QCD

Giulia Zanderighi (Max Planck Institute für Physik)

5^{th} Lecture

Joint ICTP-SAIFR school on Particle Physics — July 2026

NLO n-jet cross-section

Now consider n-jet cross-section at NLO. At scale μ the result reads

$$\sigma^{\mathrm{NLO}}_{\mathrm{njets}}(\mu) = \alpha_s(\mu)^n A(p_i, \epsilon_i, \dots) + \alpha_s(\mu)^{n+1} \left(B(p_i, \epsilon_i, \dots) - n b_0 \ln \frac{\mu^2}{Q_0^2} \right) + \dots$$

- So the NLO result compensates the LO scale dependence. The residual dependence is NNLO.

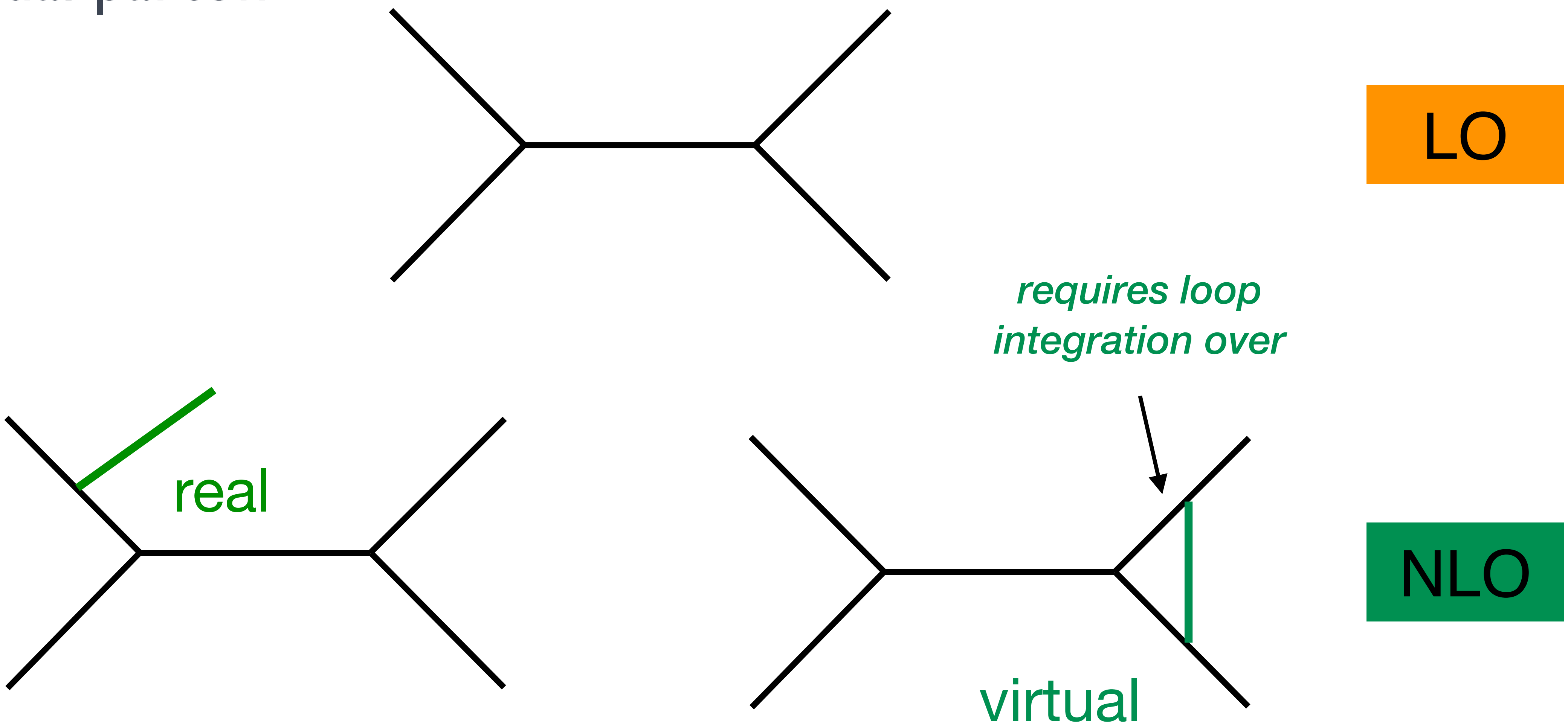
- Scale dependence and normalisation start being under control only at NLO, since compensation mechanism kicks in

- Notice also that a good scale choice automatically resums large logarithms to all orders, while a bad one spuriously introduces large logs and ruins the PT expansion

- Scale variation is conventionally used to estimate theory uncertainty, but the validity of this procedure should not be overrated (see later)

NLO calculations

NLO accuracy requires to dress a process with one real or one virtual parton



We will not have the time to do detailed NLO calculations, but let us look a bit more in detail at the issue of divergences/subtraction

Regularization

Regularisation: a way to make intermediate divergent quantities meaningful

- In QCD dimensional regularisation is today the standard procedure, based on the fact that d-dimensional integrals are more convergent if one reduces the number of dimensions.

$$\int \frac{d^4 l}{(2\pi)^4} \to \mu^{2\epsilon} \int \frac{d^d l}{(2\pi)^d}\,, \quad d = 4 - 2\epsilon < 4$$

- N.B. to preserve the correct dimensions a mass scale μ is needed
- Divergences become intermediate poles in 1/ε $\int_0^1 \frac{dx}{x} \to \int_0^1 \frac{dx}{x^{1-\epsilon}} = \frac{1}{\epsilon}$
- This procedure works both for UV divergences and IR divergences

Alternative regularization schemes: photon mass (EW), cut-offs, Pauli-Villard ...
Compared to those methods, dimensional regularizatiom has the big virtue that it leaves the regularized theory Lorentz invariant, gauge invariant, unitary etc.

Subtraction and slicing methods

- Consider e.g. an n-jet cross-section with some arbitrary infrared safe jet definition. At NLO, two divergent integrals, but the sum is finite

$$\sigma^J_{\rm NLO} = \int_{n+1} d\sigma^J_{\rm R} + \int_n d\sigma^J_{\rm V}$$

- Since one integrates over a different number of particles in the final state, real and virtual need to be evaluated first, and combined then
- This means that one needs to find a way of removing divergences before evaluating the phase space integrals

Two main ideas

- *phase space slicing* ⇒ not much used at NLO because of numerical issues, revived at NNLO
- *subtraction method* ⇒ most used at NLO, many variations at NNLO

Subtraction method

- The real cross-section can be written schematically as

$$d\sigma_R^J = d\phi_{n+1}|\mathcal{M}_{n+1}|^2 F_{n+1}^J(p_1, \ldots, p_{n+1})$$

where F^J is the arbitrary jet-definition

- The matrix element has a non-integrable divergence

$$|\mathcal{M}_{n+1}|^2 = \frac{1}{x}\mathcal{M}(x)$$

where x vanishes in the soft/collinear divergent region

- IR divergences in the loop integration regularized by taking D=4-2ε

$$2\,\mathrm{Re}\{\mathcal{M}_V \cdot \mathcal{M}_0^*\} = \frac{1}{\epsilon}\mathcal{V}$$

Subtraction method

- The n-jet cross-section becomes

$$\sigma^J_{\rm NLO} = \int_0^1 \frac{dx}{x^{1+\epsilon}} \mathcal{M}(x) F^J_{n+1}(x) + \frac{1}{\epsilon} \mathcal{V} F^J_n$$

- Infrared safety of the jet definition implies

$$\lim_{x \to 0} F^J_{n+1}(x) = F^J_n$$

- KLN cancelation guarantees that

$$\lim_{x \to 0} \mathcal{M}(x) = \mathcal{V}$$

- One can then add and subtract the analytically computed divergent part

$$\sigma^J_{\rm NLO} = \int_0^1 \frac{dx}{x^{1+\epsilon}} \mathcal{M}(x) F^J_{n+1}(x) - \int_0^1 \frac{dx}{x^{1+\epsilon}} \mathcal{V} F^J_n + \int_0^1 \frac{dx}{x^{1+\epsilon}} \mathcal{V} F^J_n + \frac{1}{\epsilon} \mathcal{V} F^J_n$$

Subtraction method

- This can be rewritten exactly as

$$\boxed{\sigma^J_{\mathrm{NLO}} = \int_0^1 \frac{dx}{x^{1+\epsilon}} \left(\mathcal{M}(x) F^J_{n+1} - \mathcal{V} F^J_n\right) + \mathcal{O}(1) \mathcal{V} F^J_n}$$

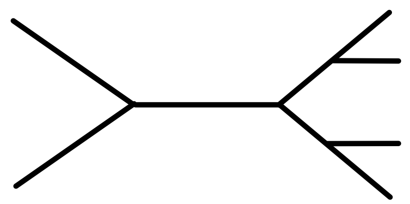
$\Rightarrow$ Now both terms are finite and can be evaluated numerically

- Subtracted cross-section must be calculated separately for each process, but automated now. It must be valid everywhere in phase space
- Systematised in the seminal papers of Catani-Seymour (dipole subtraction, '96) and Frixione-Kunszt-Signer (FKS method, '96)
- Subtraction used in all recent NLO applications and public codes (Event2, Disent, MCFM, NLOjet++, Madgraph5@aMC@NLO, POWHEG ...)

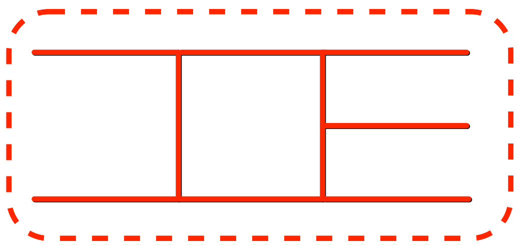
Ingredients at NLO

A full N-particle NLO calculation requires:

- [x] tree graph rates with N+1 partons
  ➔ soft/collinear divergences



- [ ] virtual correction to N-leg process
  ➔ divergence from loop integration,
  use e.g. dimensional regularization



bottleneck for a very long time

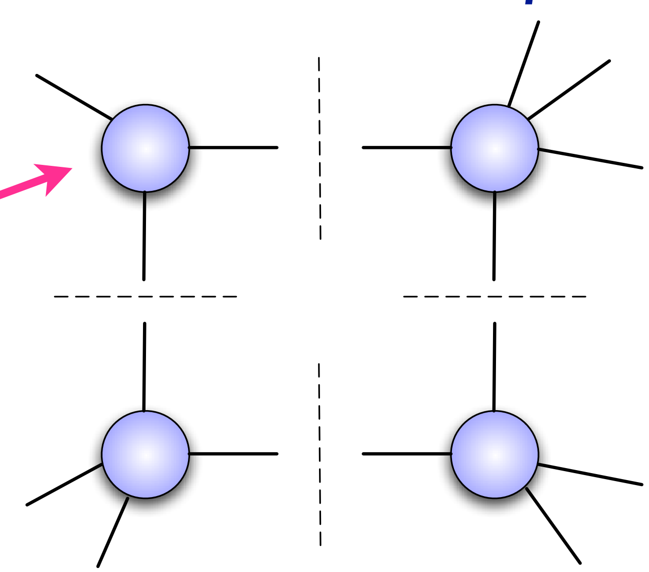
- [x] **set of subtraction terms**

Virtual one-loop: two breakthrough ideas

Aim: NLO loop integral without doing the integration

1) "... we show how to use generalized unitarity to read off the (box) coefficients. The generalized cuts we use are quadrupole cuts ..."

NB: non-zero because cut gives complex momenta



Britto, Cachazo, Feng '04

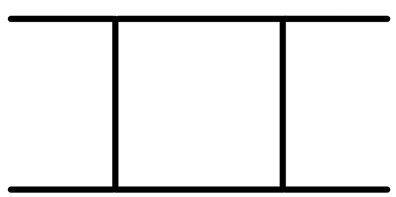
Quadrupole cuts: 4 on-shell conditions on 4 dimensional loop momentum) freezes the integration. But rational part of the amplitude, coming from D=4-2ε not 4, must be computed separately.

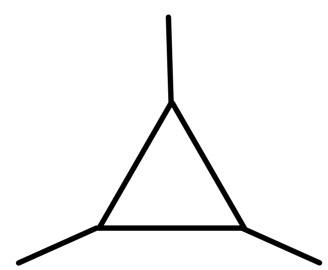
One-loop: two breakthrough ideas

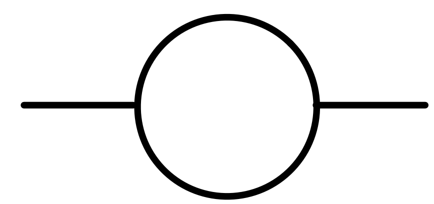
Aim: NLO loop integral without doing the integration

2) The OPP method: "We show how to extract the coefficients of 4-, 3-, 2- and 1-point one-loop scalar integrals...."

$$\mathcal{A}_N = \sum_{[i_1|i_4]} \left(d_{i_1 i_2 i_3 i_4} \; I^{(D)}_{i_1 i_2 i_3 i_4} \right) + \sum_{[i_1|i_3]} \left(c_{i_1 i_2 i_3} \; I^{(D)}_{i_1 i_2 i_3} \right) + \sum_{[i_1|i_2]} \left(b_{i_1 i_2} \; I^{(D)}_{i_1 i_2} \right)$$







Ossola, Pittau, Papadopolous '06

Coefficients can be determined by solving a purely algebraic system of equations at *integrand* level. Clever choices make the systems almost block-diagonal.

[In D-dimensions: sufficient to include also pentagons.]

NLO: one-loop amplitudes

Main **one-loop amplitude providers**:

- **BlackHat(https://blackhat.hepforge.org/)**
- **Collier (https://collier.hepforge.org)**
- **GoSam (https://gosam.hepforge.org)**
- **Golem95 (https://golem.hepforge.org/)**
- **Helac-NLO/Helac-1Loop (https://helac-phegas.web.cern.ch)**
- **Ninja (https://ninja.hepforge.org/)**
- **Njet (https://www.physik.hu-berlin.de/de/pep/tools/njet)**
- **NLOX (http://www.hep.fsu.edu/~nlox/)**
- **OpenLoops (https://openloops.hepforge.org)**
- **Recola/Recola2 (https://recola.hepforge.org)**

✓ Fast, automated generation and numerical evaluation of one-loop amplitudes

✓ Easy interface with Sherpa, Herwig, POWHEG, and others

✓ QCD only, or full SM (QCD+EW)

NLO: multi-purpose tools

1. **MadGraph5_aMC@NLO (https://launchpad.net/mg5amcnlo)**
   - Full automation of NLO QCD and EW
   - Process generation via FeynRules/UFO. Supports parton showers via aMC@NLO
2. **Sherpa+OpenLoops (https://sherpa.hepforge.org, https://openloops.hepforge.org)**
   - SHERPA handles phase space, subtraction, matching, and showering
   - OpenLoops provides fast NLO matrix elements. Efficient for multi-leg processes
3. **Herwig+Matchbox (https://herwig.hepforge.org)**
   - Herwig's Matchbox module enables automated NLO QCD corrections and matching
   - Works with external amplitude providers (OpenLoops, MadGraph, etc.)
4. **POWHEG-BOX (http://powhegbox.mib.infn.it)**
   - NLO with matching to parton showers (POWHEG method)
   - Semi-automated; requires user input for new processes
5. **MCFM (https://mcfm.fnal.gov)**
   - Parton-level code for NLO calculations (less automated)
   - Mostly SM processes. Mostly based on analytic calculations, very stable and fast

…

Automated NLO

<u>Example</u>: single Higgs production processes (similar results available for all SM processes of similar complexity, backgrounds to Higgs studies)

	Process	Syntax	Cross section (pb)			
	Single Higgs production		LO 13 TeV		NLO 13 TeV	
g.1	$pp \to H$ (HEFT)	`p p > h`	$1.593 \pm 0.003 \cdot 10^{1}$	$^{+34.8\%}_{-26.0\%}$ $^{+1.2\%}_{-1.7\%}$	$3.261 \pm 0.010 \cdot 10^{1}$	$^{+20.2\%}_{-17.9\%}$ $^{+1.1\%}_{-1.6\%}$
g.2	$pp \to Hj$ (HEFT)	`p p > h j`	$8.367 \pm 0.003 \cdot 10^{0}$	$^{+39.4\%}_{-26.4\%}$ $^{+1.2\%}_{-1.4\%}$	$1.422 \pm 0.006 \cdot 10^{1}$	$^{+18.5\%}_{-16.6\%}$ $^{+1.1\%}_{-1.4\%}$
g.3	$pp \to Hjj$ (HEFT)	`p p > h j j`	$3.020 \pm 0.002 \cdot 10^{0}$	$^{+59.1\%}_{-34.7\%}$ $^{+1.4\%}_{-1.7\%}$	$5.124 \pm 0.020 \cdot 10^{0}$	$^{+20.7\%}_{-21.0\%}$ $^{+1.3\%}_{-1.5\%}$
g.4	$pp \to Hjj$ (VBF)	`p p > h j j $$ w+ w- z`	$1.987 \pm 0.002 \cdot 10^{0}$	$^{+1.7\%}_{-2.0\%}$ $^{+1.9\%}_{-1.4\%}$	$1.900 \pm 0.006 \cdot 10^{0}$	$^{+0.8\%}_{-0.9\%}$ $^{+2.0\%}_{-1.5\%}$
g.5	$pp \to Hjjj$ (VBF)	`p p > h j j j $$ w+ w- z`	$2.824 \pm 0.005 \cdot 10^{-1}$	$^{+15.7\%}_{-12.7\%}$ $^{+1.5\%}_{-1.0\%}$	$3.085 \pm 0.010 \cdot 10^{-1}$	$^{+2.0\%}_{-3.0\%}$ $^{+1.5\%}_{-1.1\%}$
g.6	$pp \to HW^{\pm}$	`p p > h wpm`	$1.195 \pm 0.002 \cdot 10^{0}$	$^{+3.5\%}_{[illegible]}$ $^{+1.9\%}_{[illegible]}$	$1.419 \pm 0.005 \cdot 10^{0}$	$^{+2.1\%}_{[illegible]}$ $^{+1.9\%}_{-1.4\%}$
g.7	$pp \to$ [illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible] $^{+1.2\%}_{-1.0\%}$
g.8*	$pp \to$ [illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible] $^{+0.9\%}_{-0.6\%}$
g.9	$pp \to$ [illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible] $^{+1.9\%}_{-1.4\%}$
g.10	$pp \to$ [illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible] $^{+1.1\%}_{-0.9\%}$
g.11*	$pp \to$ [illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible] $^{+0.7\%}_{-0.6\%}$
g.12*	$pp \to HW^+W^-$ (4f)	`p p > h w+ w-`	$8.325 \pm 0.139 \cdot 10^{-3}$	$^{[illegible]}_{-0.3\%}$ $^{[illegible]}_{-1.6\%}$	$1.065 \pm 0.003 \cdot 10^{-2}$	$^{[illegible]}_{-1.9\%}$ $^{+2.0\%}_{-1.5\%}$
g.13*	$pp \to HW^{\pm}\gamma$	`p p > h wpm a`	$2.518 \pm 0.006 \cdot 10^{-3}$	$^{+0.7\%}_{-1.4\%}$ $^{+1.9\%}_{-1.5\%}$	$3.309 \pm 0.011 \cdot 10^{-3}$	$^{+2.7\%}_{-2.0\%}$ $^{+1.7\%}_{-1.4\%}$
g.14*	$pp \to HZW^{\pm}$	`p p > h z wpm`	$3.763 \pm 0.007 \cdot 10^{-3}$	$^{+1.1\%}_{-1.5\%}$ $^{+2.0\%}_{-1.6\%}$	$5.292 \pm 0.015 \cdot 10^{-3}$	$^{+3.9\%}_{-3.1\%}$ $^{+1.8\%}_{-1.4\%}$
g.15*	$pp \to HZZ$	`p p > h z z`	$2.093 \pm 0.003 \cdot 10^{-3}$	$^{+0.1\%}_{-0.6\%}$ $^{+1.9\%}_{-1.5\%}$	$2.538 \pm 0.007 \cdot 10^{-3}$	$^{+1.9\%}_{-1.4\%}$ $^{+2.0\%}_{-1.5\%}$
g.16	$pp \to Ht\bar{t}$	`p p > h t t~`	$3.579 \pm 0.003 \cdot 10^{-1}$	$^{+30.0\%}_{-21.5\%}$ $^{+1.7\%}_{-2.0\%}$	$4.608 \pm 0.016 \cdot 10^{-1}$	$^{+5.7\%}_{-9.0\%}$ $^{+2.0\%}_{-2.3\%}$
g.17	$pp \to Htj$	`p p > h tt j`	$4.994 \pm 0.005 \cdot 10^{-2}$	$^{+2.4\%}_{-4.2\%}$ $^{+1.2\%}_{-1.3\%}$	$6.328 \pm 0.022 \cdot 10^{-2}$	$^{+2.9\%}_{-1.8\%}$ $^{+1.5\%}_{-1.6\%}$
g.18	$pp \to Hb\bar{b}$ (4f)	`p p > h b b~`	$4.983 \pm 0.002 \cdot 10^{-1}$	$^{+28.1\%}_{-21.0\%}$ $^{+1.5\%}_{-1.8\%}$	$6.085 \pm 0.026 \cdot 10^{-1}$	$^{+7.3\%}_{-9.6\%}$ $^{+1.6\%}_{-2.0\%}$
g.19	$pp \to Ht\bar{t}j$	`p p > h t t~ j`	$2.674 \pm 0.041 \cdot 10^{-1}$	$^{+45.6\%}_{-29.2\%}$ $^{+2.6\%}_{-2.9\%}$	$3.244 \pm 0.025 \cdot 10^{-1}$	$^{+3.5\%}_{-8.7\%}$ $^{+2.5\%}_{-2.9\%}$
g.20*	$pp \to Hb\bar{b}j$ (4f)	`p p > h b b~ j`	$7.367 \pm 0.002 \cdot 10^{-2}$	$^{+45.6\%}_{-29.1\%}$ $^{+1.8\%}_{-2.1\%}$	$9.034 \pm 0.032 \cdot 10^{-2}$	$^{+7.9\%}_{-11.0\%}$ $^{+1.8\%}_{-2.2\%}$

✓ A solved problem

NLO: the present

Today focus on

- automation of NLO for BSM signals
- loop-induced processes: higher-order, but enhanced by gluon PDF
- automation of NLO electroweak corrections (necessary to match accuracy of NNLO)
- automation of NLO in SMEFT

➡ Practical limitation: high-multiplicity difficult because of numerical instabilities, long run-time on clusters to obtain stable results (edge: about 6 particles in the final state, depending on the process)

Comparison to at least NLO is the standard now in most LHC analyses

NLO revolution?

What lead to this remarkable progress?

The fact that

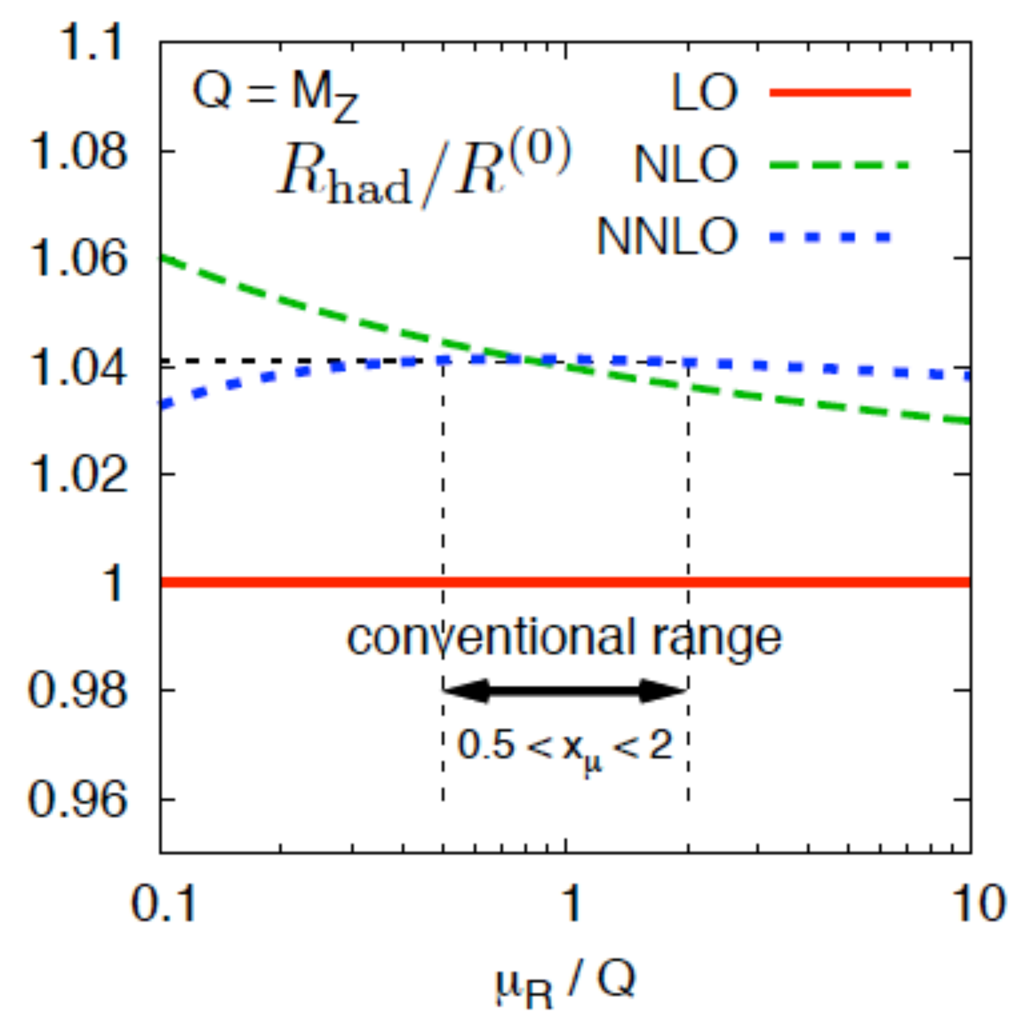
1. leading order can be computed automatically and efficiently (e.g. via recursion relations)
2. one can reduce the one-loop to product of tree-level amplitudes
3. it was well understood how to subtract singularities
4. the basis of master integrals was known

But for item 2. everything was there since the time of Passarino-Veltman (even item 2. was understood, but no efficient/practical method exited).
We will later on compare this to the current status of NNLO

Uncertainties

The seemingly unpleasant feature that cross sections depend on the renormalization and factorization scales can be turned into a useful tool: it provides an estimate of the theoretical uncertainty associated with missing higher-order corrections.

<u>Example</u>: Let us consider again the R-ratio.
Fix the renormalization scale to the scale at which the hard process occurs (Q) and vary it up and down by a factor 2

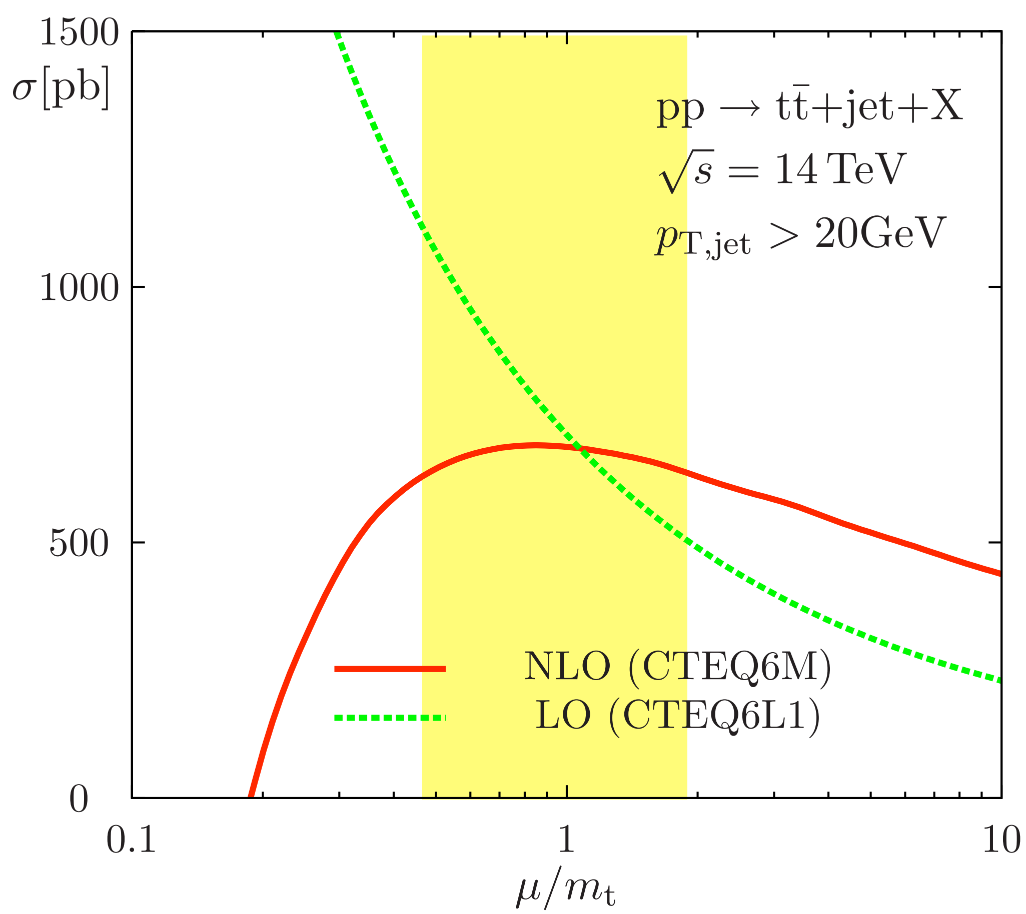


<u>NB</u>:

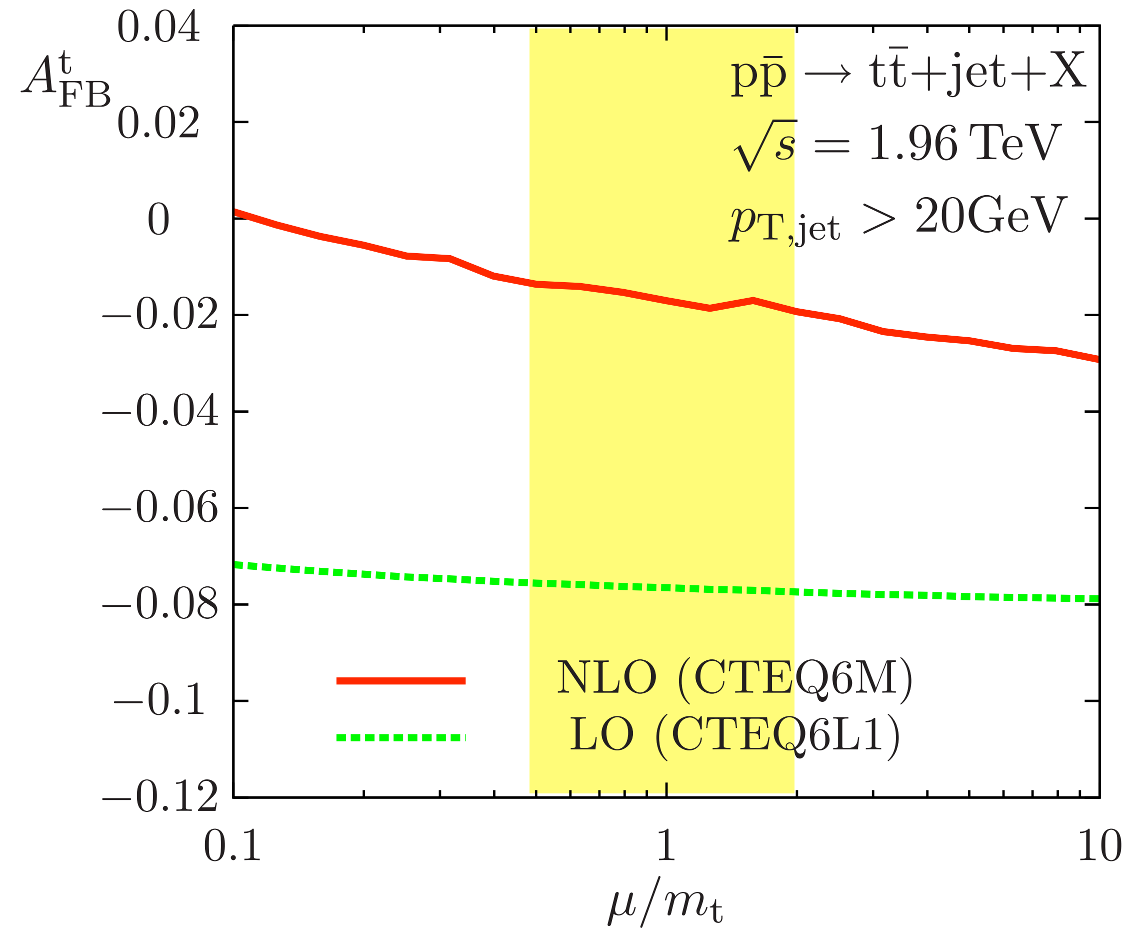
- the factor 2 is conventional
- it is a procedure that seems to work well in practice, but with limitations
- in complicated processes large degree of freedom in the choice of the central scale

1. LHC example of NLO: tt+1jet

Cross-section



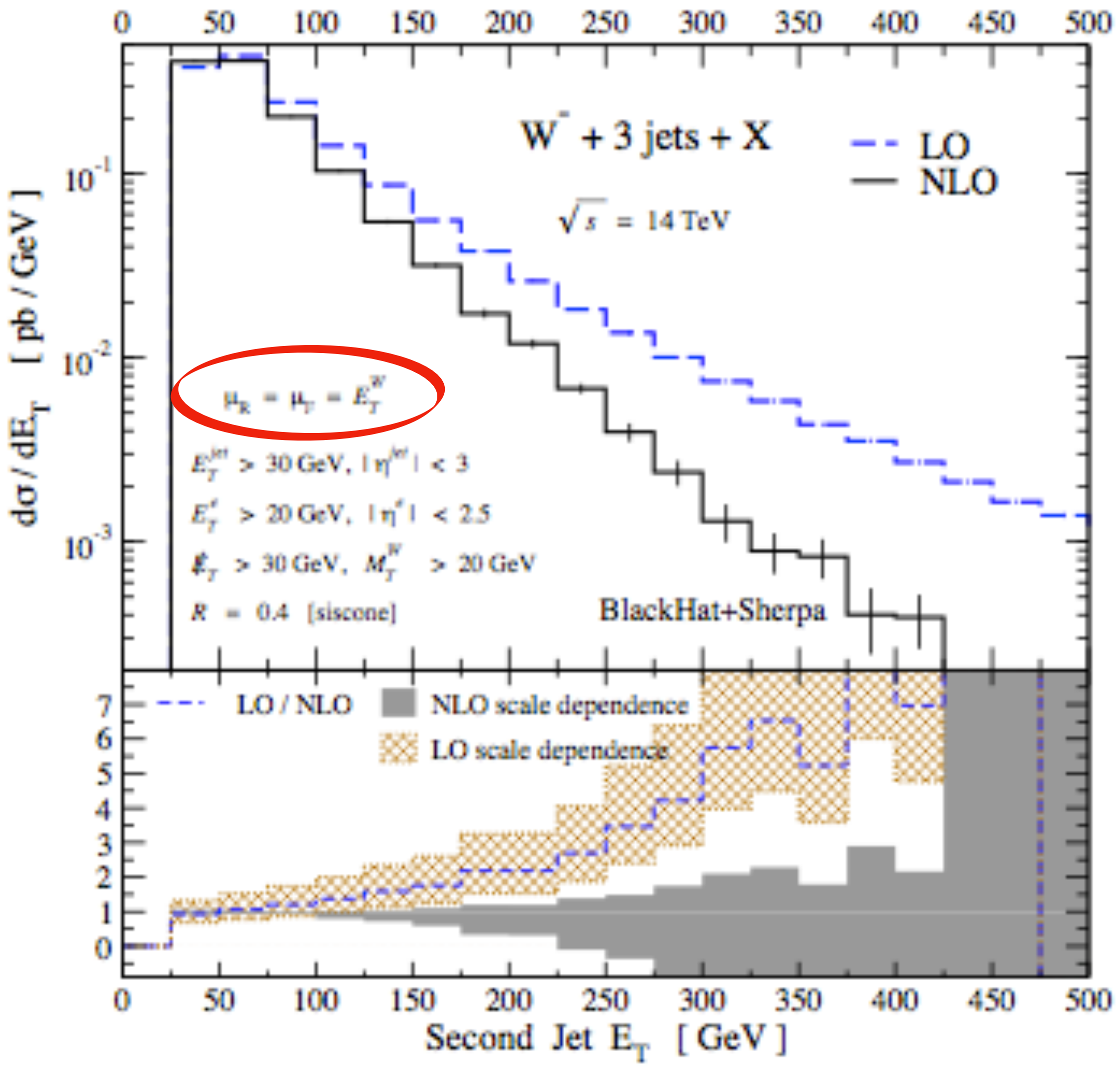
Asymmetry

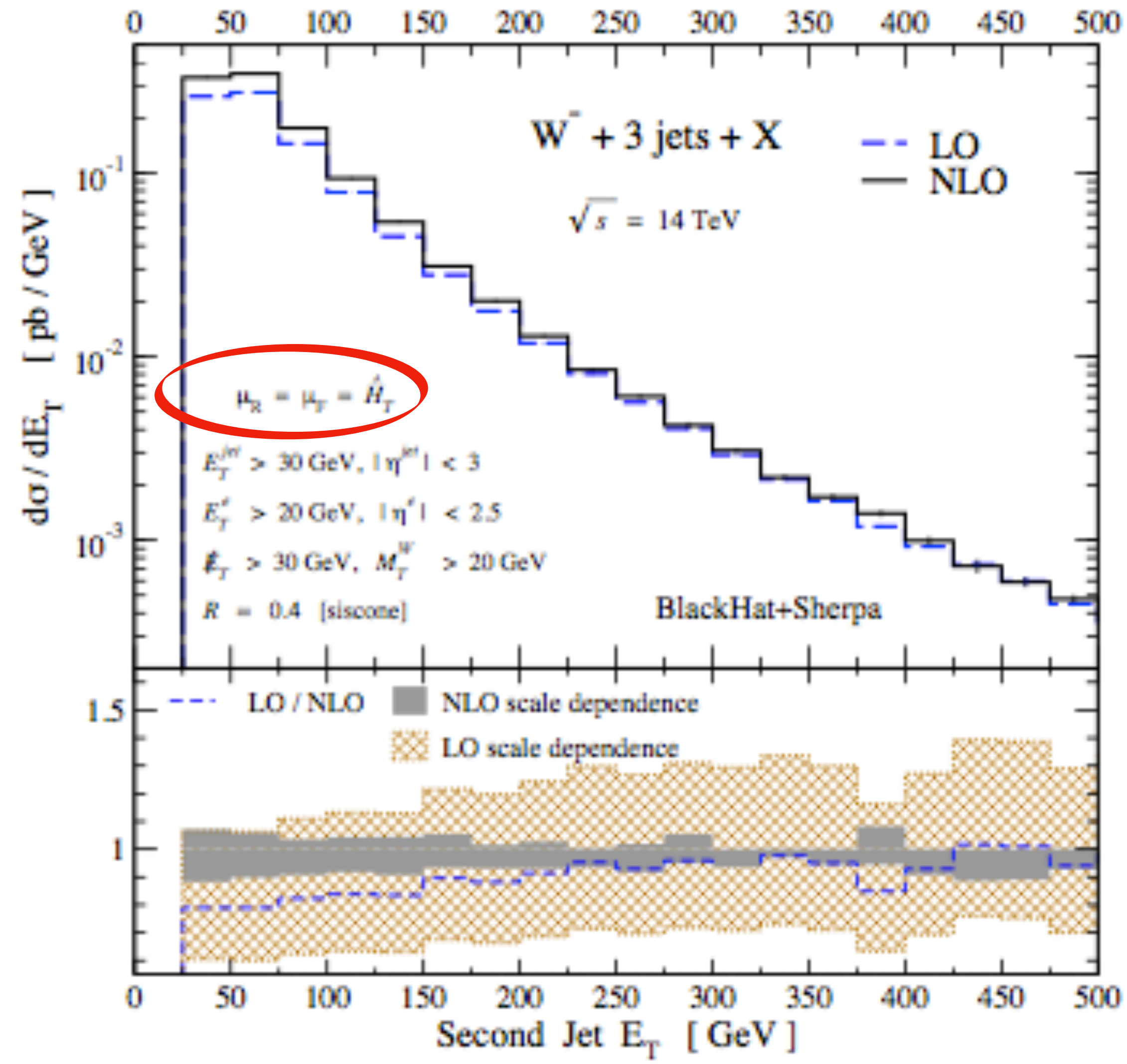


- Scale variation is not a perfect procedure to assess the theory uncertainty
- Ambiguities in the central scale choice (more so for more complicated processes)
- Scale variation for ratios (asymmetries etc.) underestimates the uncertainty

2. LHC example of NLO: W+3jets

Scale choice: example of W+3 jets

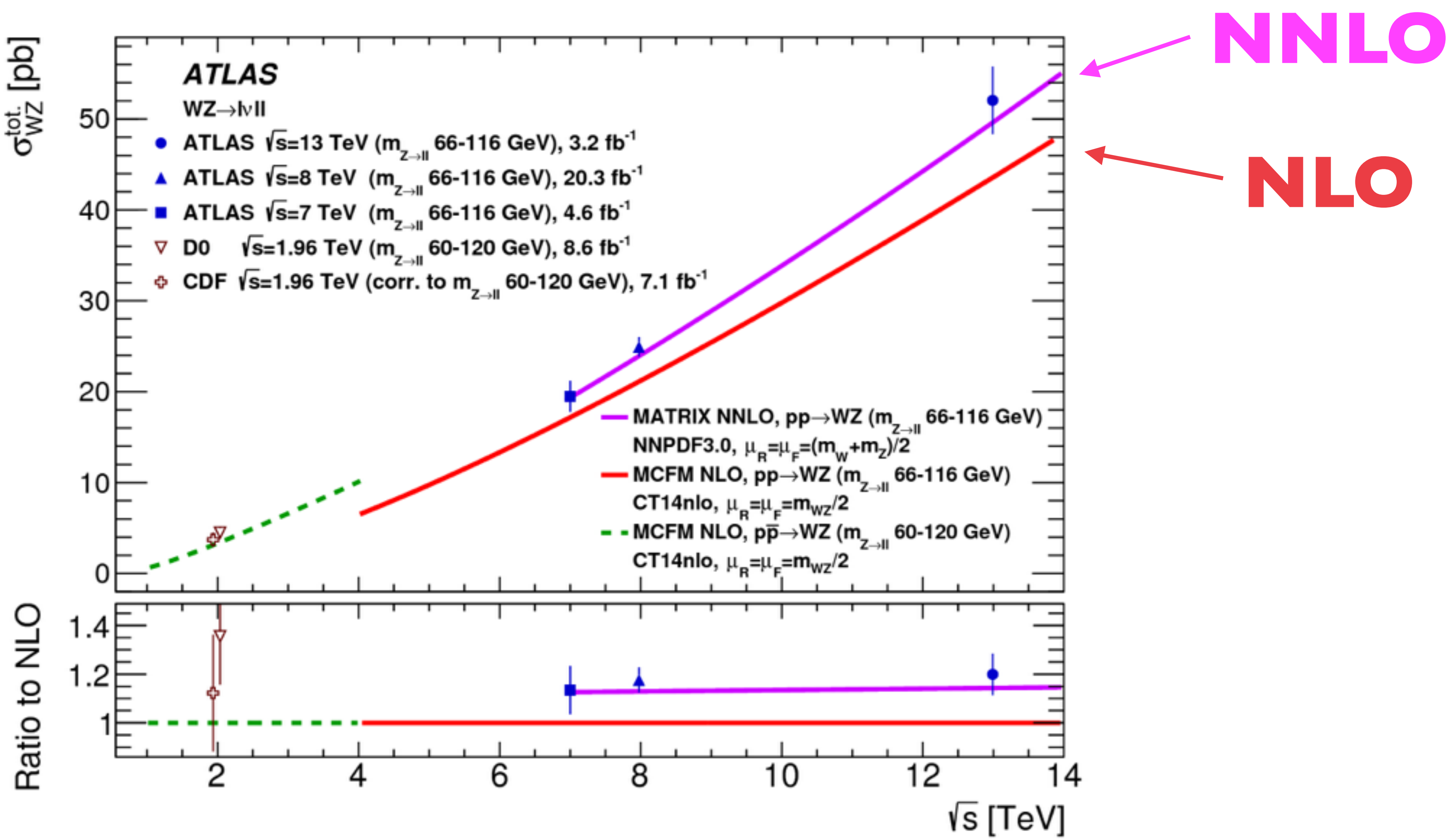




... large logarithms can appear in some distributions, invalidating even an NLO prediction.

Bern et al. '09

Is NNLO needed?



LHC data clearly already requires NNLO

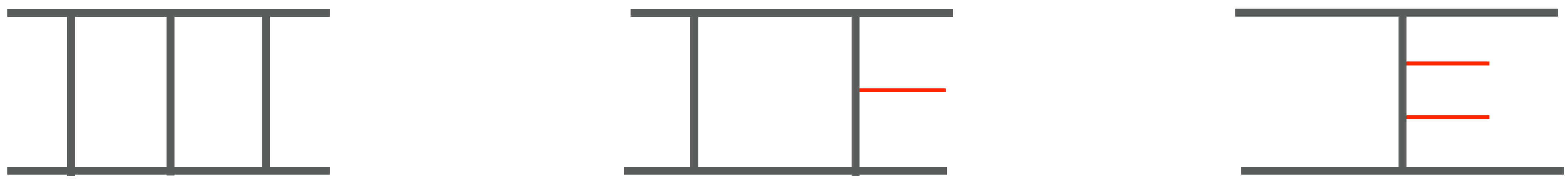
Same conclusion in all measurements examined so far

With more data NLO likely to be insufficient

Why is NNLO difficult

calculation of two-loop master integrals (when many scales are involved)

methods to cancel (overlapping) divergences before integration



$$\int d\Phi_n 2\mathrm{Re}|\mathcal{M}^{2-loop}\mathcal{M}^{tree}| \qquad \int d\Phi_n d\Phi_1 2Re|\mathcal{M}^{\mathrm{one-loop}}_{n+1}\mathcal{M}^{\mathrm{tree}}_{n+1}| \qquad \int d\Phi_n d\Phi_2 |\mathcal{M}^{\mathrm{tree}}|^2_{n+2}$$

$$\int d\Phi_n \left\{ \left(a_4 \frac{1}{\epsilon^4} + a_3 \frac{1}{\epsilon^3} + \ldots + a_0 \right) - \left(a_4 \frac{1}{\epsilon^4} + a_3 \frac{1}{\epsilon^3} + \ldots + b_0) \right) \right\}$$

Cancelation manifest after phase space integration, but to have fully differential results must achieve cancelation before integration

Ingredients for NNLO

At NNLO the situation is very different from NLO

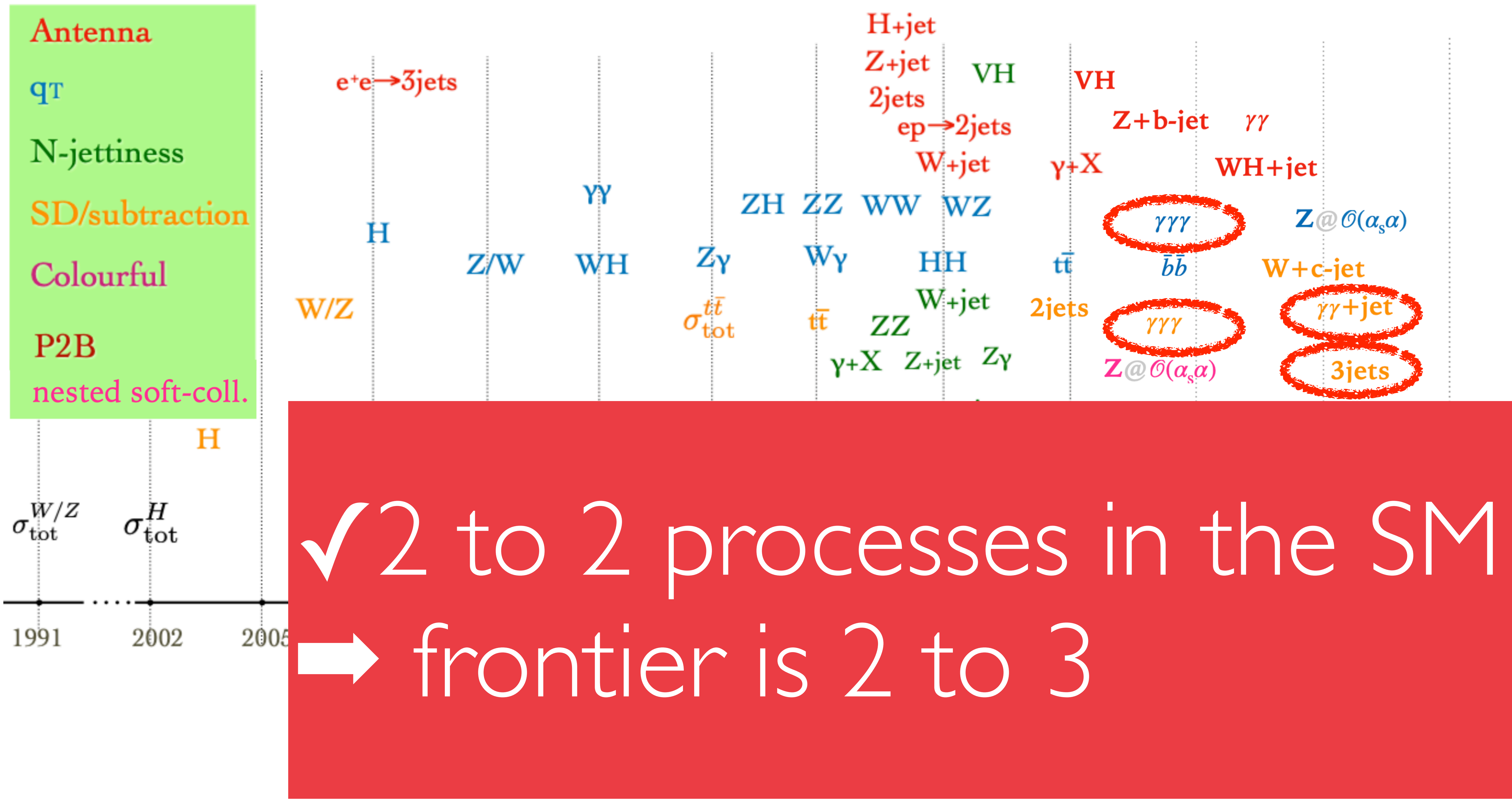
1. leading order of very limited importance
2. no procedure to reduce two-loop to tree-level (unitarity approaches still face many outstanding issues)
3. subtraction of singularities far from trivial
4. basis set of master integrals not known, integrals not all/always known analytically

And all this even for simple processes (no full result exists for any $2 \to 4$ scattering process)

What changed in the last years (and is undergoing more changes)

1. technology to compute integrals
2. extension of systematic subtraction to NNLO

NNLO: status



✓2 to 2 processes in the SM

➡ frontier is 2 to 3

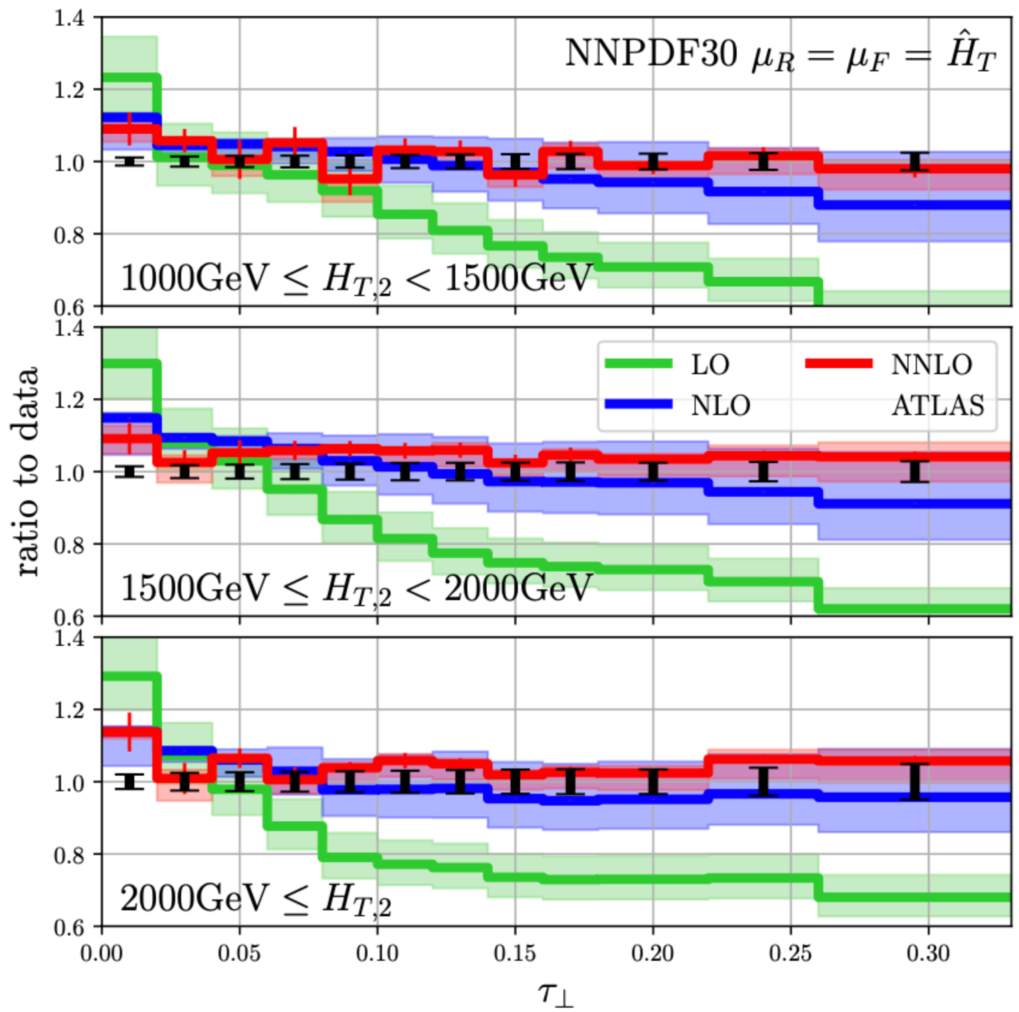
Different colour: different way to handle intermediate divergences

NNLO: one highlight

Recent milestone: NNLO calculation of three jet production at the LHC

Recently applied to compute event shapes at the LHC, which might be used to fit the strong coupling

Alvarez et al *JHEP* 03 (2023) 129



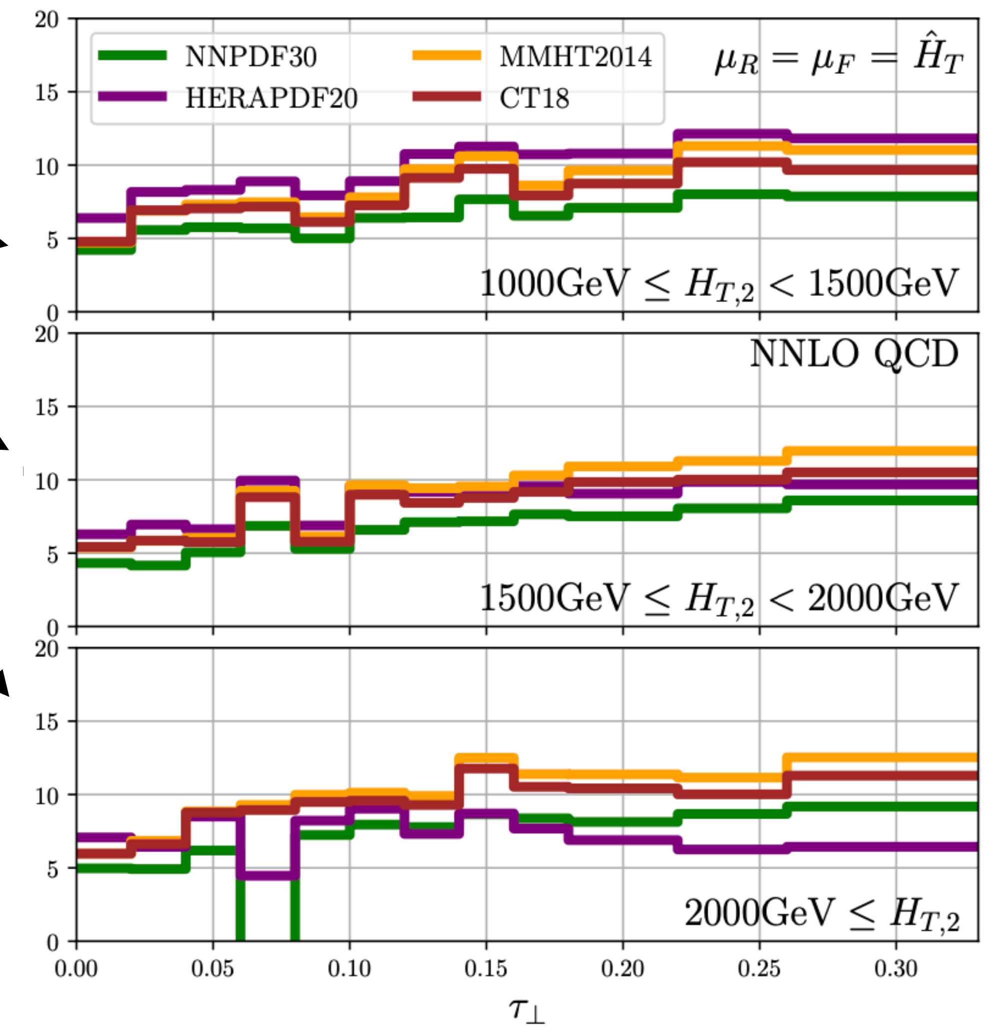
$$T_\perp = \frac{\sum_i |\vec{p}_{T,i} \cdot \hat{n}_\perp|}{\sum_i |\vec{p}_{T,i}|}$$

Shape dependent NNLO effects.
NNLO improves description of data

Application of 3jet@NNLO

$$R^{\mathrm{NNLO}}(\alpha_s) = \frac{d\sigma_3(\alpha_s(\mu),\mu)}{d\sigma_2(\alpha_s(\mu),\mu)} = R^{\mathrm{NNLO}}(\alpha_{s,0}) + R^{\mathrm{NNLO}'}(\alpha_{s,0})\,(\alpha_s - \alpha_{s,0}) + \ldots$$

$$\boxed{c = \frac{R^{\mathrm{NNLO}'}(\alpha_{s,0})}{R^{\mathrm{NNLO}}(\alpha_{s,0})}}$$



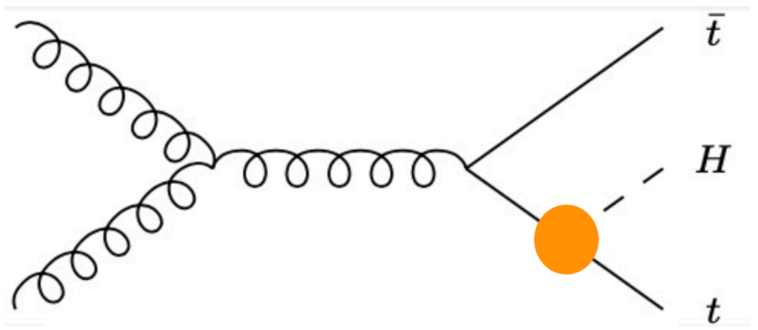
Alvarez et al *JHEP* 03 (2023) 129

Since $\alpha_s \approx 0.1$, c ≈ 10 implies that 1% shift in α_s leads to ≈ 1% shift in predictions

➠ good prospects for the precision determination of α_s and its running through TeV scales from LHC data

ttH at approximate NNLO

Two-loop pp → ttH amplitudes still missing.



Idea: approximate with amplitudes with a soft Higgs emitted off heavy quarks

	$\sqrt{s} = 13$ TeV		$\sqrt{s} = 100$ TeV	
σ [fb]	gg	$q\bar{q}$	gg	$q\bar{q}$
σ_{LO}	261.58	129.47	23055	2323.7
$\Delta\sigma_{\mathrm{NLO,H}}$	88.62	7.826	8205	217.0
$\Delta\sigma_{\mathrm{NLO,H}}\|_{\mathrm{soft}}$	61.98	7.413	5612	206.0
$\Delta\sigma_{\mathrm{NNLO,H}}\|_{\mathrm{soft}}$	$-2.980(3)$	$2.622(0)$	$-239.4(4)$	$65.45(1)$

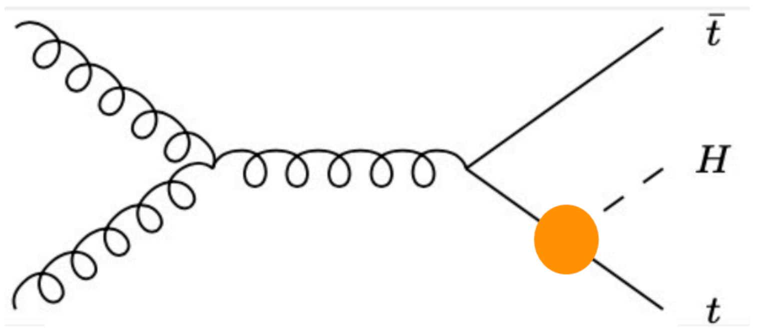
Test the procedure at NLO

- approximation not that great. It works better for the qq than the gg channel

ttH at approximate NNLO

Two-loop pp → ttH amplitudes still missing.

Idea: approximate with amplitudes with a soft Higgs emitted off heavy quarks



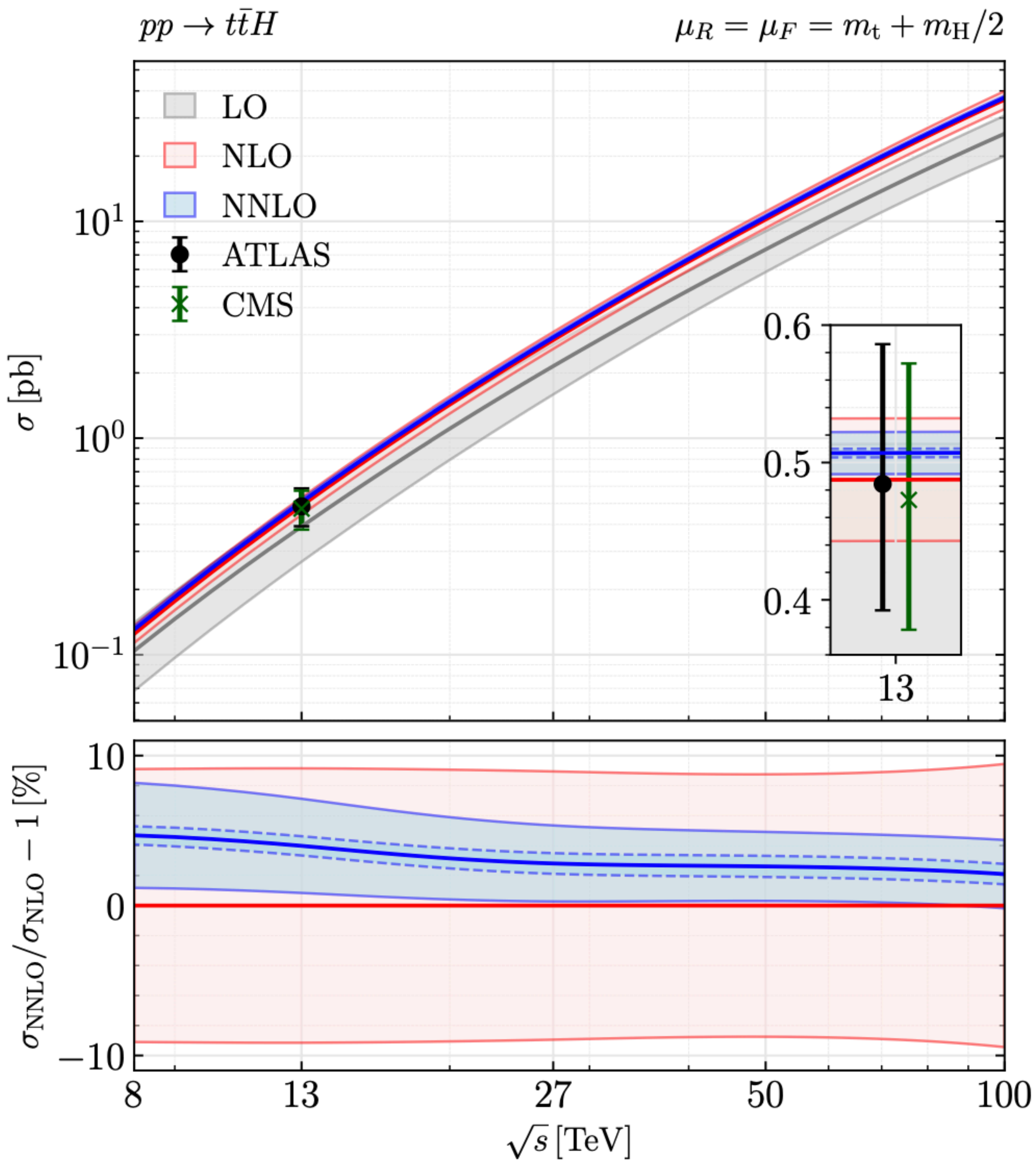
	$\sqrt{s} = 13\,\text{TeV}$		$\sqrt{s} = 100\,\text{TeV}$	
σ [fb]	gg	$q\bar{q}$	gg	$q\bar{q}$
σ_{LO}	261.58	129.47	23055	2323.7
$\Delta\sigma_{\text{NLO,H}}$	88.62	7.826	8205	217.0
$\Delta\sigma_{\text{NLO,H}}\vert_{\text{soft}}$	61.98	7.413	5612	206.0
$\Delta\sigma_{\text{NNLO,H}}\vert_{\text{soft}}$	$-2.980(3)$	$2.622(0)$	$-239.4(4)$	$65.45(1)$

Size of approx. NNLO

- approximation works better for qq then gg channel
- but two-loop corrections are very small (below a %)

27

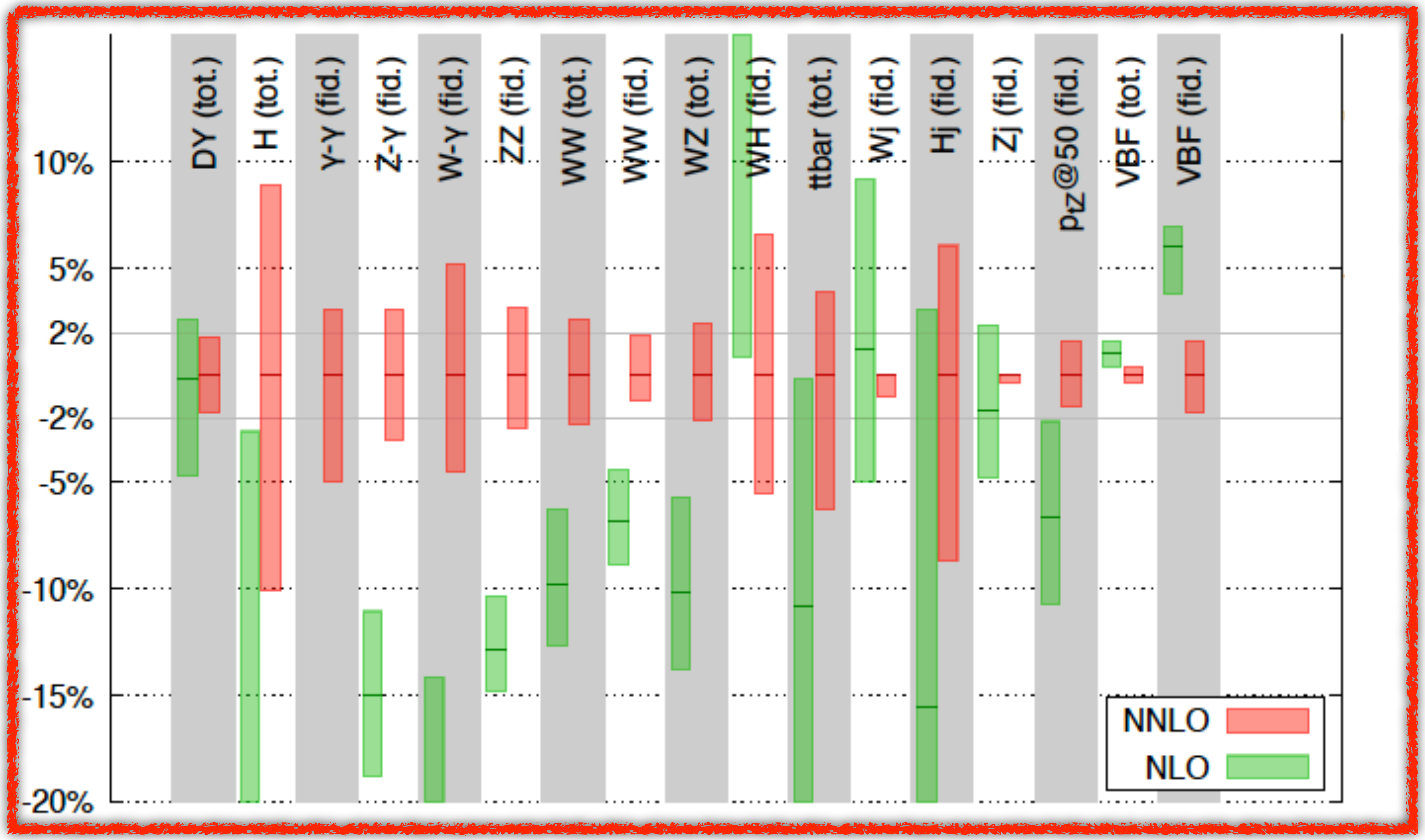
ttH at approximate NNLO



⇒ estimated uncertainty on the total cross section at the few percent level

Interesting to validate this once full NNLO is available

NNLO uncertainty?

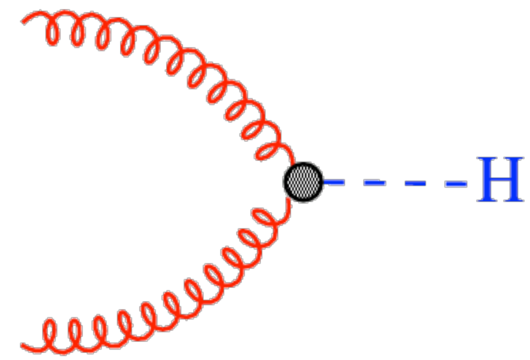


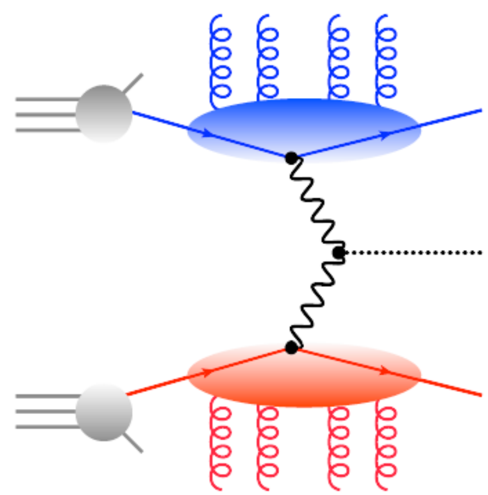
NNLO ***scale*** uncertainty bands of 1-2%.
Is the ***theory*** uncertainty indeed 1-2%?

From G. Salam

N³LO: only 3 LHC process known so far

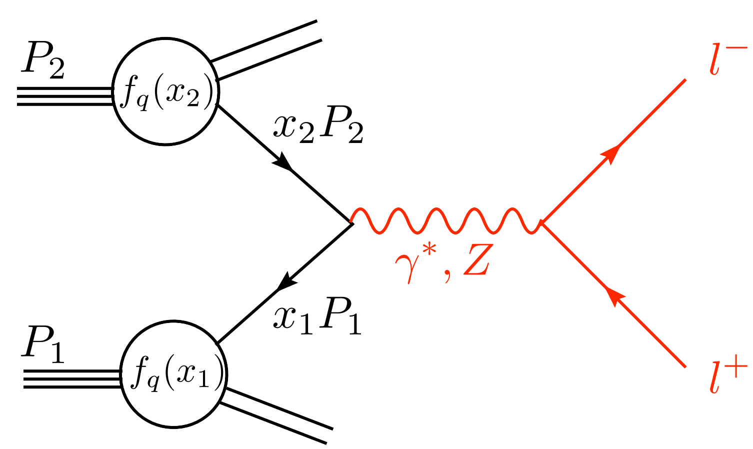
Gluon fusion Higgs production (in the large m_t effective theory)





Vector boson fusion Higgs production (in the structure function approximation, i.e. double DIS process)

Drell Yan (W and Z bosons to leptons)



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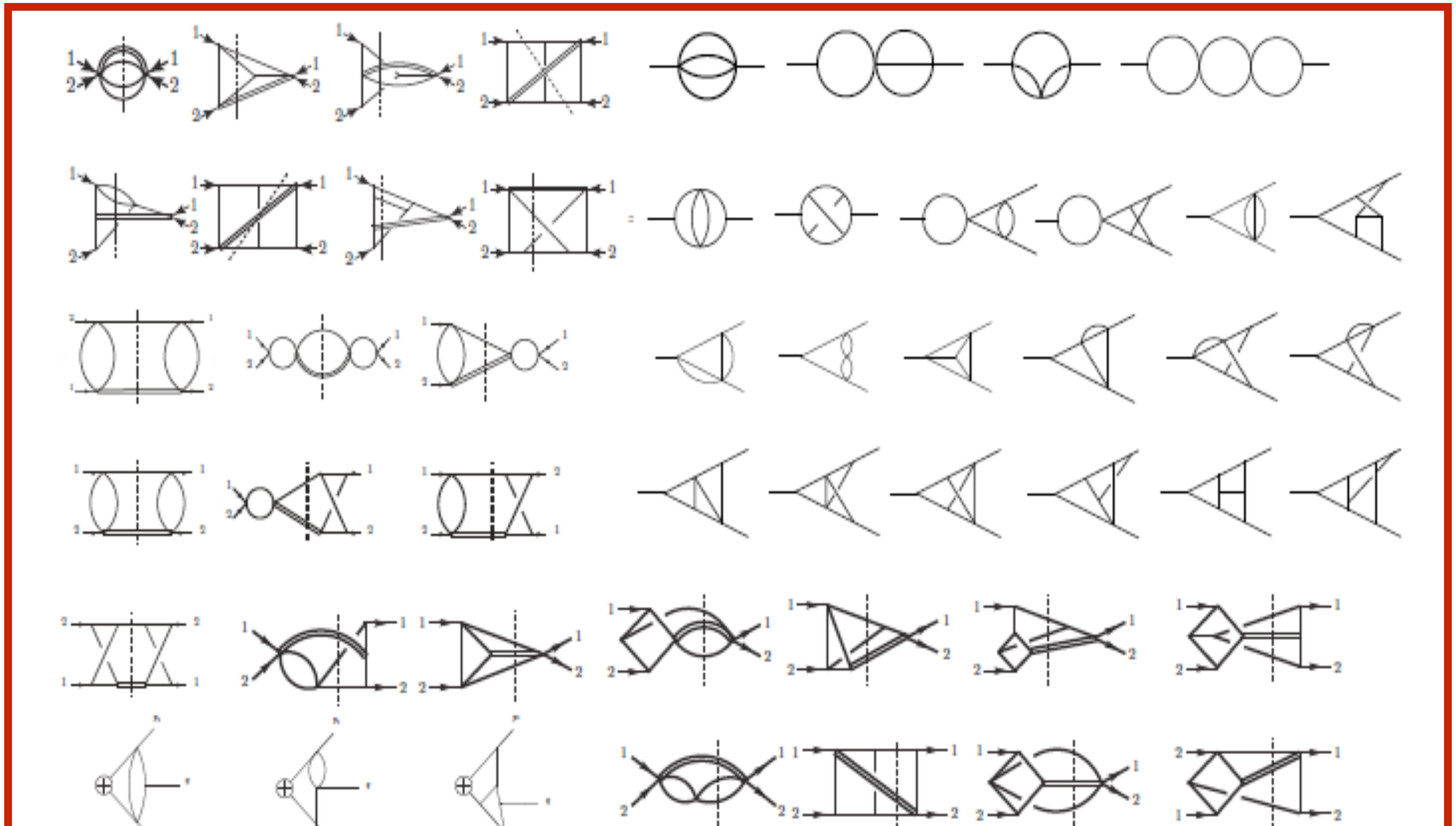
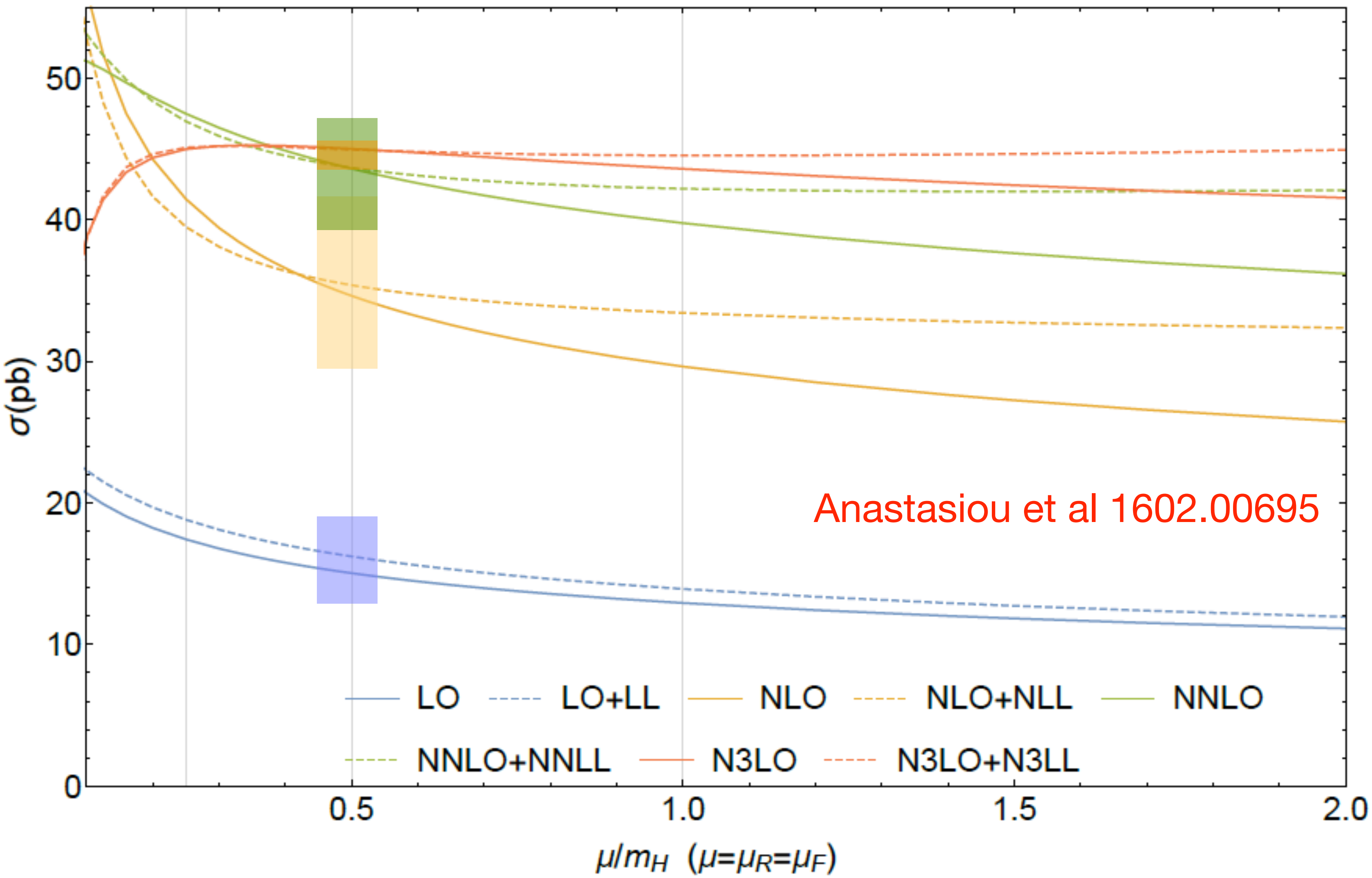
Higgs production at N^{3}LO

- O(100000) interference diagrams (1000 at NNLO)
- 68273802 loop and phase space integrals (47000 at NNLO)
- about 1000 master integrals (26 at NNLO)



Higgs production at N^3LO

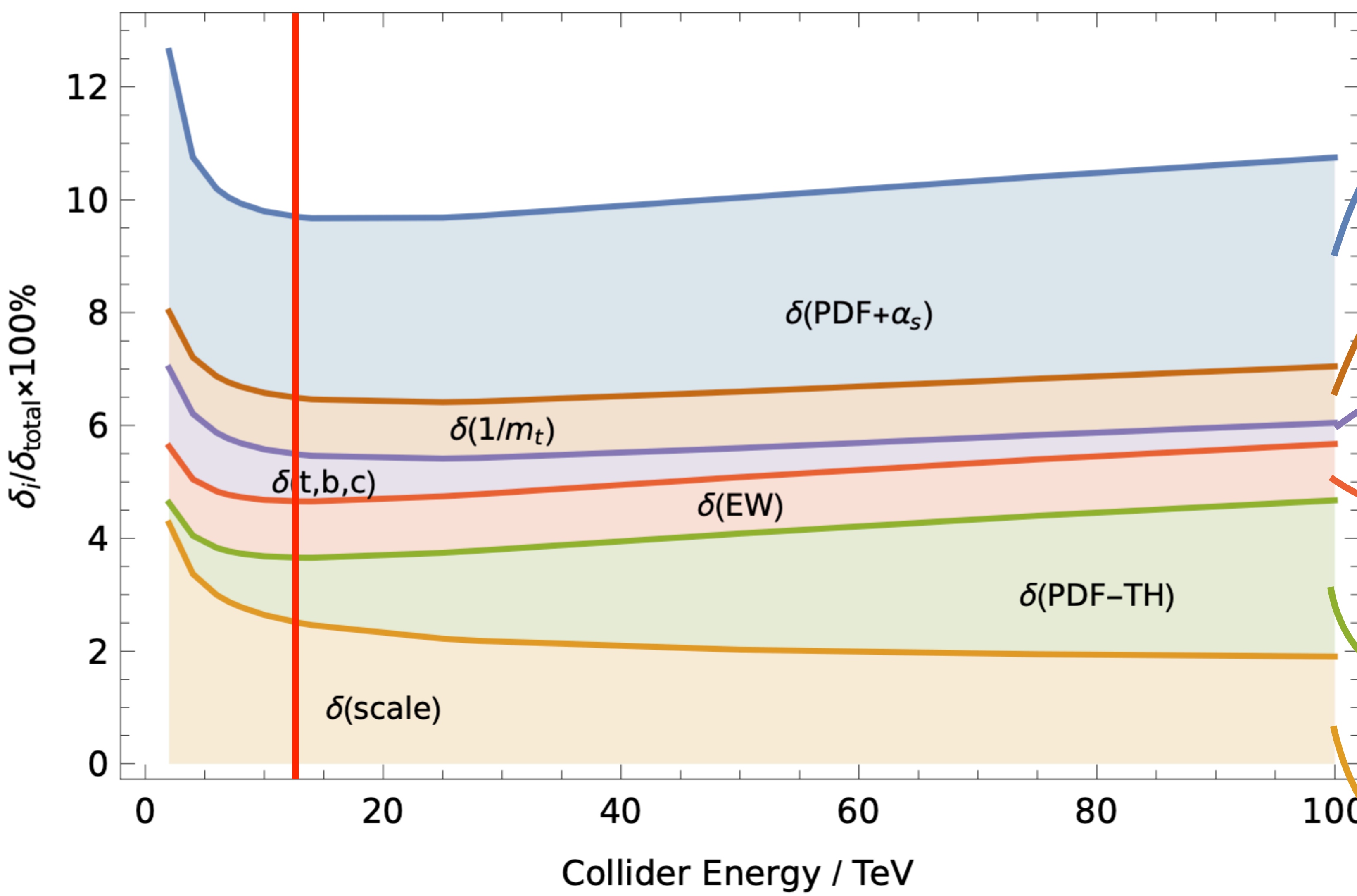


- N^3LO finally stabilizes the perturbative expansion
- also matched to resummed calculation (essentially no impact on central value at preferred scale $m_H/2$)

Error budget: one example

Gluon-fusion Higgs productions (known to N³LO fully differential)

Error budget as of 2018



More data; lattice determination α_s; progress in α_s fits

Removed by Czakon et al '21

can be removed

Reduced by factor 2 through mixed EW-QCD calculations

Missing N³LO PDFs

Missing N⁴LO

Dulat, Lazopoulos, Mistlberger '18

Summary of perturbative calculations

- LO: fully automated. Edge: up to 8-10 final-state particles
- NLO: also automated. Edge: up 4-6 final-state particles
- NNLO: the current focus. Most $2 \to 2$ processes known. First $2 \to 3$ calculations for the LHC.
- NNNLO: the frontier. Only 3 processes known: fully inclusive Higgs production via gluon fusion (large m_t effective theory), vector boson fusion (factorised approximation) and Drell Yan production

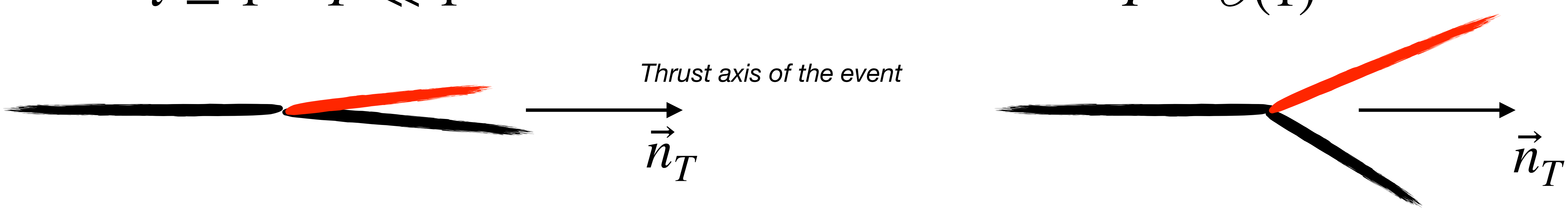
Analytic resummation

Discuss all-order resummation in the example of the thrust variable

The most popular one is the thrust in e^+e^- collisions

$$T \equiv \frac{1}{Q} \max_{n_T} \sum_i |\vec{p}_i \cdot \vec{n}_T|$$

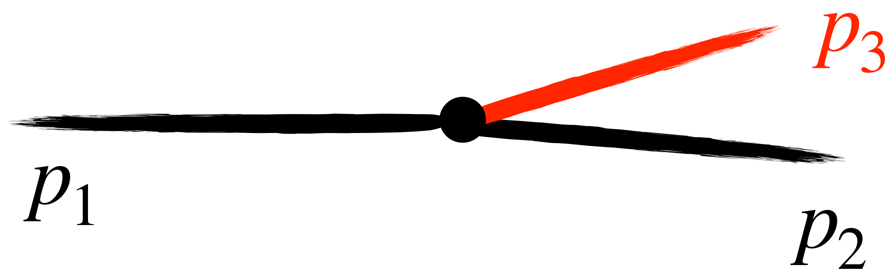
$\tau \equiv 1 - T \ll 1$ $\qquad\qquad$ $T = \mathcal{O}(1)$



Event shapes are sensitive to QCD radiation. They can be used to classify events and to measure the strong coupling constant.

Sketch of leading order calculations

For massless particles



$$T = \max\{x_1, x_2, x_3\} \qquad x_i = \frac{2E_i}{Q} \qquad 0 \leq x_i \leq 1 \qquad \sum_i x_i = 2 \qquad \tau(\{x_i\}) = 1 - \max\{x_1, x_2, x_3\}$$

The differential cross section reads with one real emission

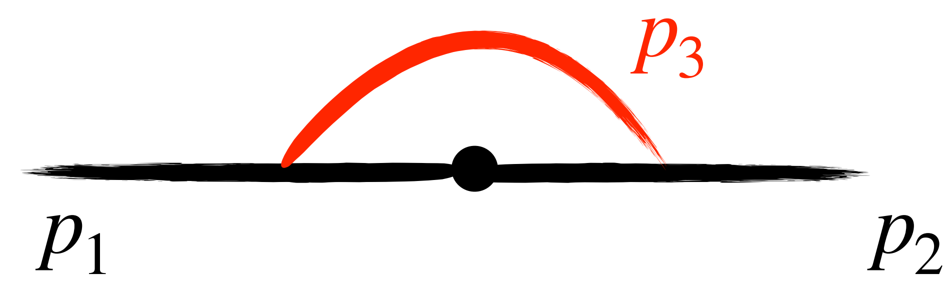
$$\Sigma(\tau) = 1 + \int [d\Phi_3] \, | M(\{x_i\} \, |^2 \, \theta(\tau - \tau(\{x_i\})$$

$$= 1 + \frac{C_F \alpha_s}{2\pi} \int_0^1 dx_1 \int_0^{x_1} dx_2 \frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)} \theta(\tau(\{x_i\}) - \tau) + \text{perm} \, .$$

➥ *divergent at x_1 = 1*

Sketch of leading order calculations

Virtual corrections



$$T = \max\{x_1, x_2, x_3\} = 1 \qquad \tau = 0$$

Including virtual corrections (same matrix element but opposite sign):

$$\Sigma(\tau) = 1 + \frac{C_F\,\alpha_s}{2\pi}\int_0^1 dx_1 \int_0^{x_1} dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}\left[\theta(\tau(\{x_i\}) - \tau) - 1\right] + \text{perm}\,.$$

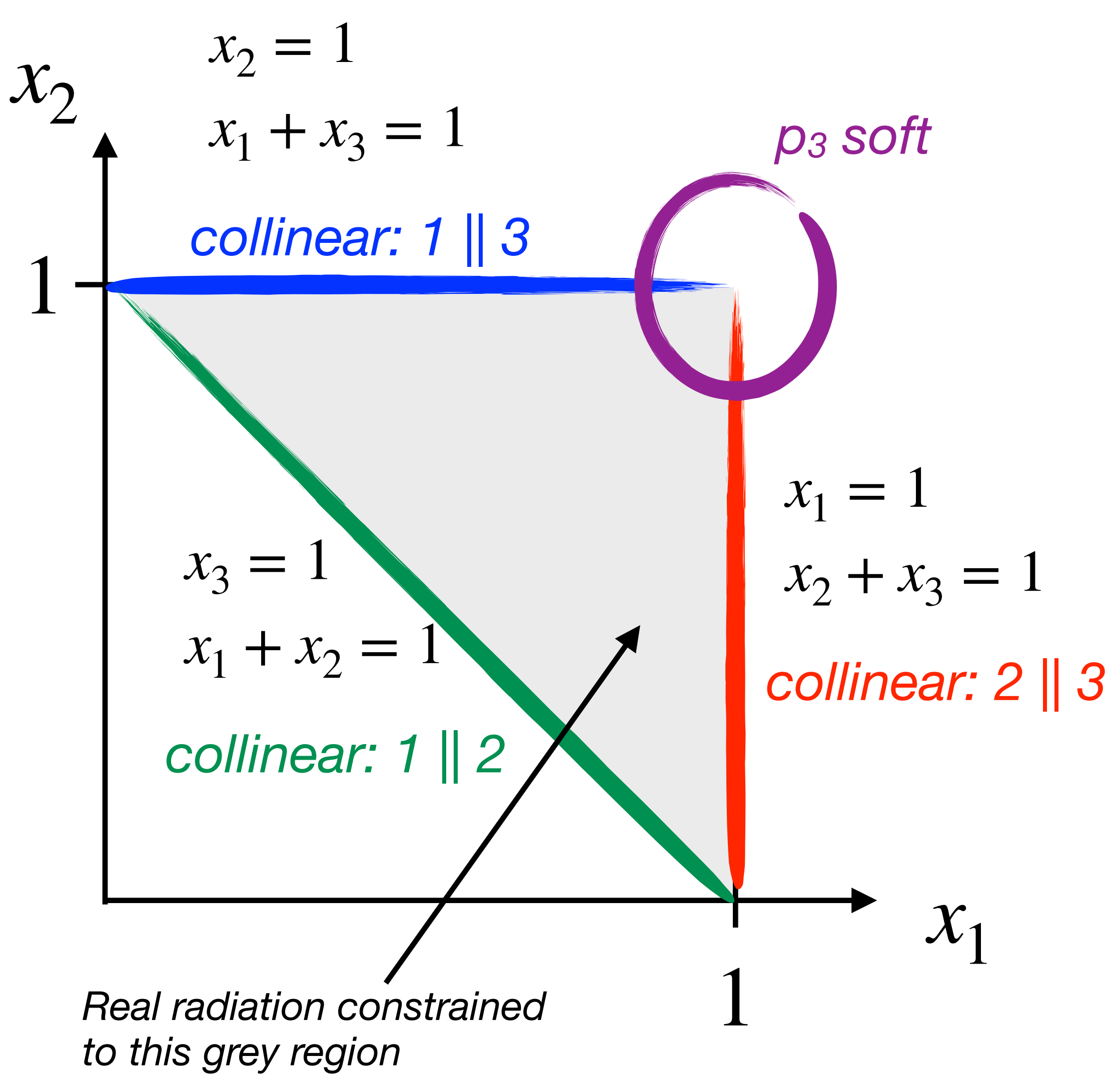
i.e.

$$\Sigma(\tau) = 1 - \frac{C_F\alpha_s}{2\pi}\int_0^{1-\tau} dx_1 \int_0^{x_1} dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} + \text{perm}\,.$$

The value of the event shape provides a bound on the integration region, which effectively shields it from singularities at $x_i = 1$.

Kinematic plane

The full phase space for real emissions

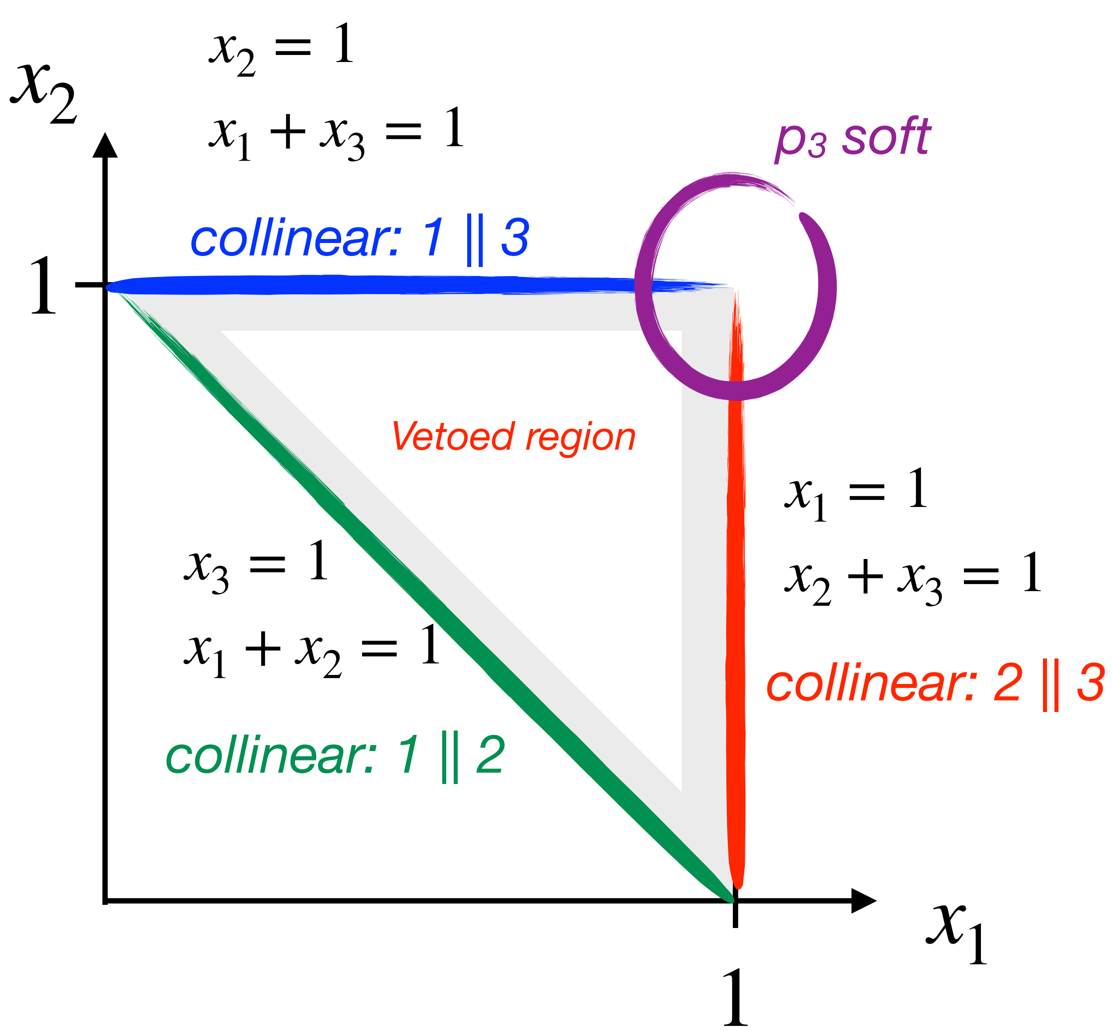


Grey: allowed integration region

Edges: of the triangle correspond to collinear regions

Vertices: of the triangle correspond to soft-collinear regions

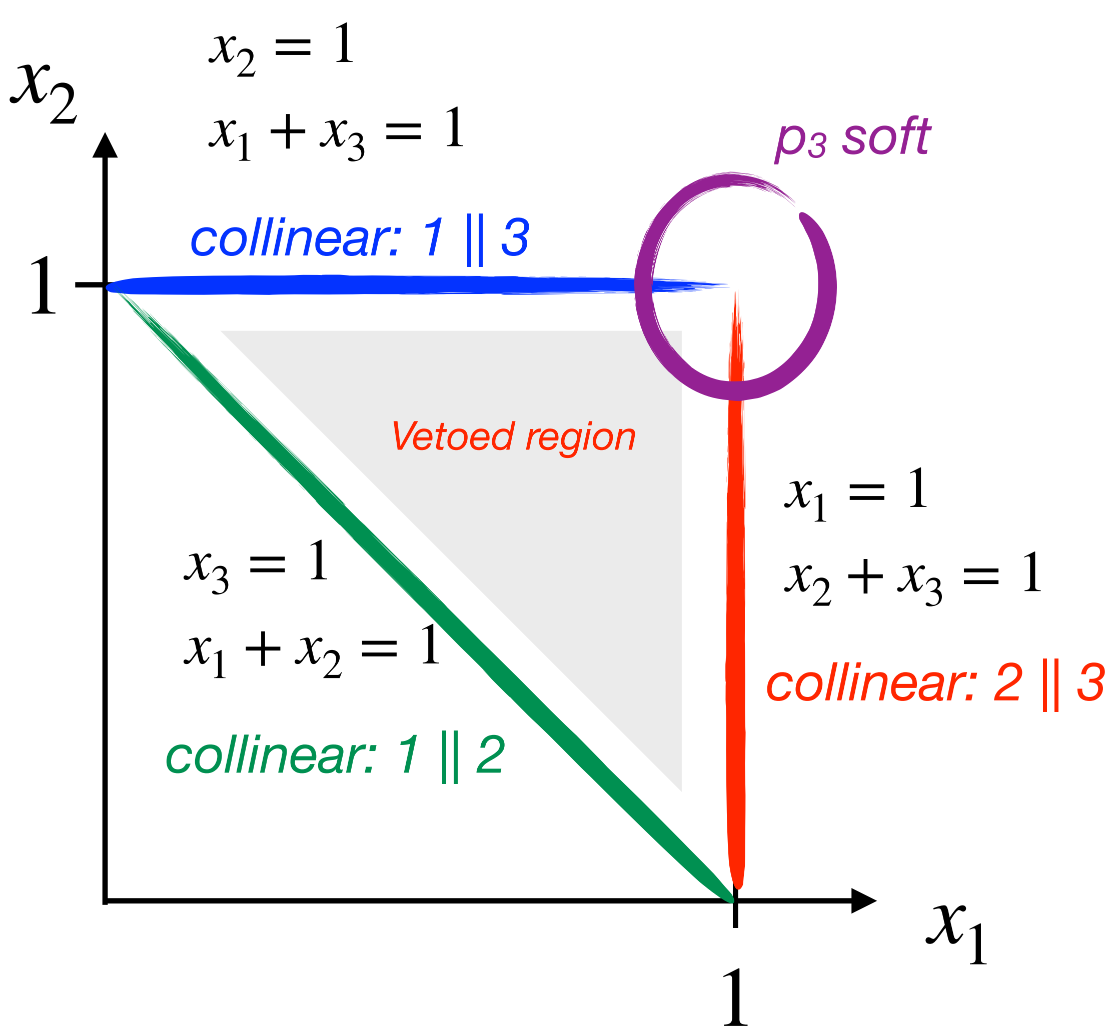
Kinematic plane



Imposing a requirement on the event shape constraints the integration region away from all singularities

→ vetoed region

Kinematic plane



Imposing a requirement on the event shape constraints the integration region away from all singularities

→ vetoed region

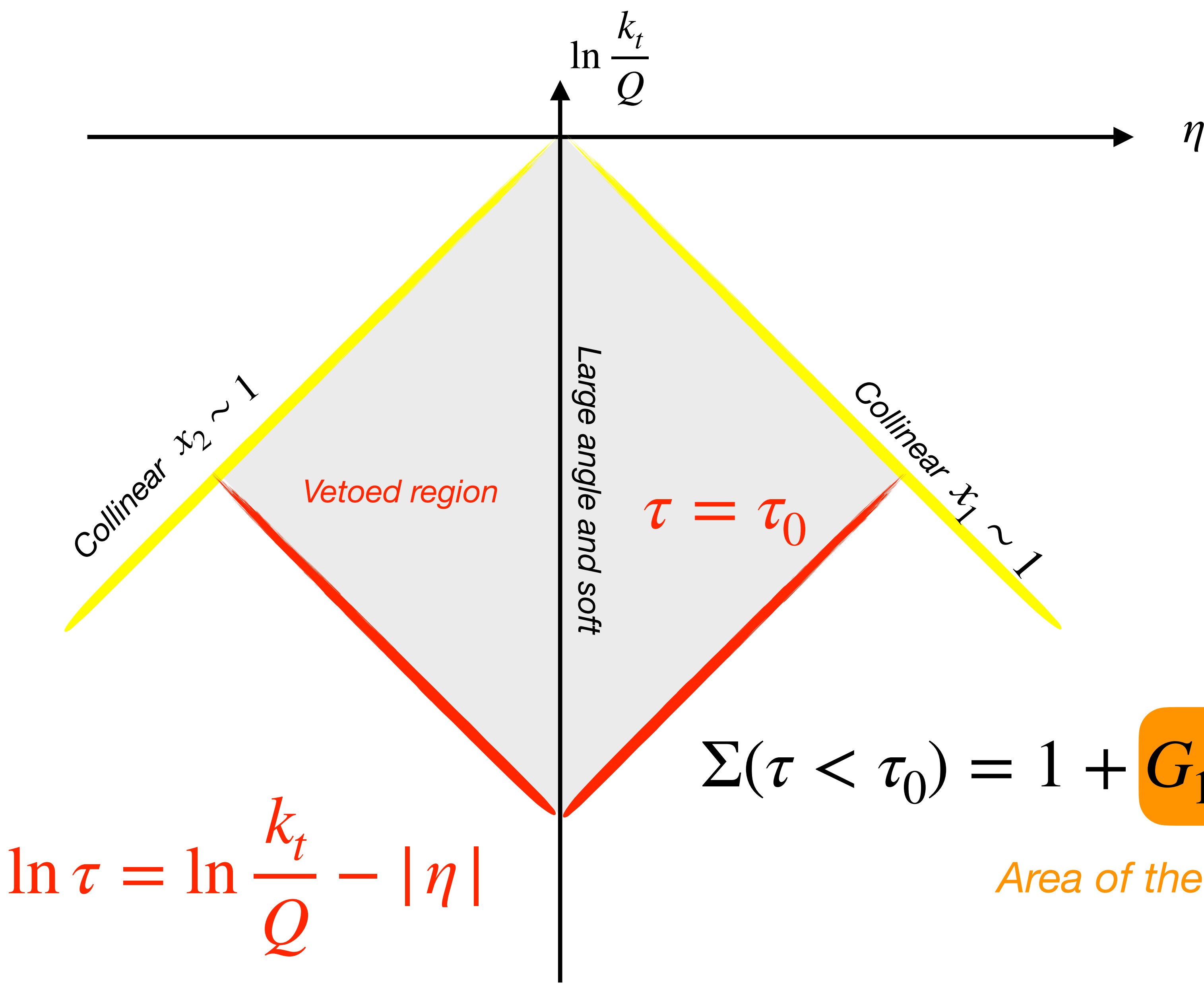
Real-virtual cancellation occurs everywhere but for the *vetoed region* which is therefore *the only one contributing* to the answer

The full calculation gives

$$\Sigma(\tau) = 1 + \frac{\alpha_s C_F}{\pi} \left\{ \left[A_1 \ln^2(1/\tau) + B_1 \ln \tau\right] + \text{regular at } \tau \to 0 \right\} + \mathcal{O}(\alpha_s^2)$$

The Lund plane

A more standard way to illustrate this relies on the Lund plane: it uses the (log of) transverse momentum and the rapidity of the emitted gluon with respect to the "Born" direction



In terms of these variables one has in the soft and collinear limit

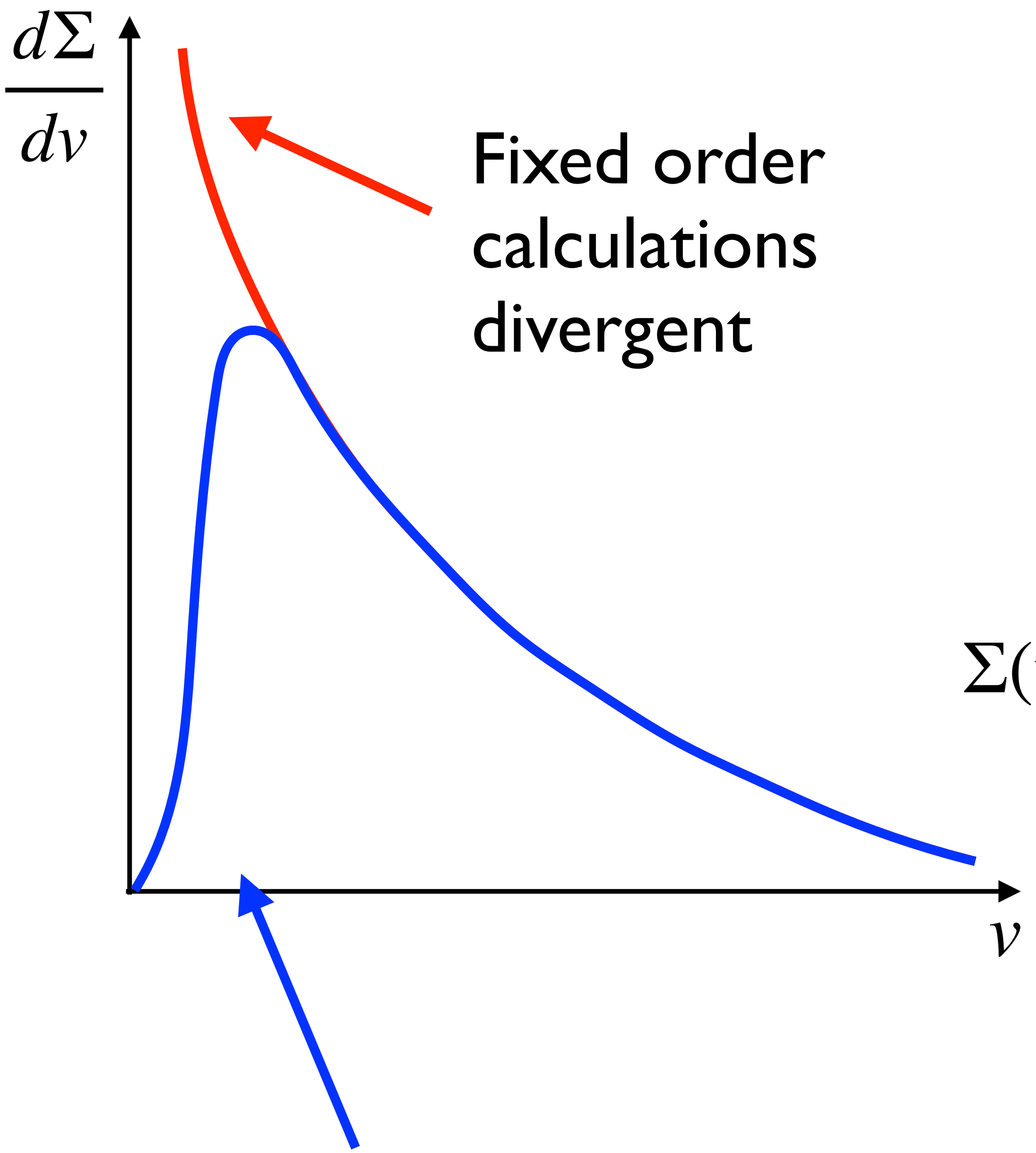
$$\tau = \frac{k_t}{Q} e^{-|\eta|}$$

$$\Sigma(\tau < \tau_0) = 1 + \boxed{G_{12} L^2} + \boxed{G_{11} L} + \ldots \qquad L \equiv \ln \tau_0$$

Area of the vetoed region

Edges of collinear regions

Breakdown of fixed order



In the region where $\alpha_s L \sim 1$fixed order perturbation theory break down: one needs to resum emissions to all orders in perturbation theory

$$\Sigma(v) \sim \exp\{Lg_1(\alpha_s L) + g_2(\alpha_s L) + \alpha_s(\alpha_s L)\}$$

Correct behaviour: exponential suppression at small v, consistent with the fact that the probability of no emitting is zero

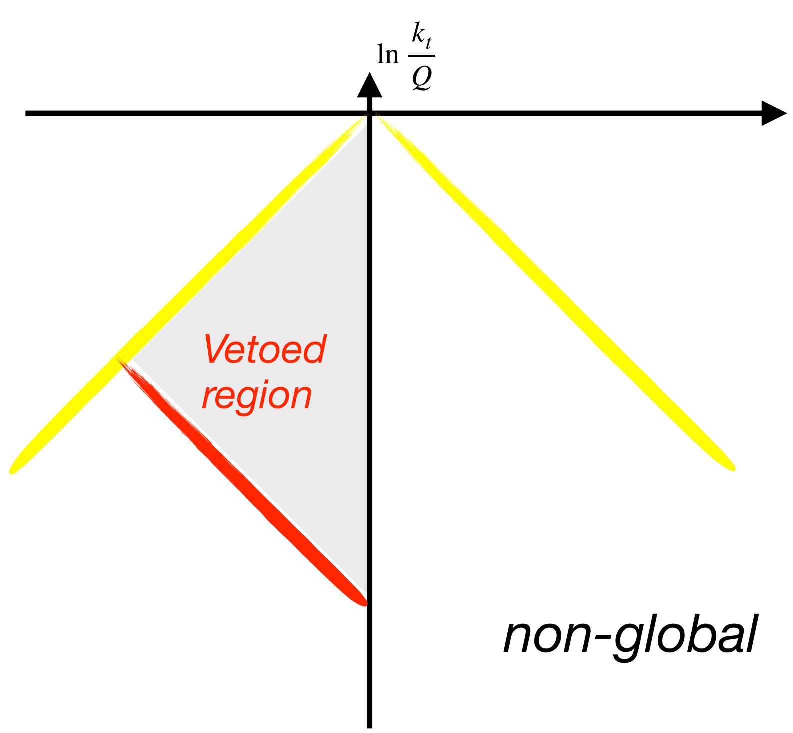
Single emission properties

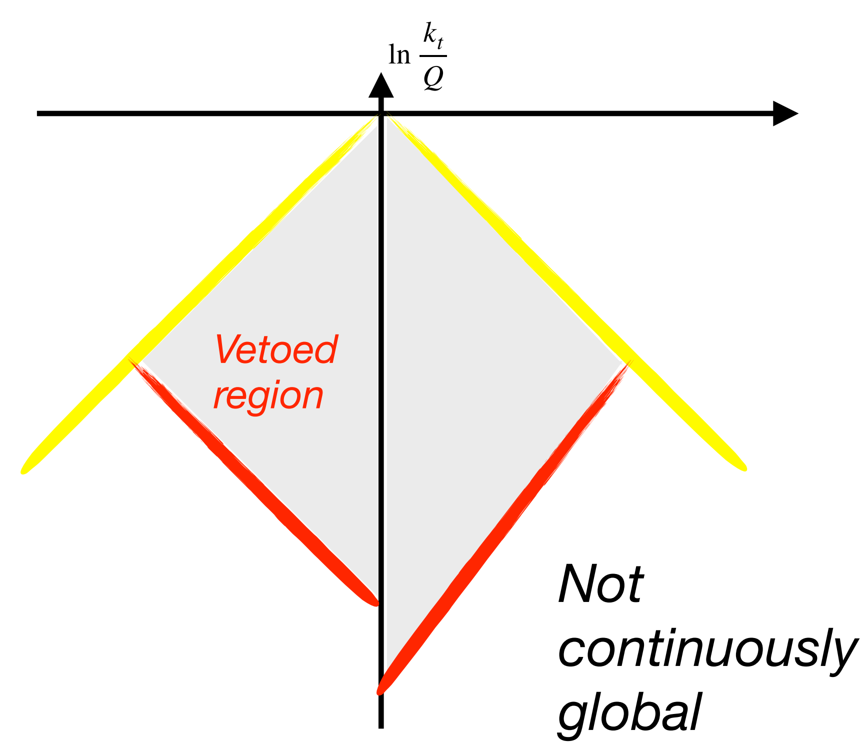
Assume that, for a single soft-collinear gluon, the following *parametrization* holds

$$V(k) = d_\ell \left(\frac{k_t}{Q}\right)^a e^{-b_\ell|\eta_\ell|} g_\ell(\phi)$$

This can be verified, and the coefficients $a_\ell, b_\ell, d_\ell, g_\ell(\phi)$ for soft emissions collinear to a "leg" ℓ , can be determined numerically

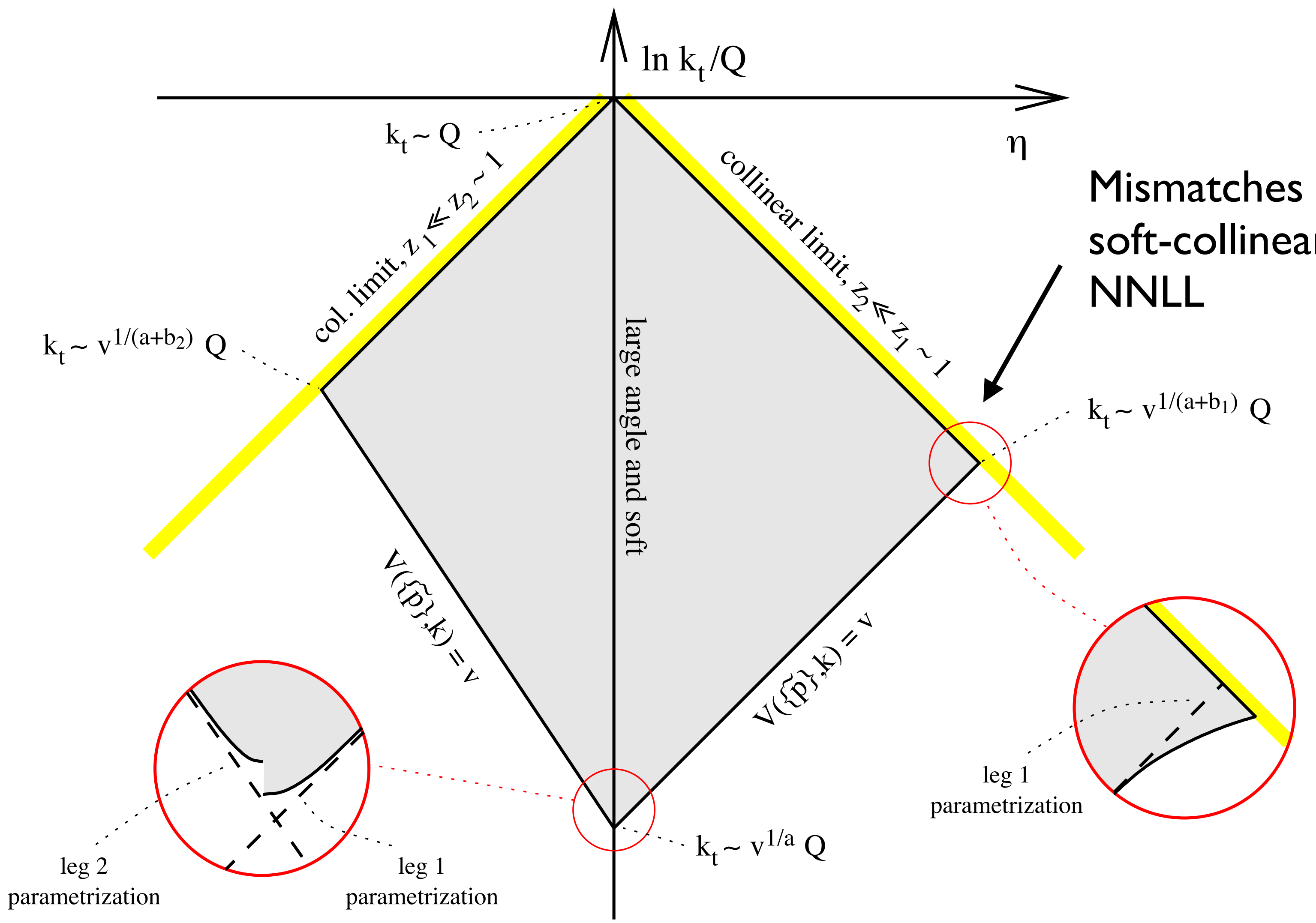
Continous globalness means: $a_1 = a_2 \ldots = a$





Single emission properties

Illustration of mismatch between exact definition and parametrization:

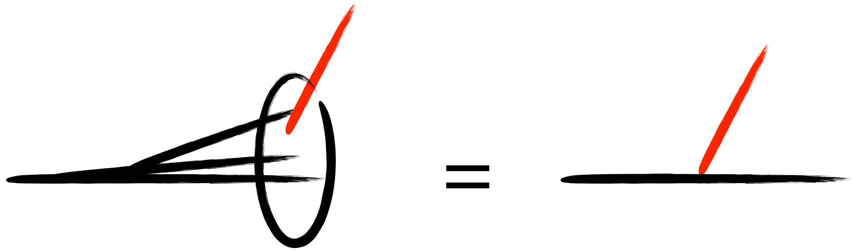


Mismatches between exact and soft-collinear approximation are NNLL

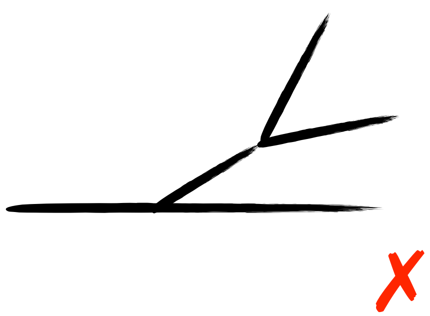
Extension to all orders?

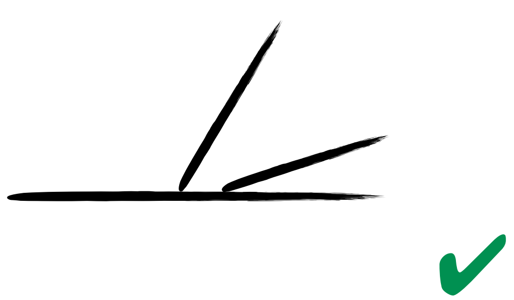
Key properties for all-order calculations: **coherence and factorisation**

- A soft gluon couples to fast partons via an eikonal current, sensing only their collective colour charge. When emitters are close together, it cannot resolve them individually — leading to **colour coherence**



- If two emissions are widely separated, their radiation patterns do not overlap, interference terms are suppressed and the emissions factorize — i.e. **they are independent**





Two emissions: non-abelian terms

For two emissions, non-abelian contributions simply

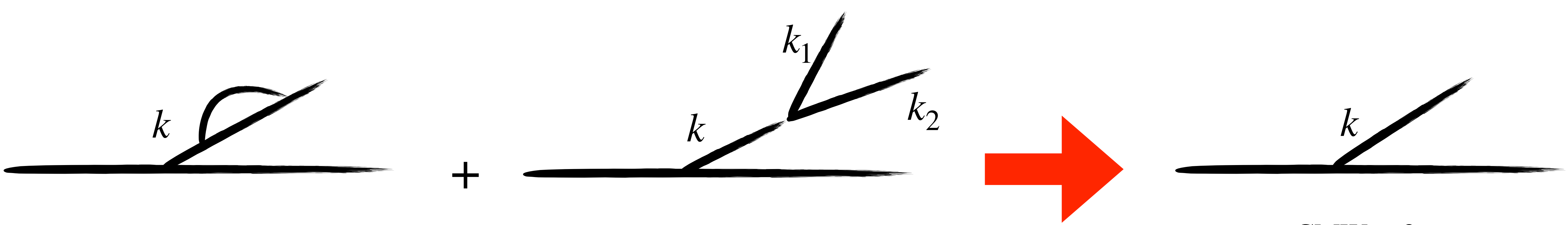
$$\alpha_s^2(\mu^2)[dk]\,|M_{1-\text{loop,N.A.}}(k) + \alpha_s^2(\mu^2)\int [dk_1][dk_2]\,|M_{N.A.}(k_1,k_2)\delta^3(k-k_1-k_2)$$

$$= \alpha_s^2(\mu^2)[dk]\,|M(k,\mu)|^2\left(\beta_0 \ln\frac{k_t^2}{\mu^2} + \frac{K^{\text{CMW}}}{2\pi}\right)$$

$$\alpha_s(k_t^2) = \alpha_s(\mu^2)\left(1+\alpha_s\beta_0\ln\frac{k_t^2}{\mu^2}\right)$$

$$\alpha_s^{\text{CMW}} = \alpha_s^{\overline{\text{MS}}}\left(1+K^{\text{CMW}}\frac{\alpha_s^{\overline{\text{MS}}}}{2\pi}\right)$$

$$\Rightarrow [dk]\alpha_s^{\text{CMW}}(k_t)\,|M(k,\mu)|^2$$



$\alpha_s^{\text{CMW}}(k_t^2)$

Catani, Marchesini, Webber 1991

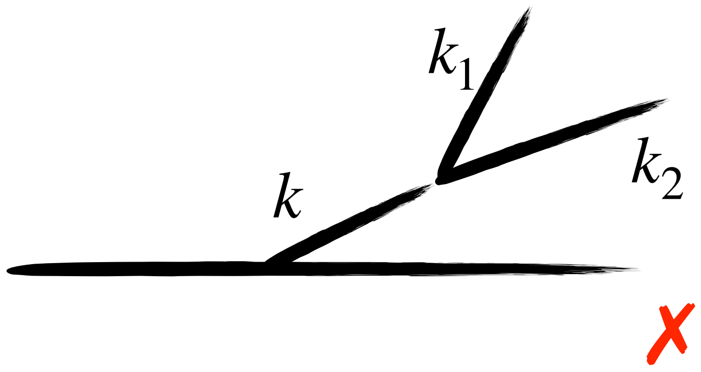
Two emissions: non-abelian terms

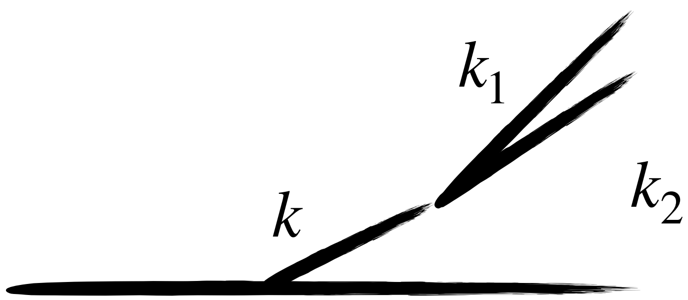
Accounting for the mismatch in the observable one obtains

$$\Sigma(v) = 1 - \int [dk]\alpha_s^{\mathrm{CMW}}(k_t^2)\,|\,M^2(k)\,|\,\Theta(V - v(k))$$

$$- \int [dk_1][dk_2]\,|\,M_{\mathrm{NA}}(k_1,k_2)\,|^2 \left[\Theta(V - v(k_1,k_2) - \Theta(V - v(k))\right]$$

$\mathcal{O}(\alpha_s)$ correction relative to single gluon emission (formally NNLL correction)



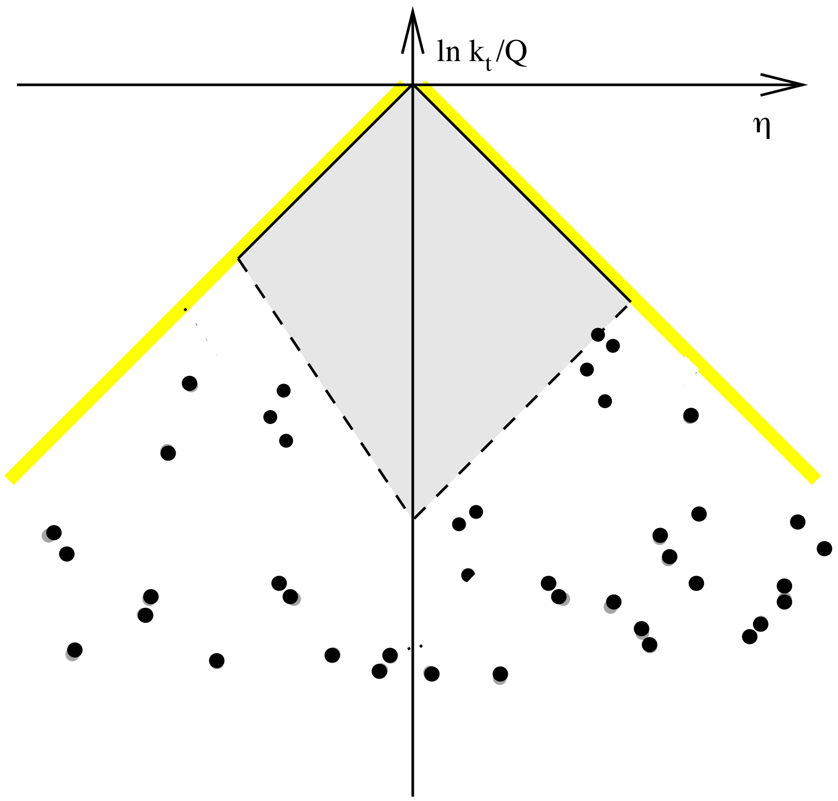


$$\left[\Theta(V - v(k_1,k_2) - \Theta(V - v(k))\right] \to 0$$

Generalising to many emissions

Coherence to all orders implies that emissions can be treaded independently if they are sufficiently separated in rapidity

However, emissions are not well-separated: e.g. there are about 10 units of rapidity at the LHC and typical events have ≫ 10 particles



Generalising to many emissions

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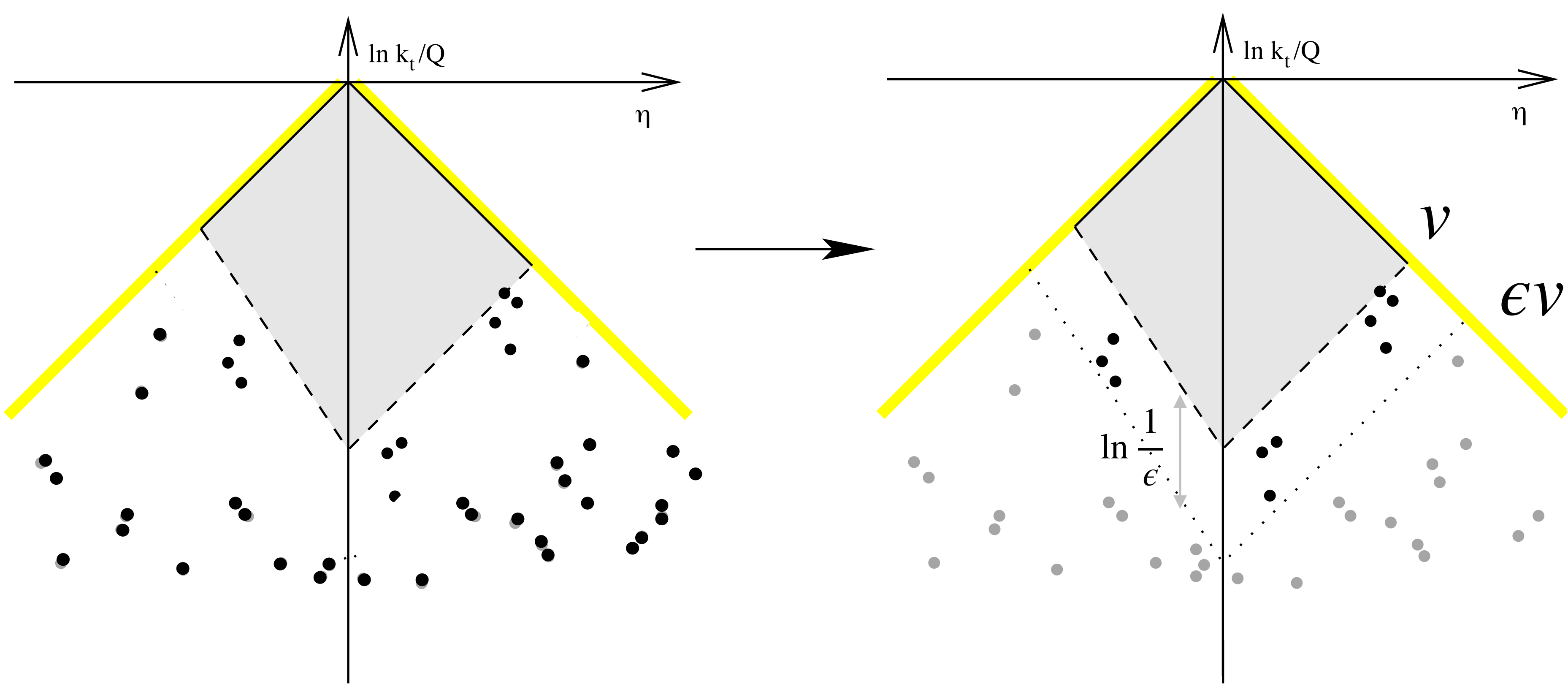
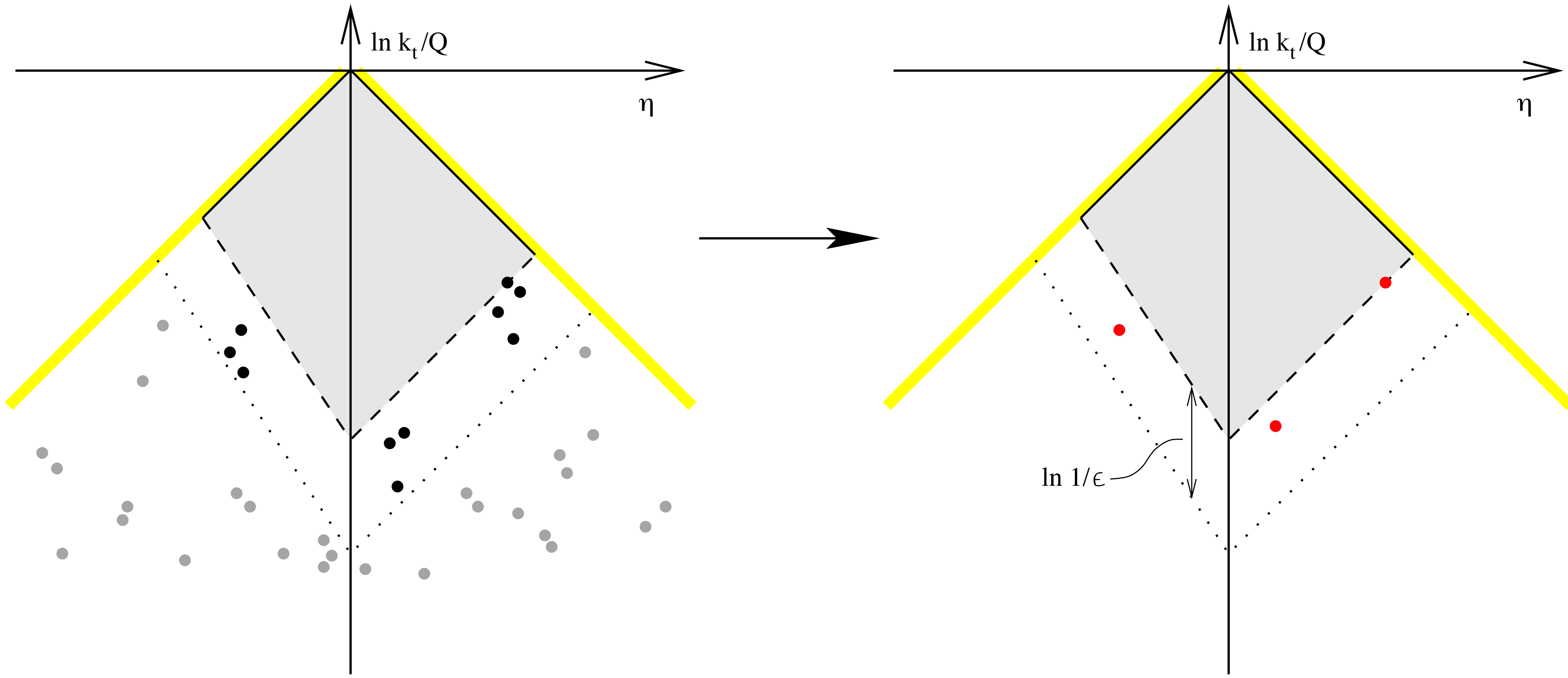


Step 1: Emissions which lead to an observable ϵv (ϵ <<1) can be neglected *for recursive infrared safe observables* → *ignore grey emissions*

Generalising to many emissions



<u>Step 2</u>: emissions "clusters" can be replaced by "clustered" emissions (red dots) *for recursive collinear safe observables*

Now the emissions of clusters can be considered independent

The answer

Putting together these ideas one obtains a resummed result of the form

$$\Sigma(v) = (1 + C(\alpha_s)\, e^{\overbrace{Lg_1(\alpha_s L)}^{\text{LL}}+\overbrace{g_2(\alpha_s L)}^{\text{NLL}}+\overbrace{\alpha_s g_3(\alpha_s L)}^{\text{NNLL}}}$$

Almost all of the result comes from the exponentiation of the one gluon result, i.e. depends only on $\{a_\ell, b_\ell, d_\ell, g_\ell(\phi)\}$ (plus the coupling in the right scheme and at the right scale)

All effects that are not a pure exponentiation of the one-gluon result come from "multiple emission" corrections and can be compute through finite integrals (difference between many emissions and the sum of them)

Parton showers

- Despite the high sophistication of exact higher-order calculations, these are limited to low number of particles in the final state

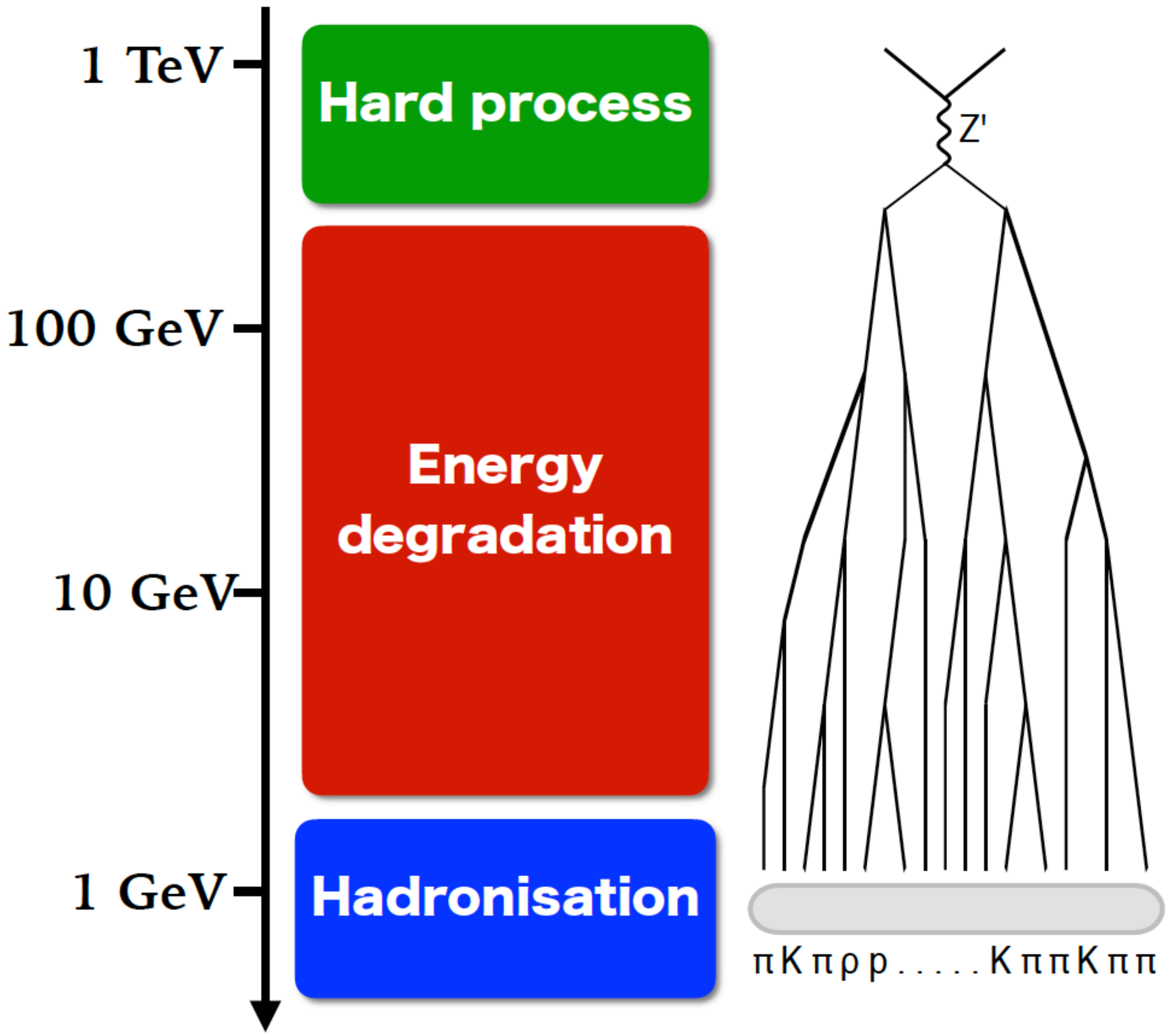
- we have also seen that sometimes large logarithms can spoil the convergence of fixed-order calculations (series not convergent)

- now we adopt a different approach: we seek for an approximate result such that enhanced terms are taken into account to all orders

- this leads to a parton shower picture, which can be implemented in computer simulations known as Monte Carlo event generators

Parton showers enter essentially any experimental study at current colliders

Parton showers



Parton shower:
Energy degradation of particles from the hard collision, producing more particles during evolution

Parton showers are ubiquitous at the LHC

Modelling QCD processes, event simulation, background estimates, unfolding, detector simulation, …

Core ingredient

Probability, in the soft-collinear limit, of *NOT* emitting a gluon above transverse momentum k_t:

$$P(\text{no single gluon above } k_t) = 1 - \frac{\alpha_s C_F}{\pi} \int^Q \frac{dE}{E}\frac{d\theta}{\theta}\Theta(E\theta - k_t)$$

In the soft-collinear limit (gluons emitted independently), the probability of nothing happening is the exponential of the one gluon result

$$P(\text{no emission above } k_t) \equiv \Delta(Q, k_t) = \exp\left[-\frac{\alpha_s C_F}{\pi} \int^Q \frac{dE}{E}\frac{d\theta}{\theta}\Theta(E\theta - k_t)\right]$$

This is called Sudakov form factor: it corresponds to the probability of evolving from scale Q to k_t without emissions

Core ingredient

In terms of the Sudakov form factor, the probability of the first emission becomes

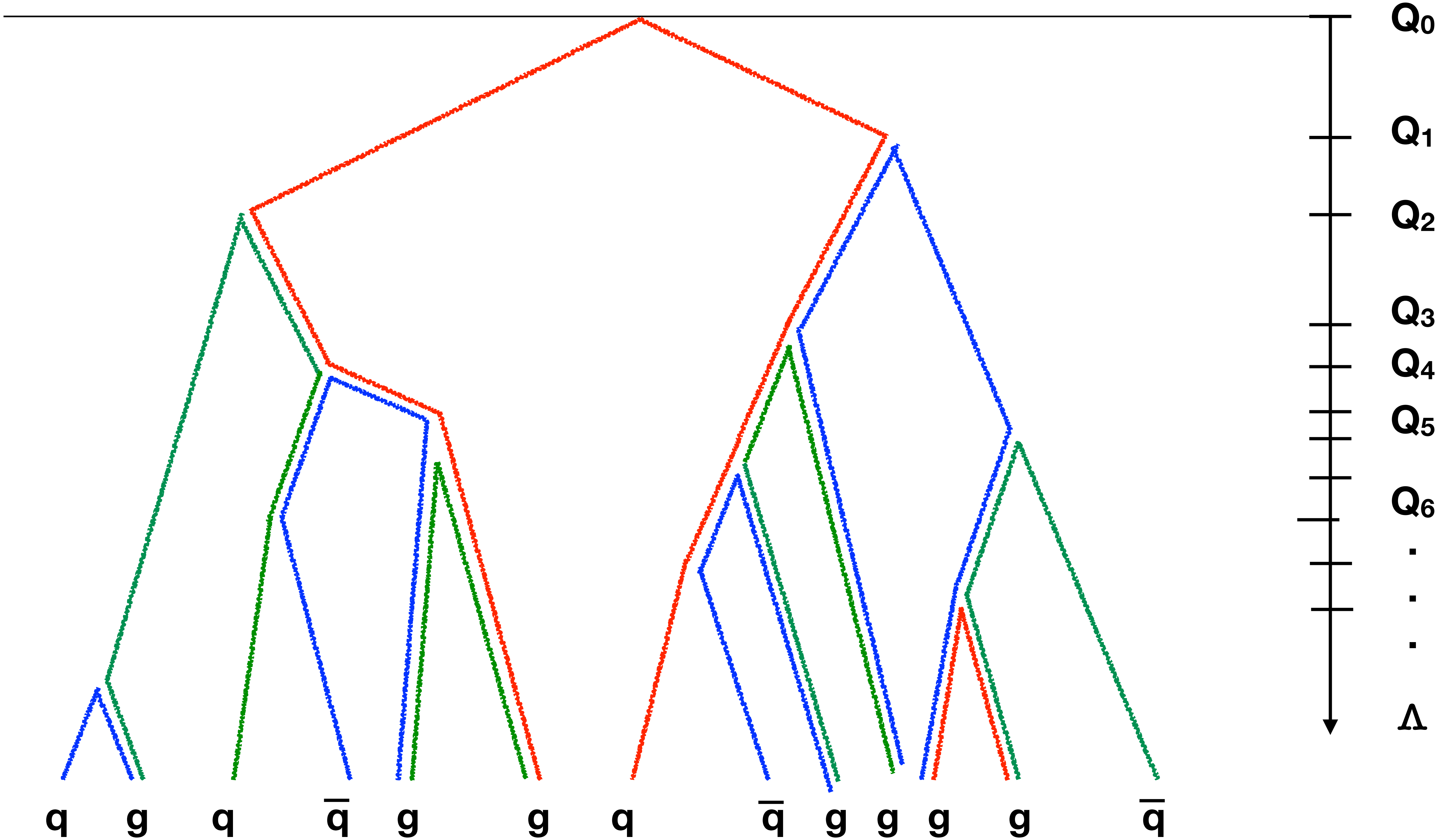
$$\frac{dP}{dk_{t,1}} = \frac{d}{dk_{t,1}} \Delta(Q, k_{t,1})$$

This probability can be generated easily numerically: generate a flat random number $0 < r < 1$ and find $k_{t,1}$ such that $r = \Delta(Q, k_{t,1})$

If $k_{t,1}$ is smaller than a fixed-cut-off stop, otherwise allow each new emission to branch further. This gives rise to a "shower" of soft-collinear emissions.

NB: All parton showers employ an explicit infrared cutoff, which marks the scale below which perturbative evolution is terminated and hadronization effects take over.

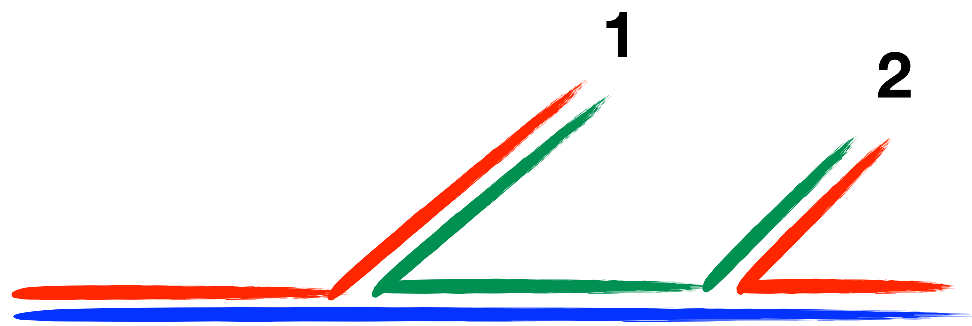
Illustration of shower evolution



Key elements of parton showers

<u>Ingredients/ambiguities</u>:

- Splitting kernels / matrix elements
- Ordering variable
- Recoil
- Vetoed region
- Colour/spin treatments
- …



<u>Dipole recoil in k_t-ordered shower:</u> emission 2 takes recoil from 1. This is wrong in a large phase space region

<u>Design principle for modern parton showers:</u>

All elements of the shower should respect the absence of crosstalk between disparate scales, i.e. it should **respect QCD factorisation**

Key elements of parton showers

The development of parton shower has seen a dramatic change in recent years. Key new elements of modern parton showers include

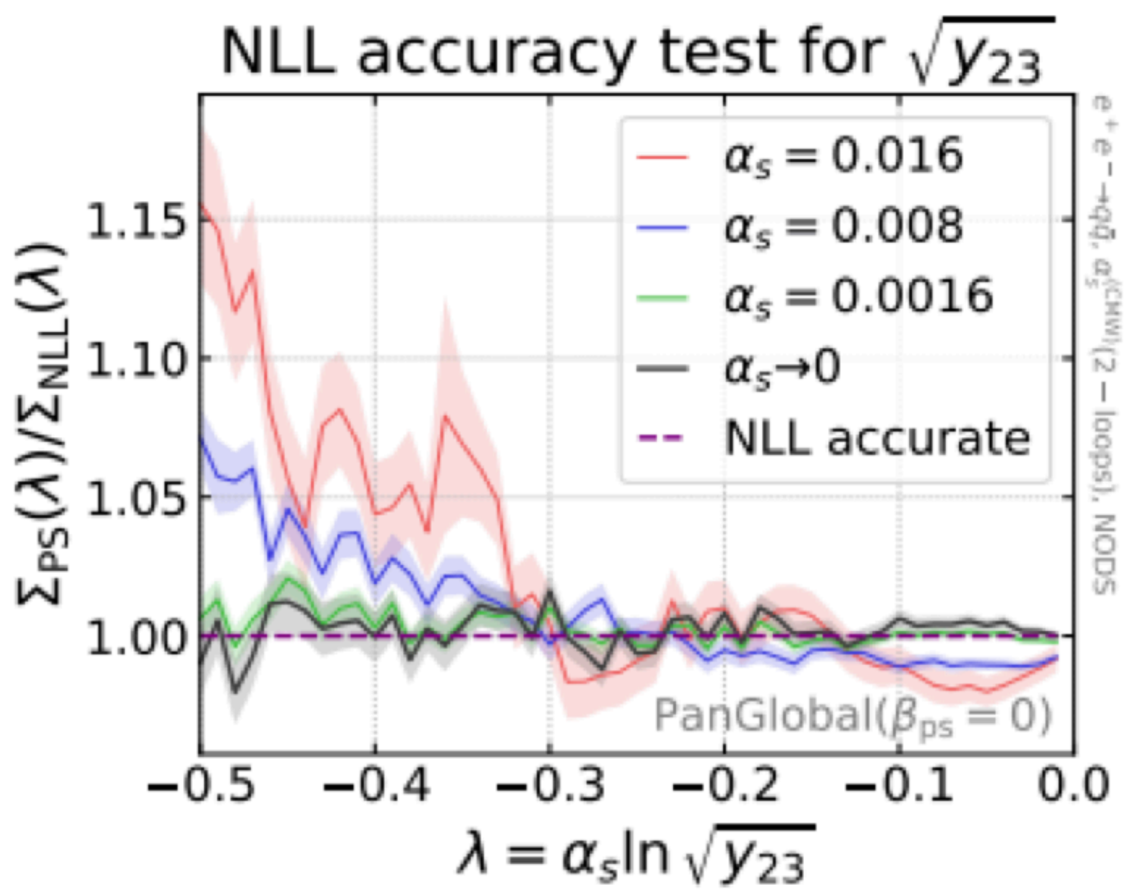
- Improvements in the accuracy of the parton shower
- Numerical procedure to validate the accuracy
- Understanding that some parton showers have lower accuracy
  $\Rightarrow$ can be disregarded when assessing theory uncertainties

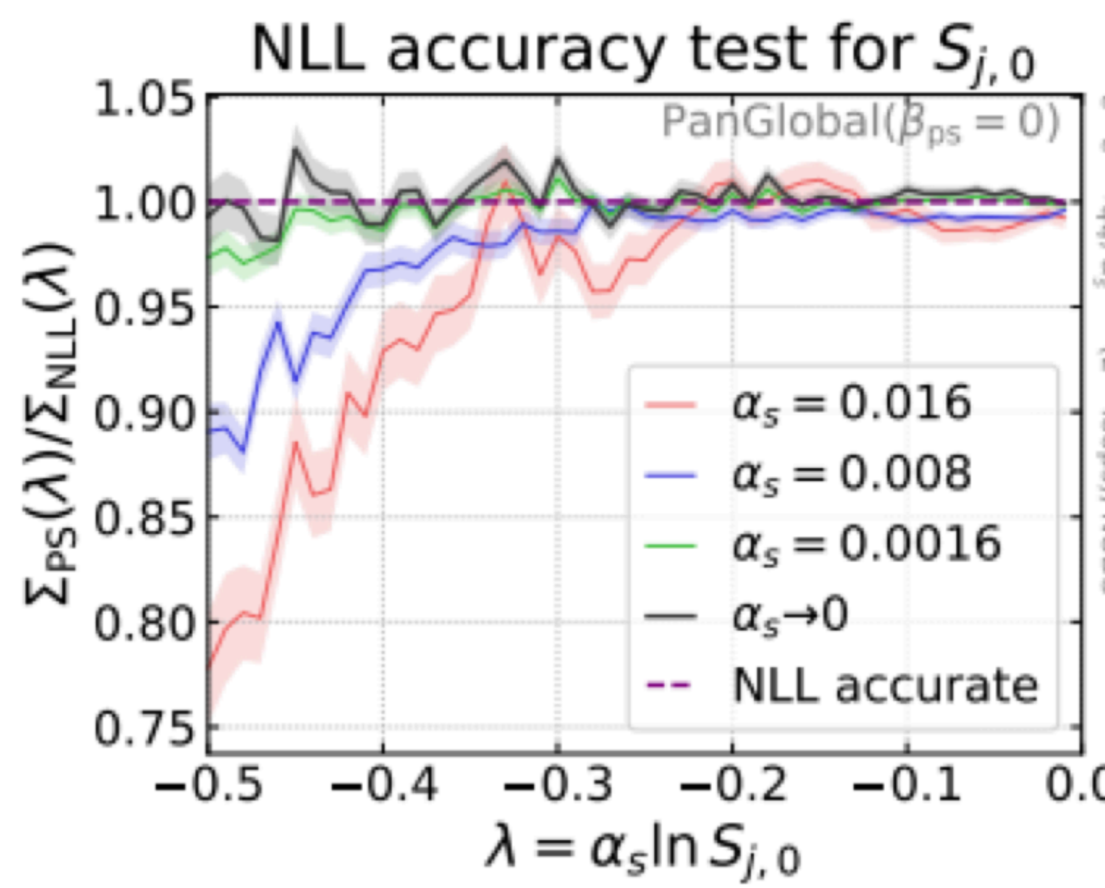
ALARIC, DEDUCTOR, PANSCALES, HERWIG7, ...

A revolution in parton shower developments is ongoing.
It will be crucial for Run 3, HL-LHC and FCC.

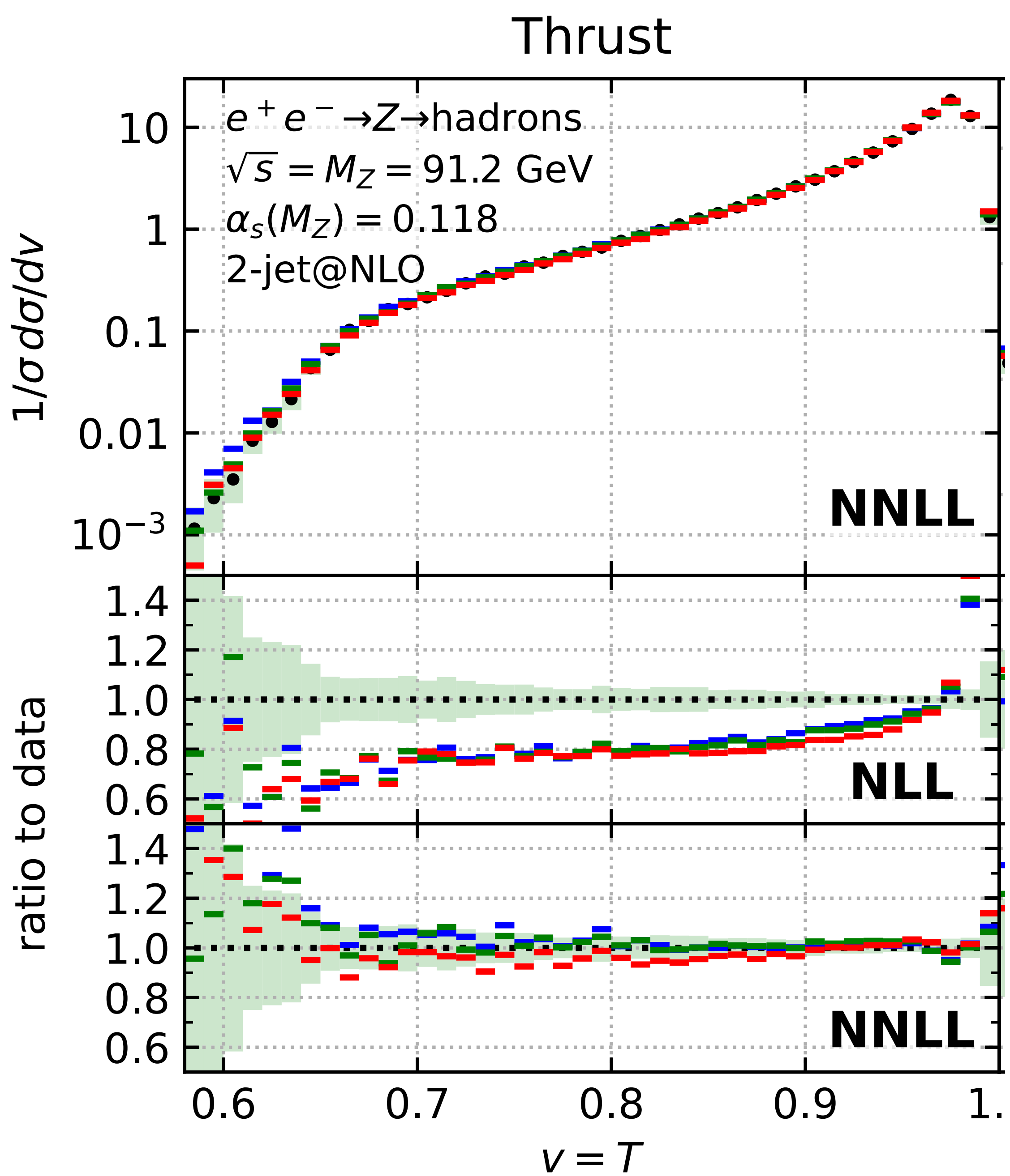
Tests of parton shower accuracy

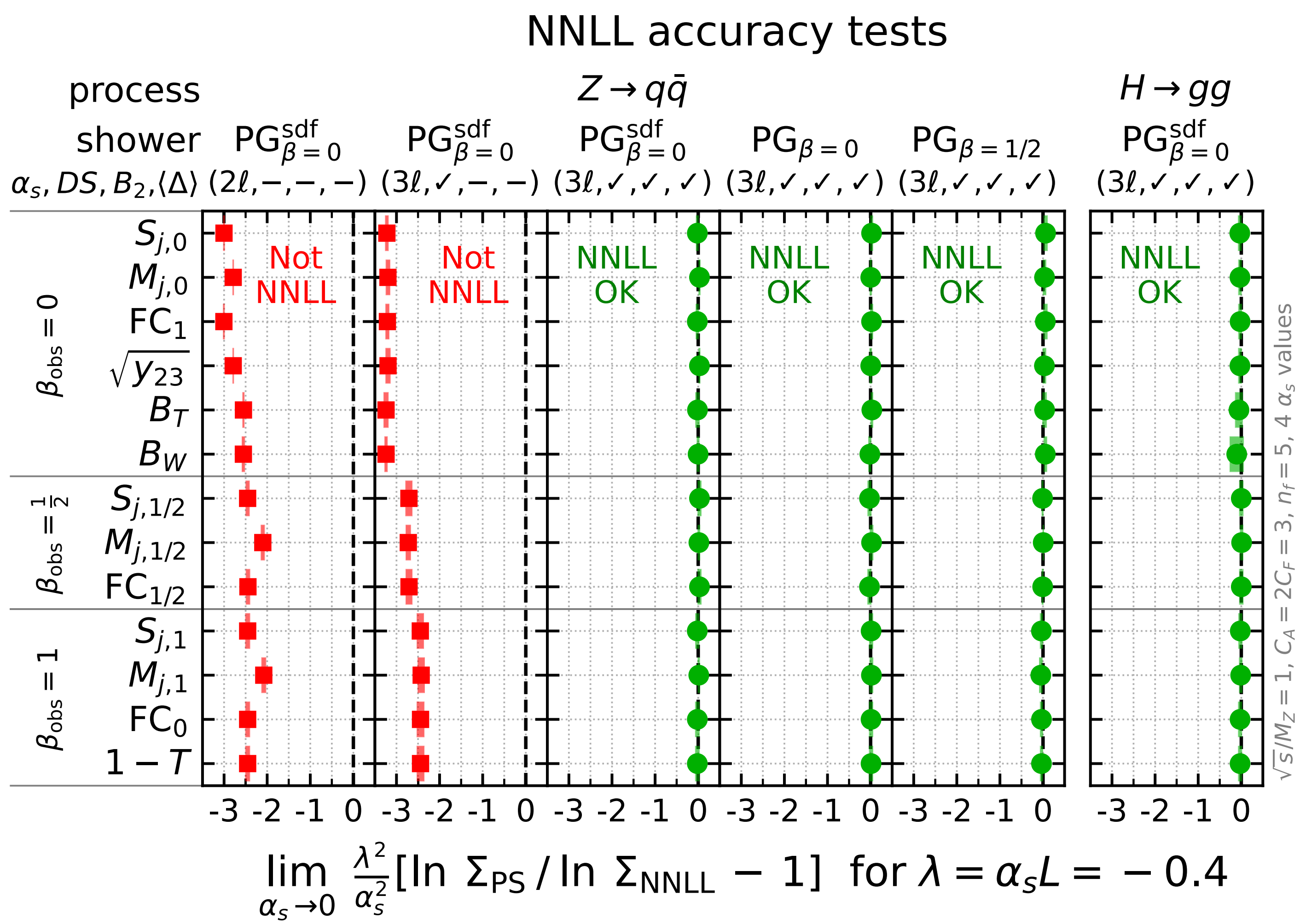
Formally test the logarithmic accuracy of parton showers by comparing to analytic calculations in the limit of very small coupling





Tests of parton shower accuracy





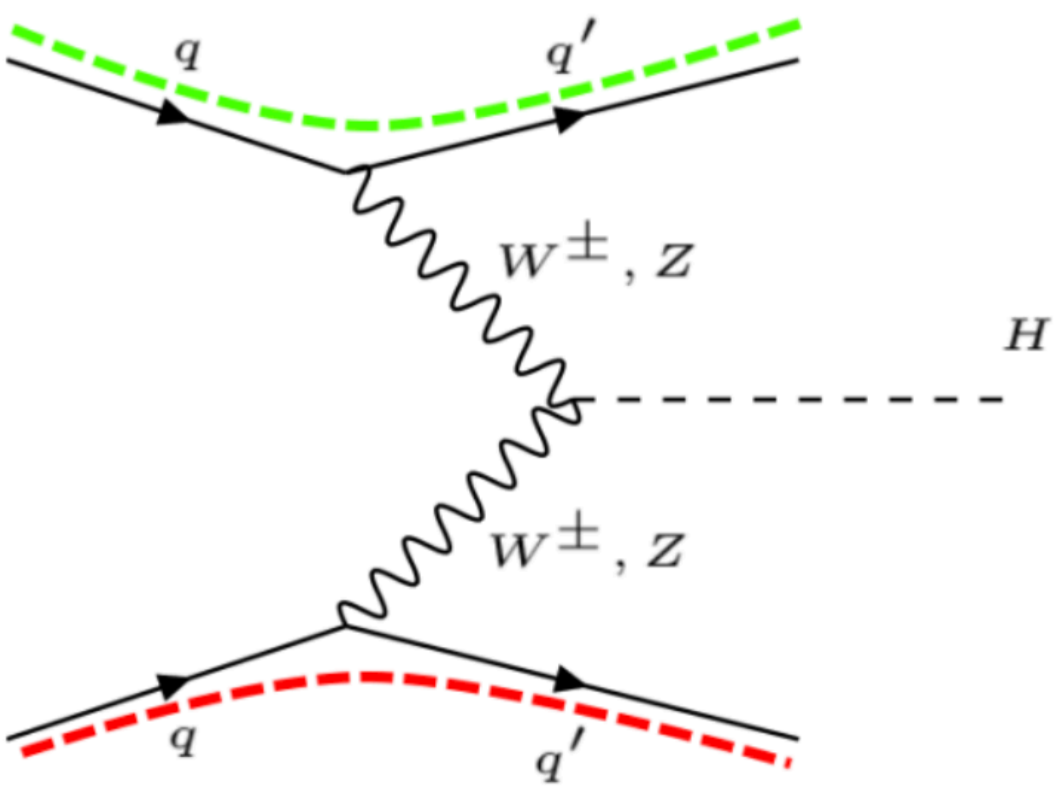
Novel parton showers reaching NLL and even NNLL accuracy for e^+e^- collisions

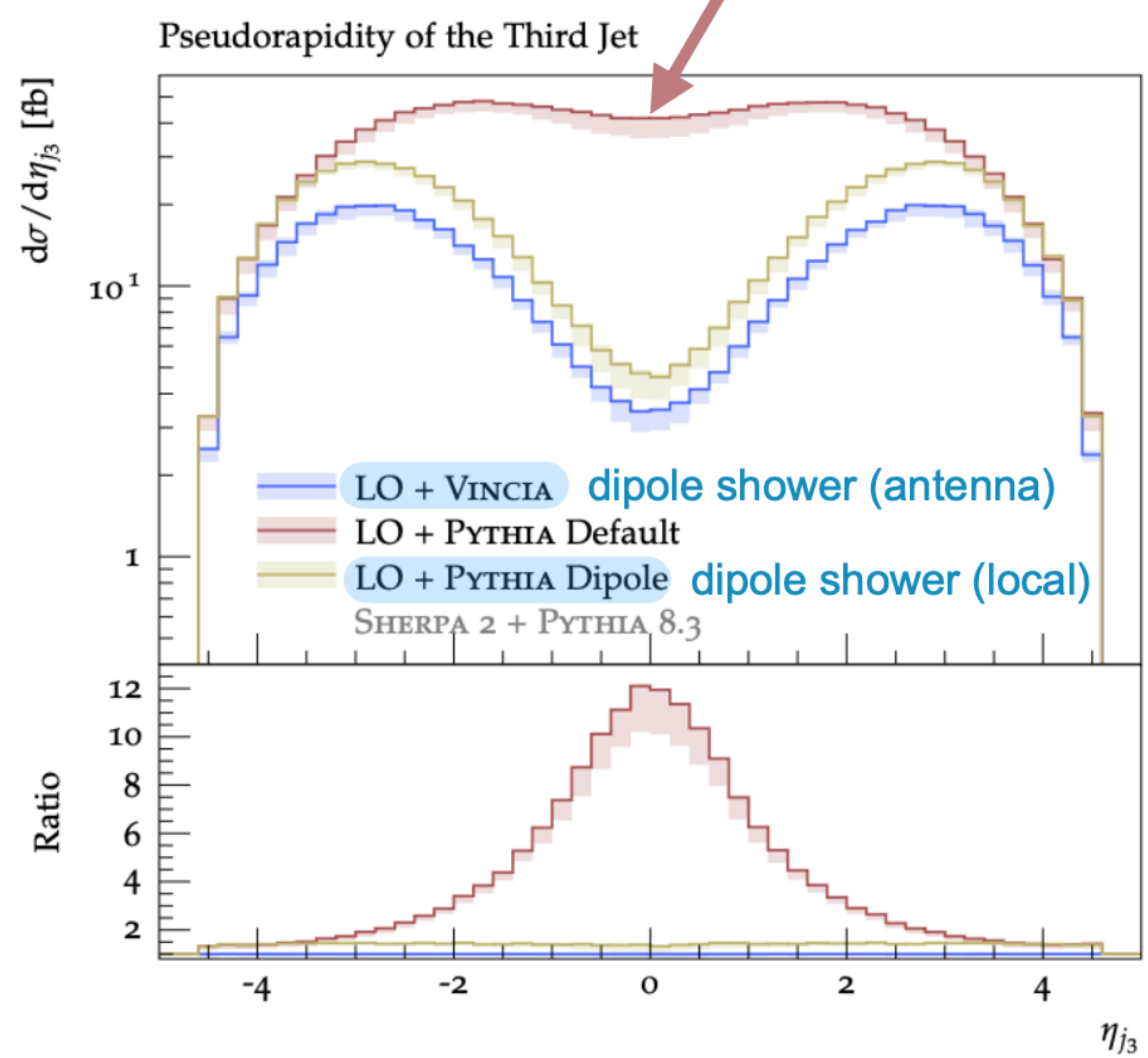
Knowledge from resummations of event shapes crucial to achieve these goals and validate them

Choices do matter

Example: vector boson fusion Higgs production

Default Pythia shower incorrectly fills the central region

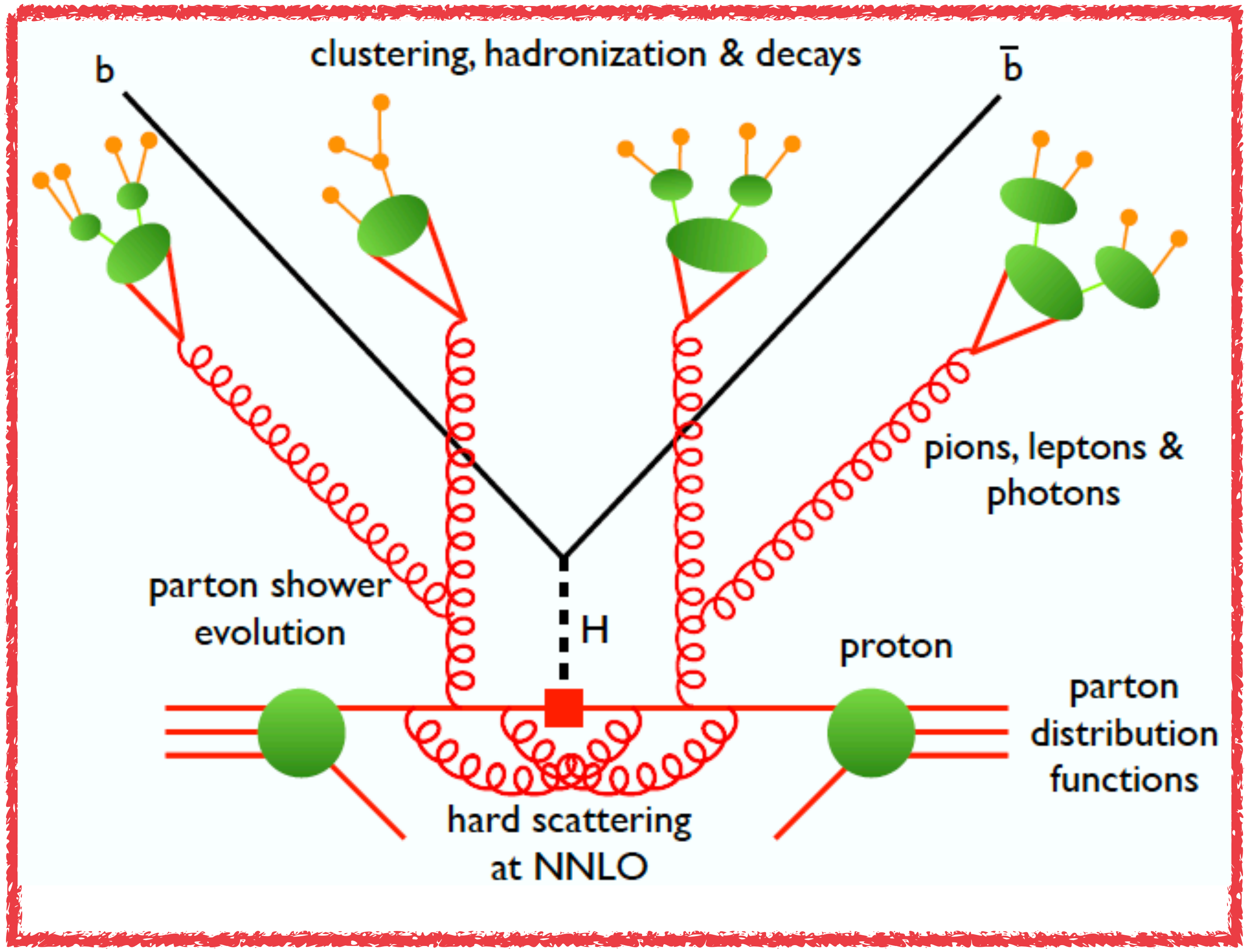




Colour coherence strongly suppresses activity in the central region

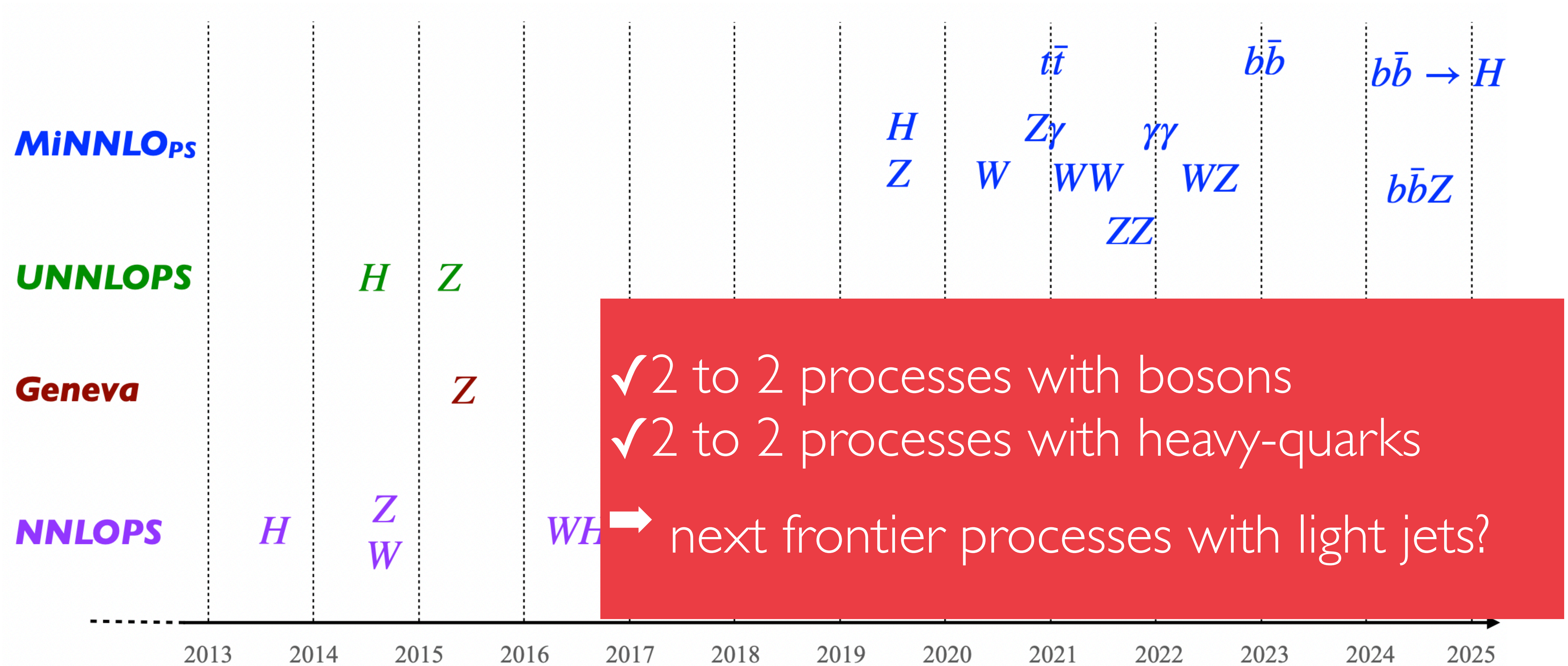
Combining NNLO + parton shower

Merging NNLO and parton shower (NNLOPS) is a must to have the best perturbative accuracy with a realistic description of final state



Matching NNLO and parton showers

Matching NNLO and parton showers done for selected processes and preserving the leading logarithmic accuracy of the shower



✓2 to 2 processes with bosons

✓2 to 2 processes with heavy-quarks

➡ next frontier processes with light jets?

Not yet clear how to preserve accuracy of more accurate showers in the matching

Summary progress in parton showers

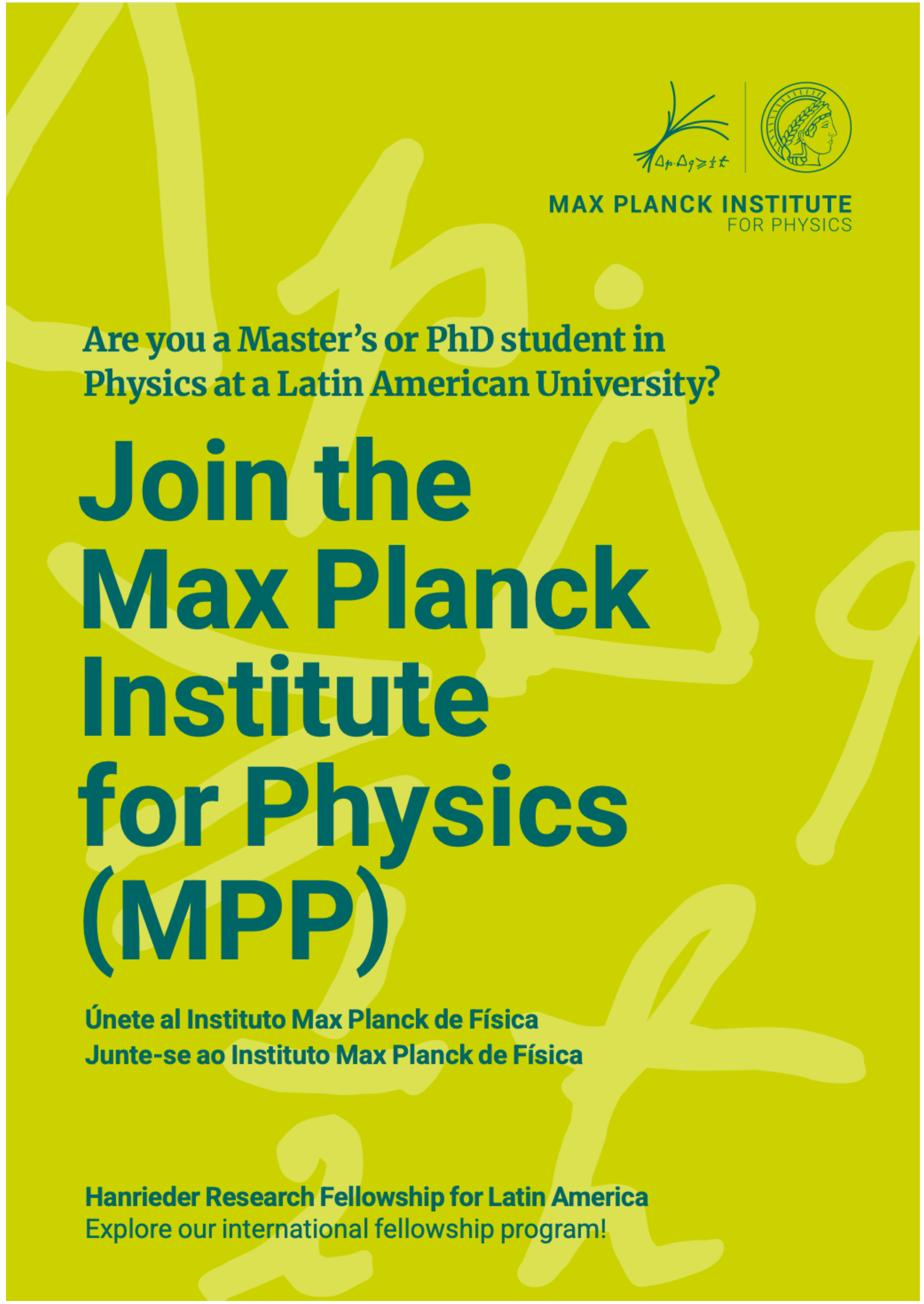
Recent progress includes

- Higher perturbative accuracy: NLL and beyond for selected observables thanks to improved shower algorithms, improved treatment of recoil, and of soft-collinear subleading effects
- Development of formal tests of accuracy
- First NNLO + PS for increasing number of processes

Summary progress in parton showers

<ins>Still many open questions</ins>

- Inclusion of yet higher logarithmic terms
- Inclusion of sub-leading color correlations and spin-correlation effects
- Better combination of QCD + QED + EW shower (EW Sudakov at high energies)
- Systematic uncertainty estimates through shower variations
- Machine learning assisted tuning



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