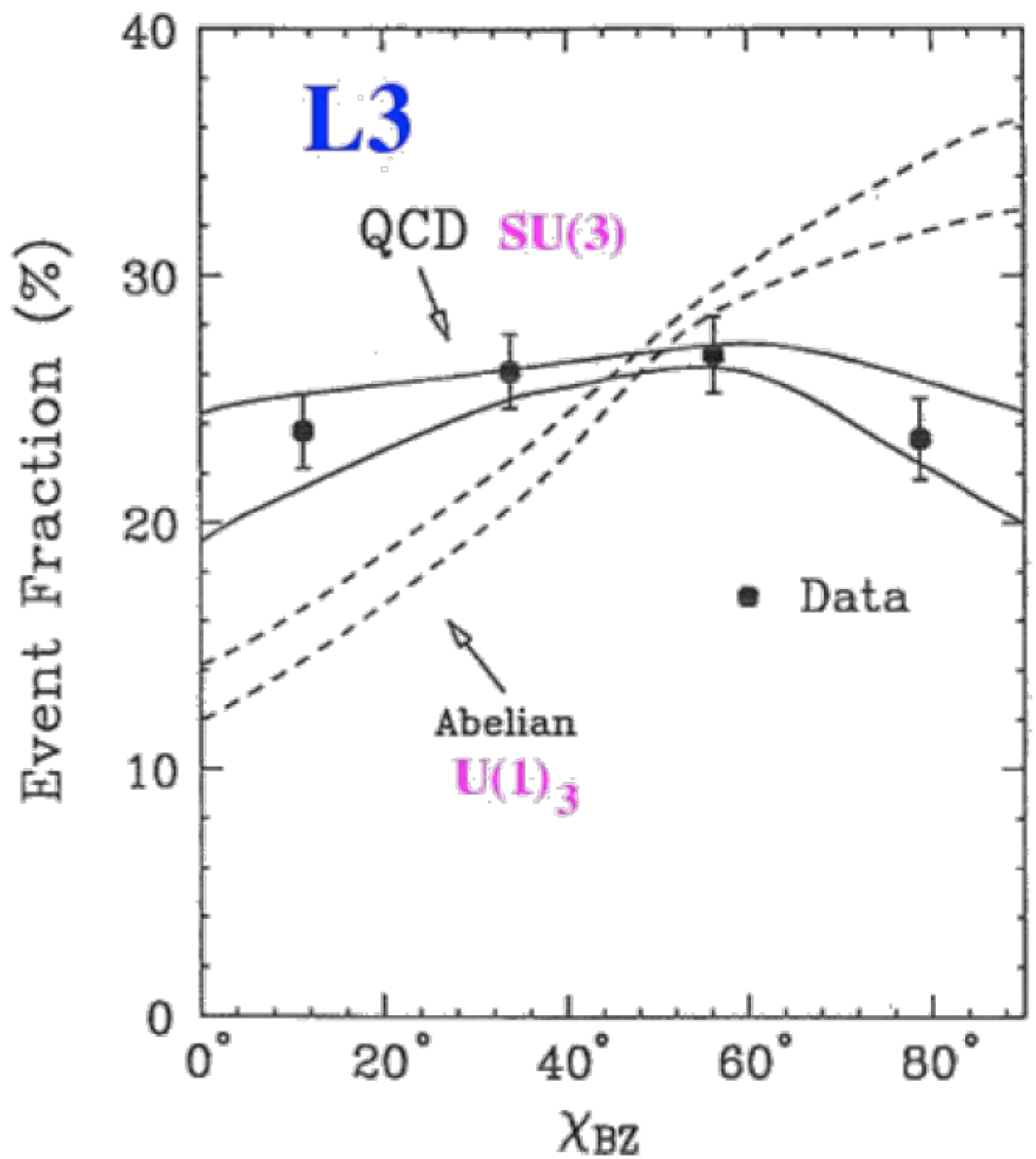


Name of Observable	Definition	Typical Value for: 			QCD calculation
Thrust	$T = \max_{\vec{n}} \left(\frac{\sum_i \lvert \vec{p}_i \vec{n} \rvert}{\sum_i \lvert \vec{p}_i \rvert} \right)$	1	≥2/3	≥1/2	(resummed) $O(\alpha_s^2)$
Thrust major	Like T, however T_{maj} and $\vec{n}_{maj}$ in plane $\perp \vec{n}_T$	0	≤1/3	$\leq 1/\sqrt{2}$	$O(\alpha_s^2)$
Thrust minor	Like T, however T_{min} and $\vec{n}_{min}$ in direction $\perp$ to $\vec{n}_T$ and $\vec{n}_{maj}$	0	0	≤1/2	$O(\alpha_s^2)$
Oblateness	$O = T_{maj} - T_{min}$	0	≤1/3	0	$O(\alpha_s^2)$
Sphericity	$S = 1.5\,(Q_1 + Q_2)$; $Q_1 \leq ... \leq Q_3$ are Eigenvalues of $S^{\alpha\beta} = \frac{\sum_i p_i^\alpha p_i^\beta}{\sum_i p_i^2}$	0	≤3/4	≤1	none (not infrared safe)
Aplanarity	$A = 1.5\,Q_1$	0	0	≤1/2	none (not infrared safe)
Jet (Hemisphere) masses	$M_\pm^2 = \left(\sum_i E_i^2 - \sum_i \vec{p}_i^{\,2}\right)_{i \in S_\pm}$ ($S_\pm$: Hemispheres $\perp$ to $\vec{n}_T$) $M_H^2 = \max(M_+^2, M_-^2)$ $M_D^2 = \lvert M_+^2 - M_-^2 \rvert$	 0 0	 ≤1/3 ≤1/3	 ≤1/2 0	(resummed) $O(\alpha_s^2)$
Jet broadening	$B_\pm = \frac{\sum_{i \in S_\pm} \lvert \vec{p}_i \times \vec{n}_T \rvert}{2 \sum_i \lvert \vec{p}_i \rvert}$; $B_T = B_+ + B_-$ $B_w = \max(B_+, B_-)$	0 0	$\leq 1/(2\sqrt{3})$ $\leq 1/(2\sqrt{3})$	$\leq 1/(2\sqrt{2})$ $\leq 1/(2\sqrt{3})$	(resummed) $O(\alpha_s^2)$
Energy-Energy Correlations	$EEC(\chi) = \sum_{events} \sum_{i,j} \frac{E_i E_j}{E_{vis}^2} \int_{\chi - \frac{\Delta\chi}{2}}^{\chi + \frac{\Delta\chi}{2}} \delta(\chi - \chi_{ij})$	 0 π 0 π 0 π			(resummed) $O(\alpha_s^2)$
Asymmetry of EEC	$AEEC(\chi) = EEC(\pi - \chi) - EEC(\chi)$	 0 π/2 0 π/2 0 π/2			$O(\alpha_s^2)$
Differential 2-jet rate	$D_2(y) = \frac{R_2(y - \Delta y) - R_2(y)}{\Delta y}$				(resummed) $O(\alpha_s^2)$

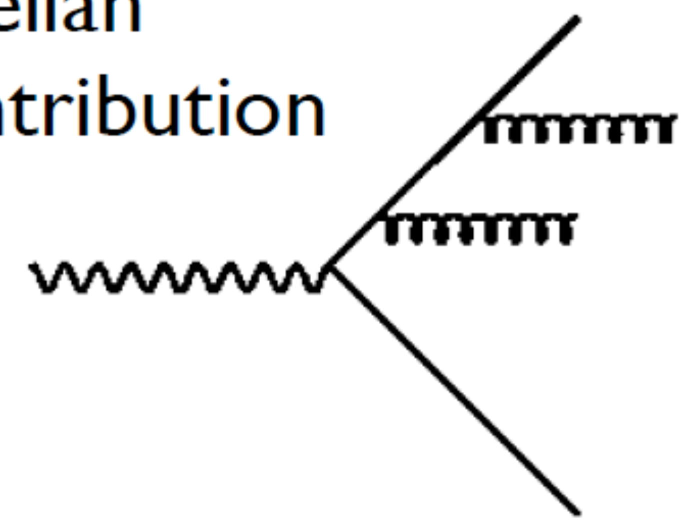
Example: spin of the gluon



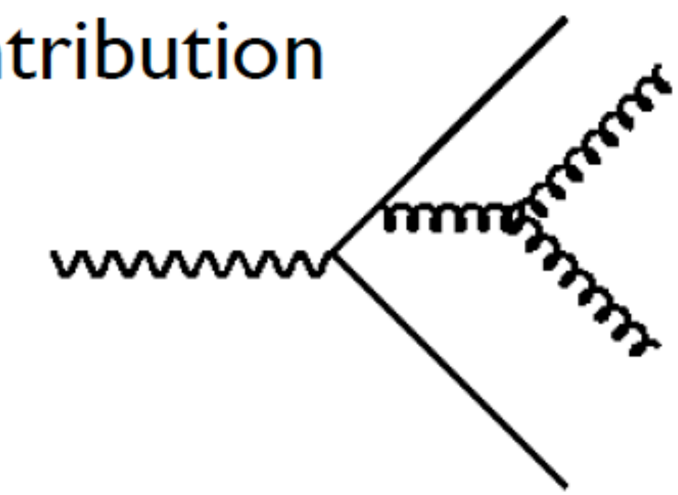
Example: non-abelian nature of QCD



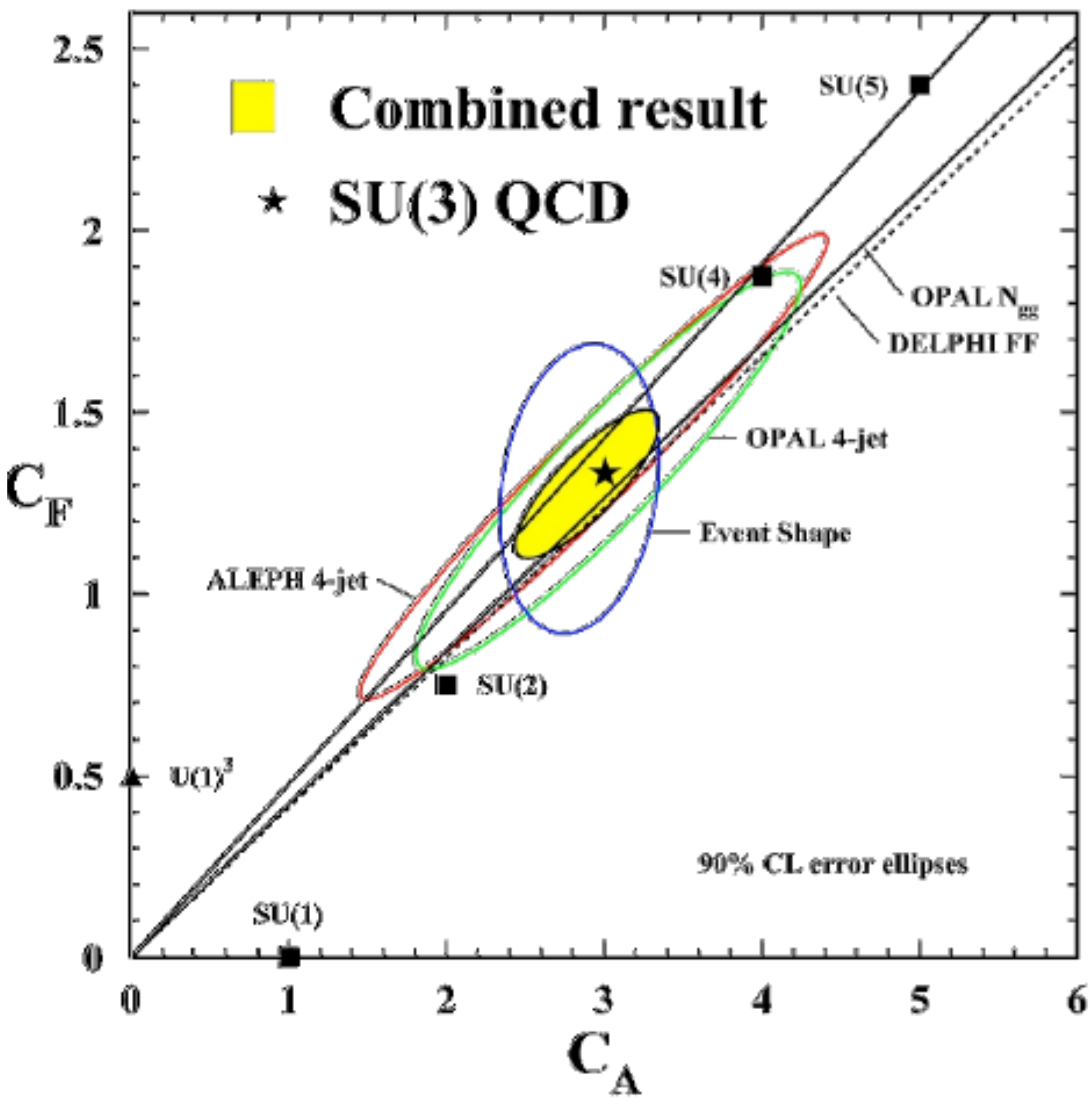
Abelian contribution



Non-Abelian contribution



Example: fits of colour fators



Fits of colour factors from 4-jet rates and event shapes

$$C_A = 2.89 \pm 0.21$$
$$C_F = 1.30 \pm 0.09$$

Well compatible with QCD:

$$C_A = 3$$
$$C_F = \frac{4}{3}$$

Recap

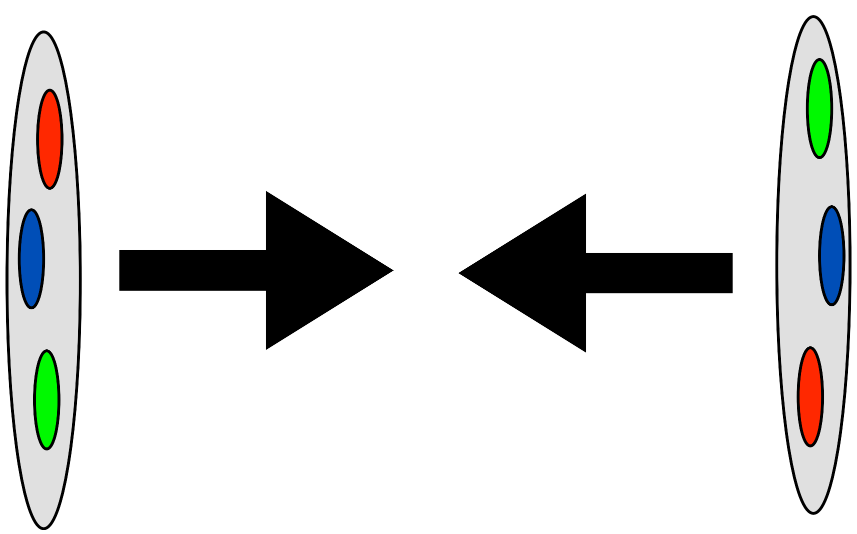
Brief recap on the infrared behaviour of QCD

- we have seen that soft and collinear divergences arise universally in QCD calculations
- these divergences cancel in e^+e^- observables in inclusive observables (KLN theorem)
- we have performed a first genuine QCD calculation: the cross-section for Sterman Weinberg jets in e^+e^- collisions
- perturbative QCD can be used to compute jet-cross section and other infrared-safe event shape variables
- comparison of theory and calculations provide stringent tests of QCD

Partons in the initial state

- We talked a lot about final state QCD effects
- This is the only thing to worry about at e^+e^- colliders (LEP)
- Hera/Tevatron/LHC involve protons in the initial state
- Proton are made of QCD constituents

Next we will focus mainly on aspects related to initial state effects



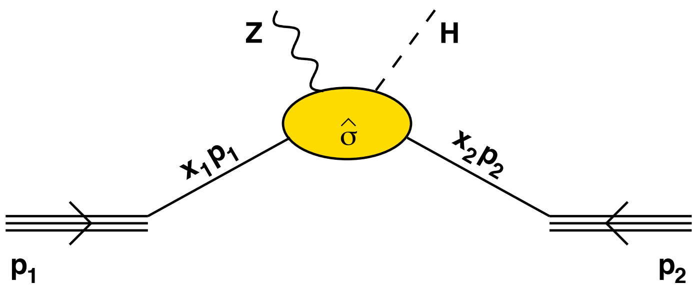
The parton model

Basic idea of the parton model: intuitive picture where in a high transverse momentum scattering partons behave as quasi free in the collision

$\Longrightarrow$ cross section is the incoherent sum of all partonic cross-sections

$$\sigma = \int dx_1 dx_2 f_1^{(P_1)}(x_1) f_2^{(P_2)}(x_2) \hat{\sigma}(x_1 x_2 s) \qquad \hat{s} = x_1 x_2 s$$

NB: This formula is wrong/incomplete (see later)



$f_i^{(P_j)}(x_i)$: parton distribution function (PDF) is the probability to find parton i in hadron j with a fraction x_i of the longitudinal momentum (transverse momentum neglected), extracted from data

$\hat{\sigma}(x_1 x_2 s)$: partonic cross-section for a given scattering process, computed in perturbative QCD

Sum rules

Momentum sum rule: conservation of incoming total momentum

$$\int_0^1 dx \sum_i x f_i^{(p)}(x) = 1$$

Conservation of flavour: e.g. for a proton

$$\int_0^1 dx \left(f_u^{(p)}(x) - f_{\bar{u}}^{(p)}(x) \right) = 2$$

$$\int_0^1 dx \left(f_d^{(p)}(x) - f_{\bar{d}}^{(p)}(x) \right) = 1$$

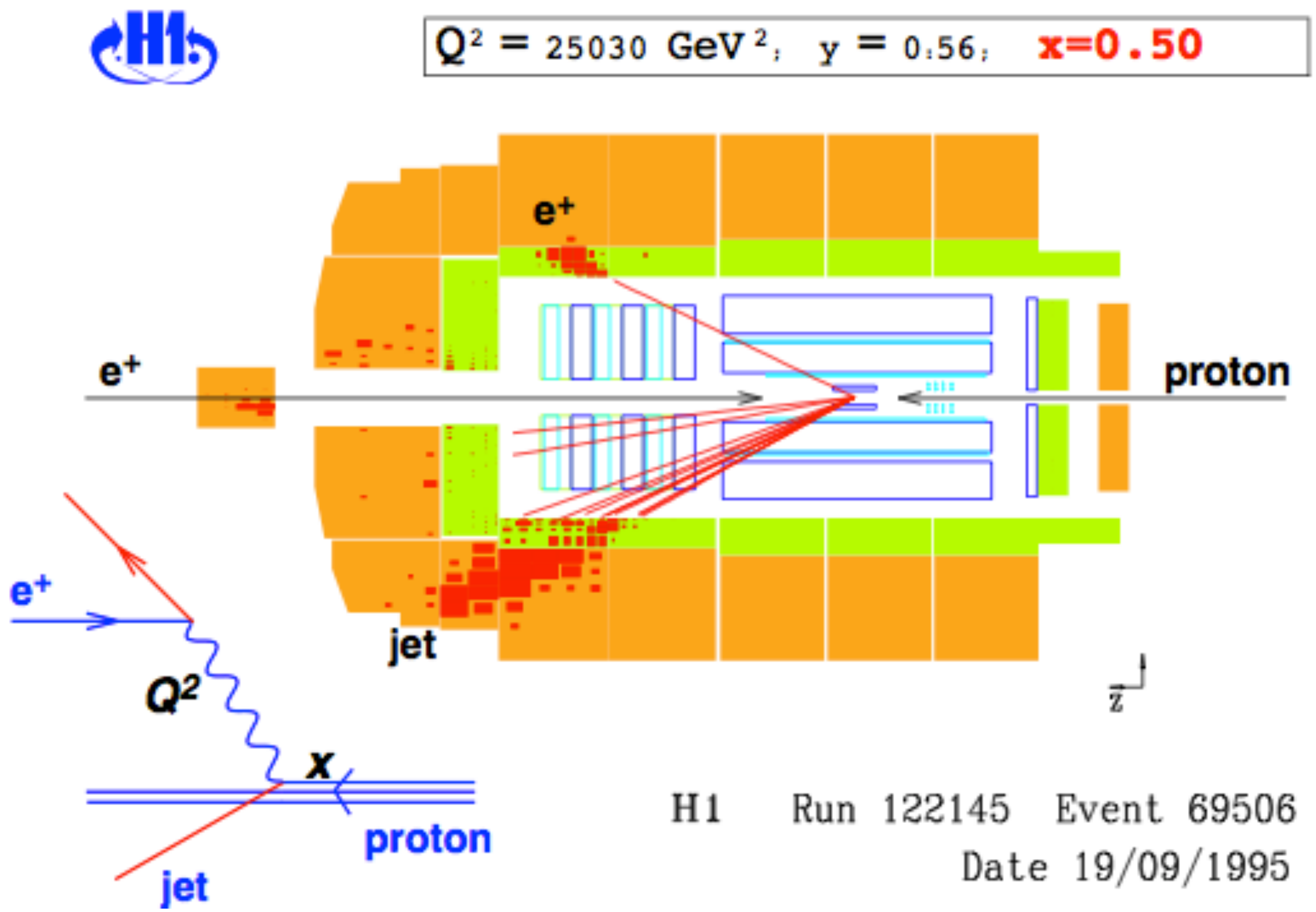
$$\int_0^1 dx \left(f_s^{(p)}(x) - f_{\bar{s}}^{(p)}(x) \right) = 0$$

In the proton: u, d valence quarks, all other quarks are called sea-quarks

How can parton densities be extracted from data?

Deep inelastic scattering

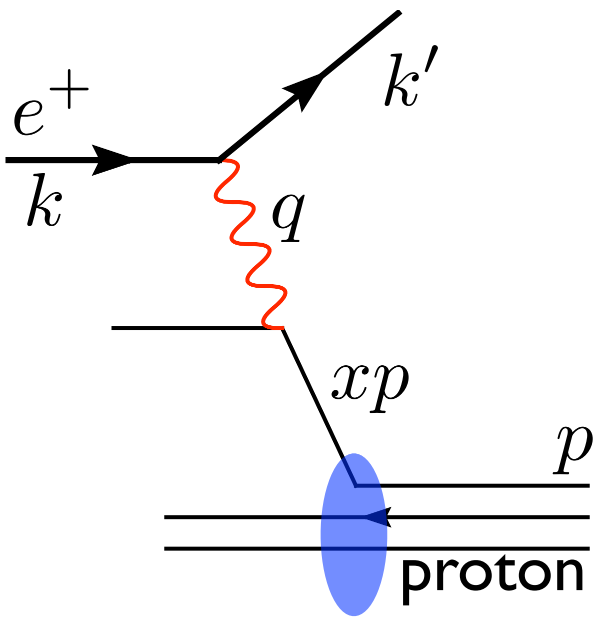
Easier than processes with two incoming hadrons is the scattering of a lepton on a (anti)-proton



Deep inelastic scattering

Protons made up of point-like quarks.
Different momentum scales involved:

- hard photon virtuality (sets the resolution scale) Q
- hard photon-quark interaction Q
- soft interaction between partons in the proton $m_p \ll Q$



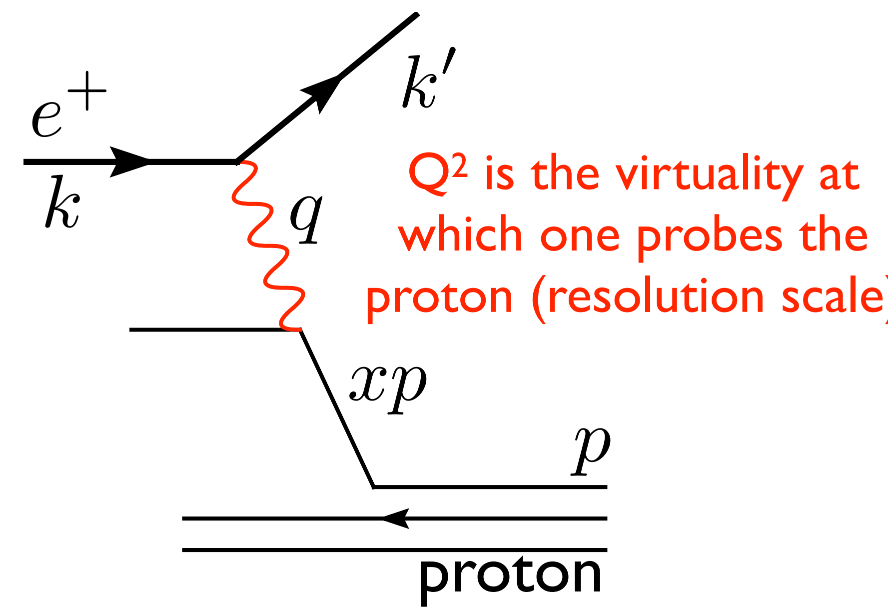
During the hard interaction, partons do not have time to interact among them, they behave as if they were free

⇒ approximate as incoherent scattering on single partons

Deep inelastic scattering

Kinematics:

$$Q^2 = -q^2 \quad s = (k+p)^2 \quad x_{Bj} = \frac{Q^2}{2p\cdot q} \quad y = \frac{p\cdot q}{k \cdot p}$$



Partonic variables:

$$\hat{p} = xp \quad \hat{s} = (k+\hat{p})^2 = 2k\cdot\hat{p} \quad \hat{y} = \frac{\hat{p}\cdot q}{k\cdot \hat{p}} = y \quad (\hat{p}+q)^2 = 2\hat{p}\cdot q - Q^2 = 0$$

$$\Rightarrow x = x_{Bj}$$

Hence at leading order, the experimentally accessible x_{Bj} coincides with the momentum fraction carried by the quark in the proton

Partonic cross section:

(apply QED Feynman rules and add phase space)

$$\boxed{\frac{d\hat{\sigma}}{d\hat{y}} = q_l^2 \frac{\hat{s}}{Q^4} 2\,\pi\,\alpha_{em}\left(1+(1-\hat{y})^2\right)}$$

Exercise: show that in the CM frame of the electron-quark system y is given by $(1-\cos\theta_{\rm el})/2$, with $\theta_{\rm el}$ the scattering angle of the electron in this frame

Exercise:

- show that the two particle phase space is $\dfrac{d\phi}{16\pi}$
- show that the squared matrix element is $\dfrac{16\pi\alpha q_l^2}{Q^4}\hat{s}xpk\left(1+(1-y)^2\right)$
- show that the flux factor is $\dfrac{1}{4xpk}$

Hence derive that

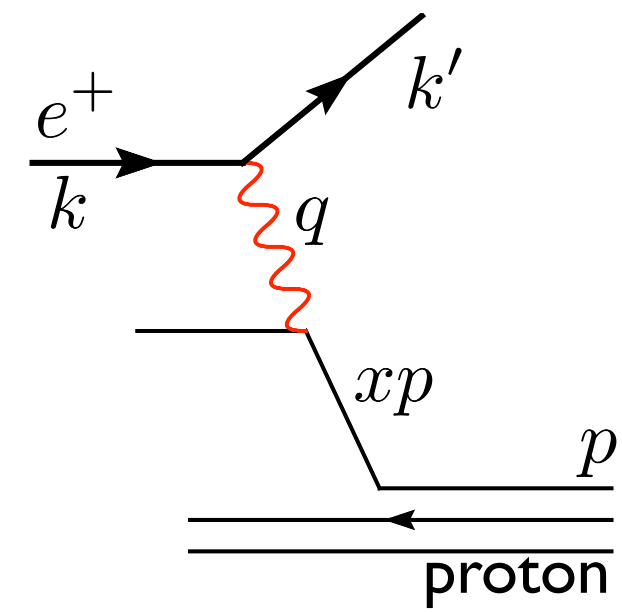
$$\frac{d\hat{\sigma}}{d\hat{y}} = q_l^2\,\frac{\hat{s}}{Q^4}\,2\,\pi\,\alpha_{em}\left(1+(1-\hat{y})^2\right)$$

Deep inelastic scattering

<u>Hadronic cross section (factorization):</u>

$$\frac{d\sigma}{dy} = \int dx \sum_l f_l^{(p)}(x) \frac{d\hat{\sigma}}{d\hat{y}}$$

Using x = x_{BJ}



$$\boxed{\begin{aligned} \frac{d\sigma}{dy\, dx_{Bj}} &= \sum_l f_l^{(p)}(x) \frac{d\hat{\sigma}}{d\hat{y}} \\ &= \frac{2\pi\, \alpha_{em}^2 s x_{Bj}}{Q^4} \left(1 + (1-y)^2\right) \sum_l q_l^2 f_l^{(p)}(x_{Bj}) \end{aligned}}$$

1. at fixed x_{Bj} and y the cross-section scales with s
2. the y-dependence of the cross-section is fully predicted and is typical of vector interaction with fermions ⇒ Callan-Gross relation
3. can access (sums of) parton distribution functions
4. Bjorken scaling: pdfs depend on x and not on Q^2 (violated by logarithmic radiative corrections, see later)

The structure function F_2

$$\frac{d\sigma}{dydx} = \frac{2\pi\alpha_{em}^2 s}{Q^4}\left(1 + (1-y^2\right) F_2(x) \qquad F_2(x) = \sum_l x q_l^2 f_l^{(p)}(x)$$

F_2 is called structure function (describes structure/constituents of nucleus)

For electron scattering on proton

$$F_2(x) = x\left(\frac{4}{9}u(x) + \frac{1}{9}d(x)\right)$$

NB: use perturbative language of quarks and gluons despite the fact that parton distribution are non-perturbative

Bjorken scaling: the fact the structure functions are independent of Q is a direct evidence for the existence of point-like quarks in the proton (violated by logarithmic corrections)

The structure function F_2

$$\frac{d\sigma}{dydx} = \frac{2\pi\alpha_{em}^2 s}{Q^4}\left(1+\left(1-y^2\right)F_2(x)\right) \qquad F_2(x) = \sum_l x q_l^2 f_l^{(p)}(x)$$

F_2 is called structure function (describes structure/constituents of nucleus)

For electron scattering on proton

$$F_2(x) = x\left(\frac{4}{9}u(x) + \frac{1}{9}d(x)\right)$$

NB: use perturbative language of quarks and gluons despite the fact that parton distribution are non-perturbative

Question: F_2 gives only a linear combination of u and d. How can they be extracted separately?

Isospin

Neutron is like a proton with u & d exchanged

For electron scattering on a proton

$$F_2^p(x) = x\left(\frac{4}{9}u_p(x) + \frac{1}{9}d_p(x)\right)$$

For electron scattering on a neutron

$$F_2^n(x) = x\left(\frac{1}{9}d_n(x) + \frac{4}{9}u_n(x)\right) = x\left(\frac{4}{9}d_p(x) + \frac{1}{9}u_p(x)\right)$$

F_2^n and F_2^p allow determination of u_p and d_p separately

NB: experimentally get F_2^n from deuteron: $F_2^d(x) = F_2^p(x) + F_2^n(x)$

Sea quark distributions

Inside the proton there are fluctuations, and pairs of $u\bar{u}$,$d\bar{d}$,$c\bar{c}$,$s\bar{s}$ ... can be created

An infinite number of pairs can be created as long as they have very low momentum, because of the momentum sum rules.

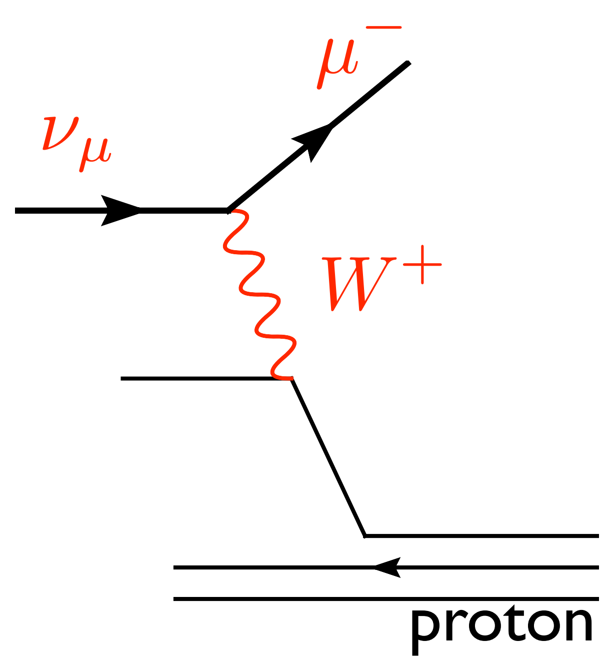
We saw before that when we say that the proton is made of uud what we mean is

$$\int_0^1 dx\,(u_p(x) - \bar{u}_p(x)) = 2 \qquad \int_0^1 dx\,\left(d_p(x) - \bar{d}_p(x)\right) = 1$$

Photons interact in the same way with u(d) and $\bar{u}$($\bar{d}$)

How can one measure the difference?

Question: What interacts differently with particle and antiparticle? W^+/W^- from neutrino scattering



Check of the momentum sum rule

$$\int_0^1 dx \sum_i x f_i^{(p)}(x) = 1$$

u_v	0,267
d_v	0,111
u_s	0,066
d_s	0,053
s_s	0,033
c_c	0,016
total	0,546

➠ *half of the longitudinal momentum carried by gluons*

$\gamma/W^{+/-}$ don't interact with gluons

How can one measure gluon parton densities?

We need to discuss radiative effects first