Causal Inference: Classical and Quantum

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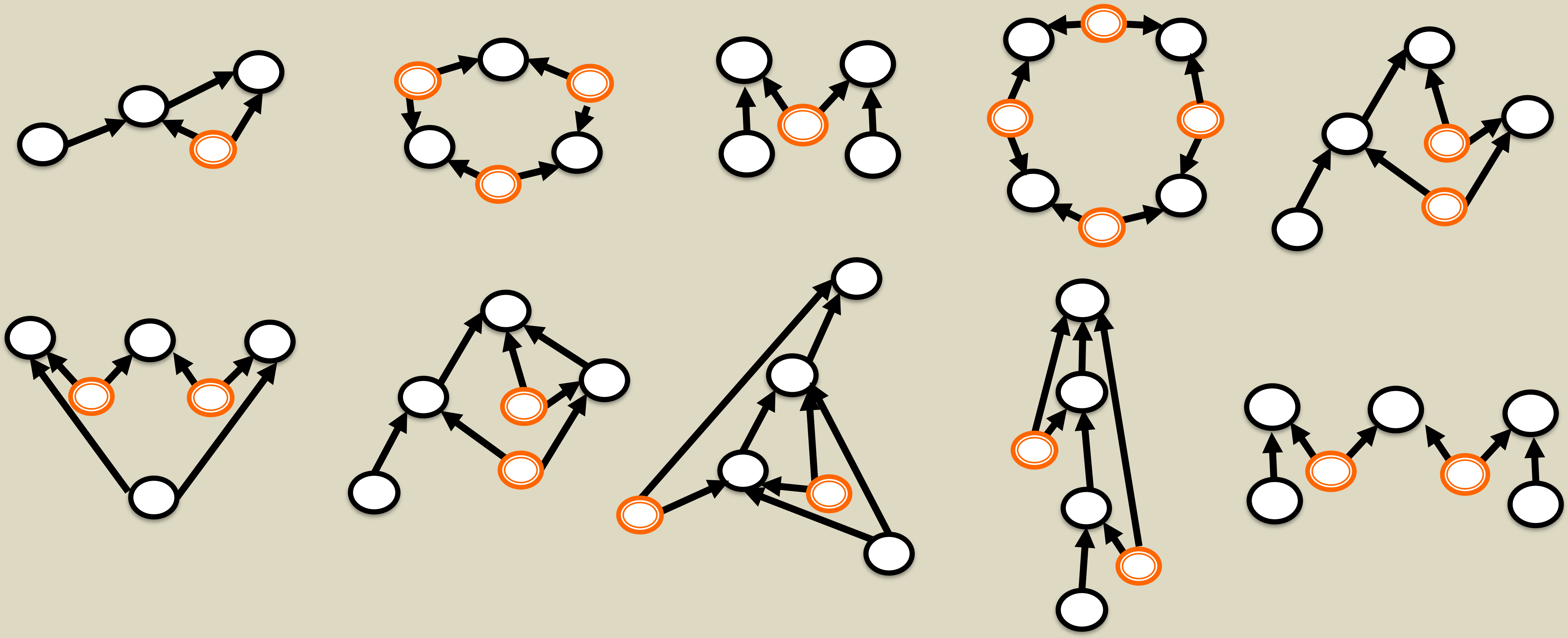
Causarum Investigatio
"Investigate the causes"

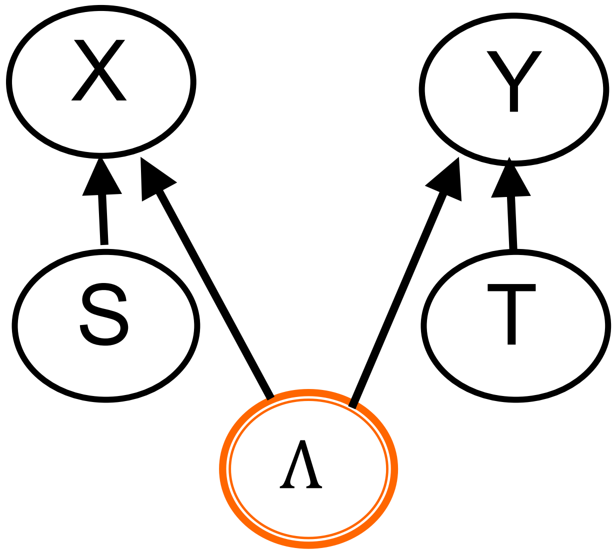
Inequality constraints

Some causal structures that admit of inequality constraints



Inequality constraints by brute-force quantifier elimination

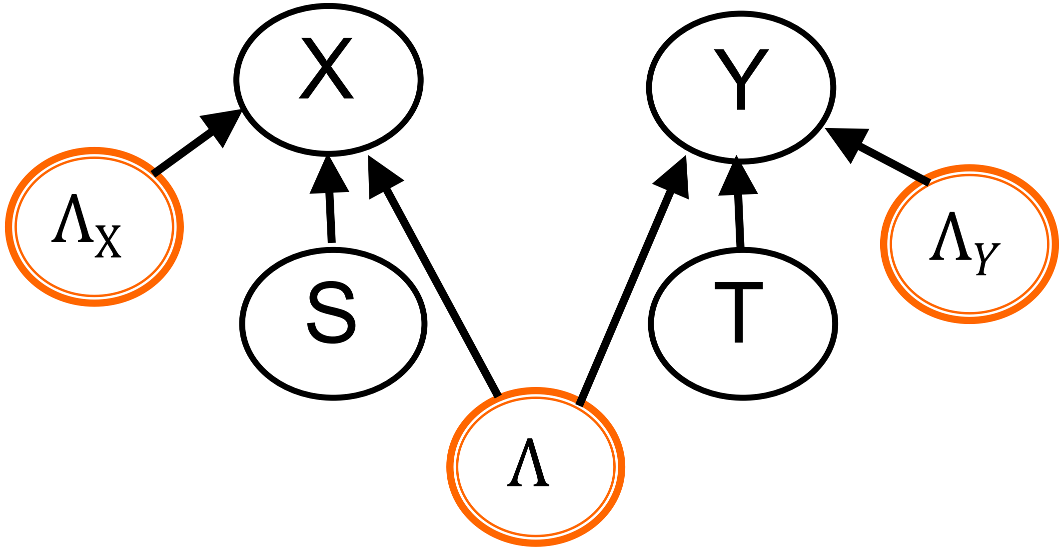
Bell scenario



$P_{X|S\Lambda}$
$P_{Y|T\Lambda}$
P_{Λ}

$$P_{XY|ST} = \Sigma_{\Lambda}\, P_{Y|T\Lambda} P_{X|S\Lambda} P_{\Lambda}$$

Bell scenario

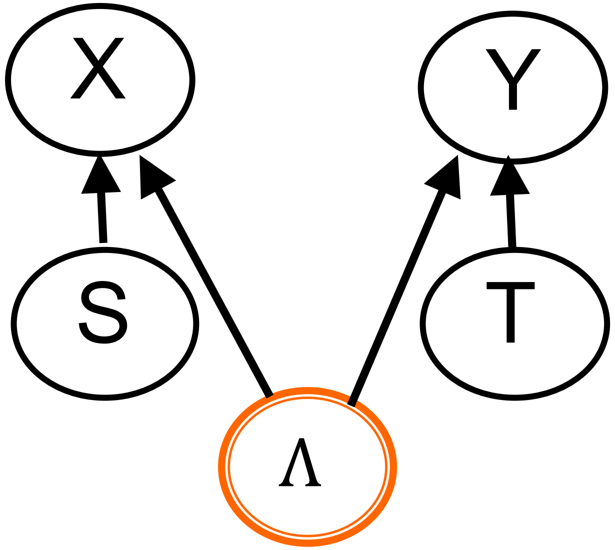


$$X = f_X(S, \Lambda, \Lambda_X)$$
$$Y = f_Y(T, \Lambda, \Lambda_Y)$$
$$P_\Lambda$$
$$P_{\Lambda_X}$$
$$P_{\Lambda_Y}$$

$$P_{XY|ST} = \sum_{\Lambda,\Lambda_X,\Lambda_Y} \delta_{X, f_X(S,\Lambda,\Lambda_X)} \delta_{Y, f_Y(T,\Lambda,\Lambda_Y)} P_\Lambda P_{\Lambda_X} P_{\Lambda_Y}$$

$$P_{X|S\Lambda} = \sum_{\Lambda_X} \delta_{X, f_X(S,\Lambda,\Lambda_X)} P_{\Lambda_X}$$
$$P_{Y|T\Lambda} = \sum_{\Lambda_Y} \delta_{Y, f_Y(T,\Lambda,\Lambda_Y)} P_{\Lambda_Y}$$

Bell scenario

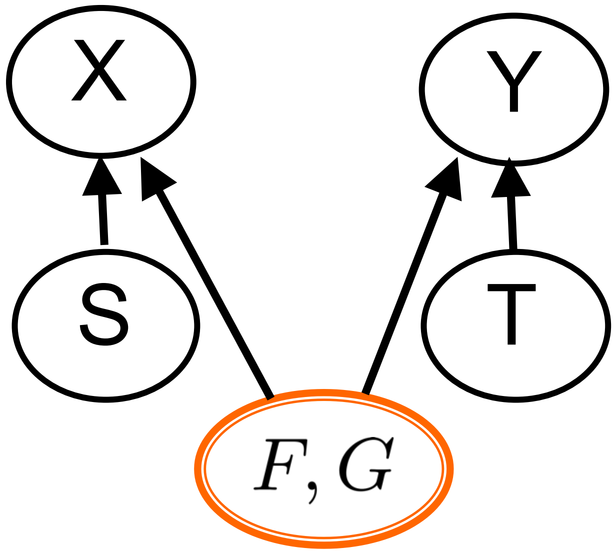


$$X = f_X(S, \Lambda)$$
$$Y = f_Y(T, \Lambda)$$
$$P_\Lambda$$

$$P_{XY|ST} = \Sigma_\Lambda \delta_{X, f_X(S,\Lambda)} \delta_{Y, f_Y(T,\Lambda)} P_\Lambda$$

One can absorb local hidden variables into the common cause hidden variable

Bell scenario



$P_{F,G}$

$$P_{XY|ST} = \sum_{f,g} \delta_{X,f(S)} \delta_{Y,g(T)} P_{F,G}(f,g)$$

Gearing: The hidden variable can be taken to stipulate the functional dependences X=f(S) and Y=g(T)

$$P_{XY|ST} = \sum_{f,g} \delta_{X,f(S)} \delta_{Y,g(T)} P_{F,G}(f,g)$$

If X,Y,S,T are binary, Λ can have cardinality 16

$$p_{xy|st} := P_{XY|ST}(xy|st) \qquad q_{fg} := P_{F,G}(f,g)$$

$$x, y, s, t \in \{0, 1\} \qquad f, g \in \{\mathbb{I}, \mathbb{F}, \mathbb{R}_0, \mathbb{R}_1\}$$

$$p_{00|00} = q_{\mathbb{R}_0,\mathbb{R}_0} + q_{\mathbb{R}_0,\mathbb{I}} + q_{\mathbb{I},\mathbb{R}_0} + q_{\mathbb{I},\mathbb{I}}$$

$$p_{00|01} = q_{\mathbb{R}_0,\mathbb{R}_0} + q_{\mathbb{R}_0,\mathbb{F}} + q_{\mathbb{I},\mathbb{R}_0} + q_{\mathbb{I},\mathbb{F}}$$

$$\vdots$$

$$0 \leq q_{fg} \leq 1 \; \forall f, g$$

16 linear equalities + inequalities

Do linear quantifier elimination on the 16 q's.

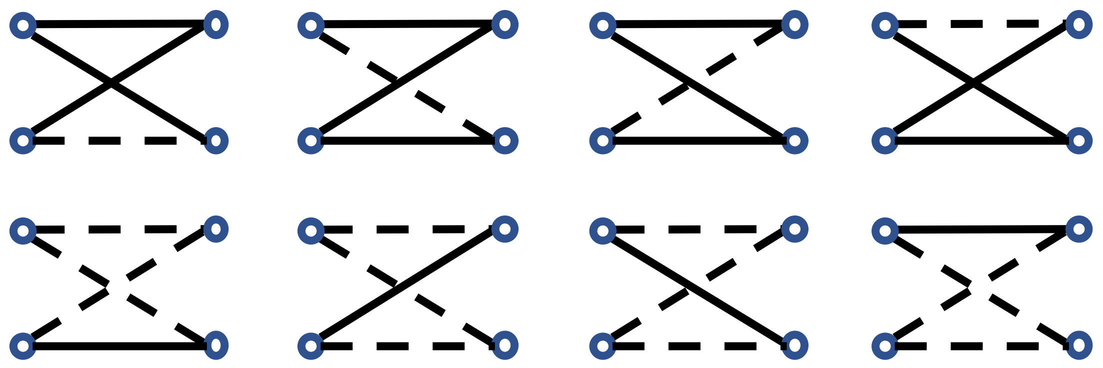
This yields the conditional independence relations and the 8 CHSH inequalities

$$P_{X|ST} = P_{X|S}$$
$$P_{Y|ST} = P_{Y|T}$$

$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \frac{3}{4}$$

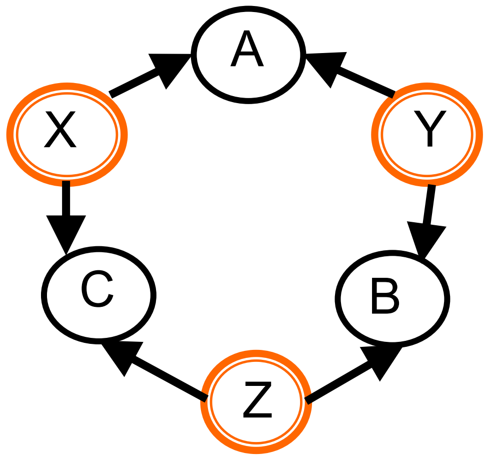
+ 7 others

Corresponding to the 8 frustrated four-node networks



Clauser, Horne, Shimony and Holte, Phys. Rev. Lett.23, 880 (1967)

DAGs for which one can deduce the cardinalities of latent variables in this manner are termed **gearable**
(R. Evans, Annals of Statistics, **46**, 2623 (2018))



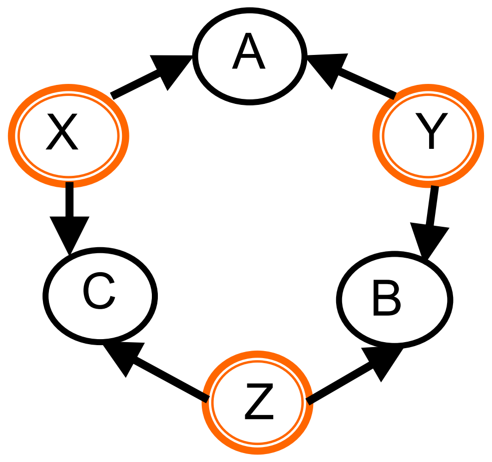
Triangle

The triangle scenario is an example of a DAG that is not gearable

Techniques for determining upper bounds on cardinalities of the latent variables in more general causal structures

R. Evans, Annals of Statistics, 46, 2623 (2018)

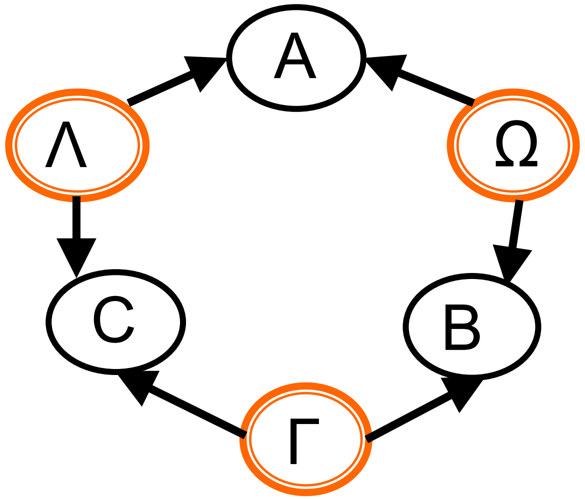
D. Rosset, N. Gisin, and E. Wolfe. Quantum Inf. & Comp. **18,** 0910 (2018)



A, B, C binary →

Sufficient for
X, Y, Z to be 6-valued

Causal structure



Parameters

$\mathrm{P}_{A|\Lambda\Omega}$

$\mathrm{P}_{B|\Omega\Gamma}$

$\mathrm{P}_{C|\Gamma\Lambda}$

P_{Λ}

P_{Ω}

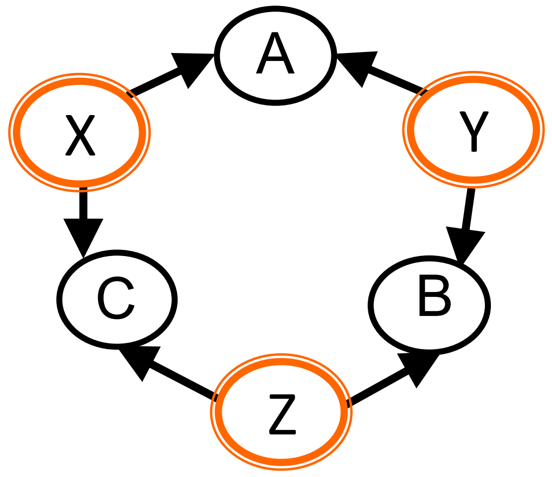
P_{Γ}

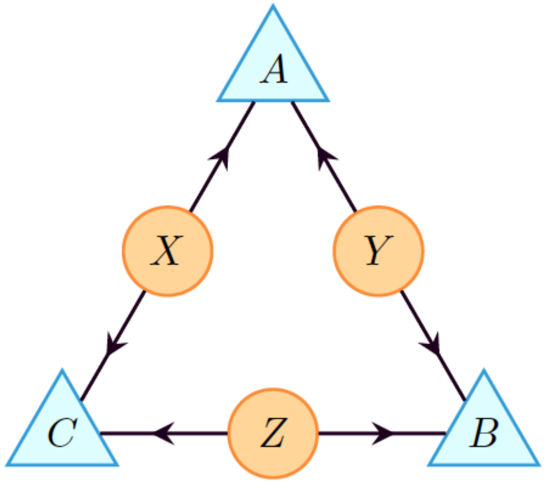
$$P_{ABC} = \sum_{\Lambda\Omega\Gamma} \mathrm{P}_{A|\Lambda\Omega}\mathrm{P}_{B|\Omega\Gamma}\mathrm{P}_{C|\Gamma\Lambda}\mathrm{P}_{\Lambda}\mathrm{P}_{\Omega}\mathrm{P}_{\Gamma}$$

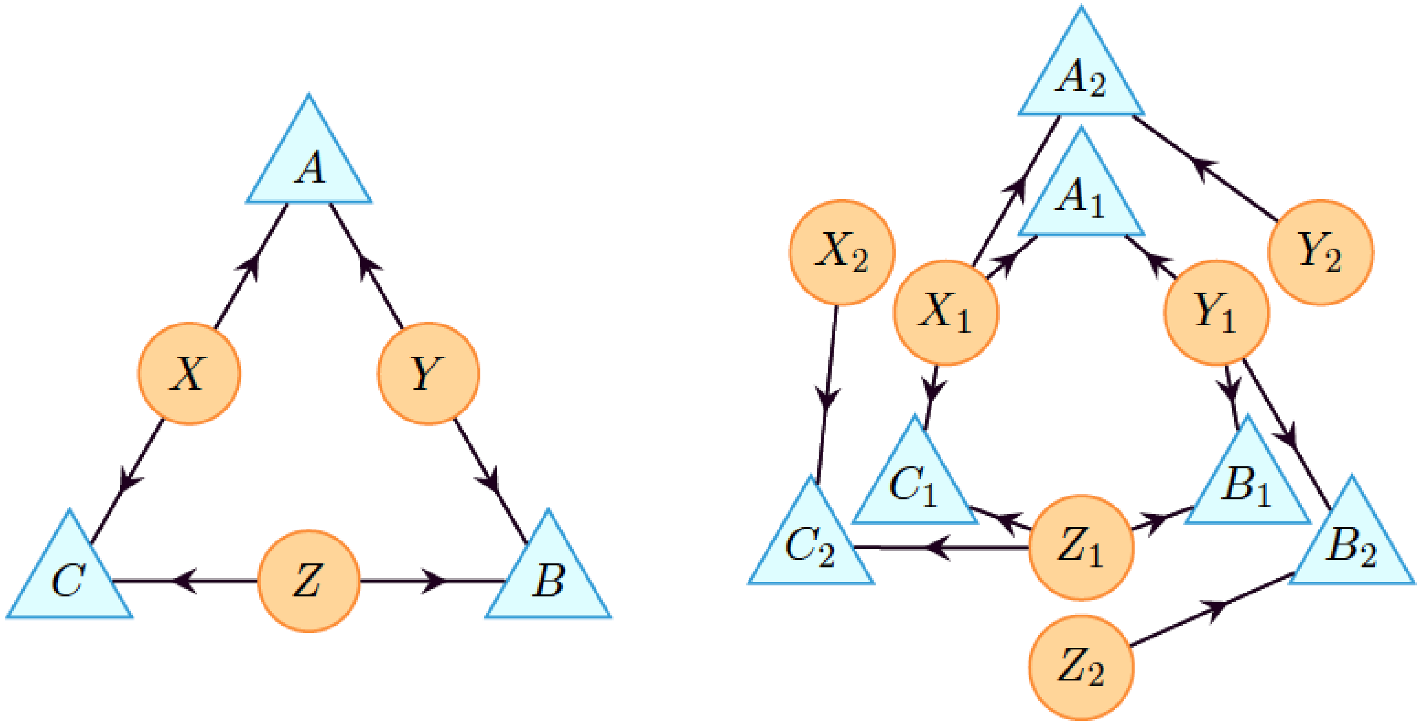
With more than one latent variable,
we require **nonlinear** quantifier elimination
which scales badly

The inflation technique for deriving inequality constraints

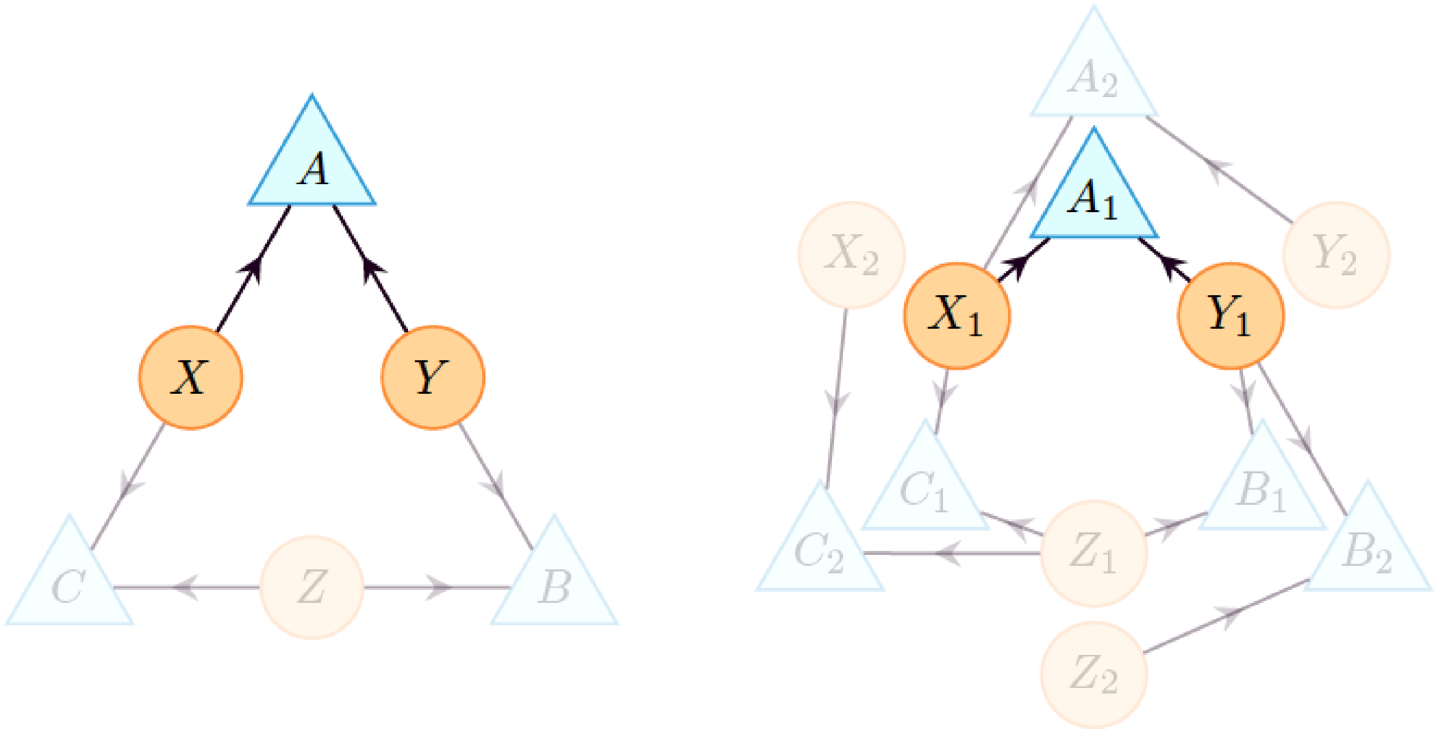
Inflation DAGs


A
X
Y
C
B
Z

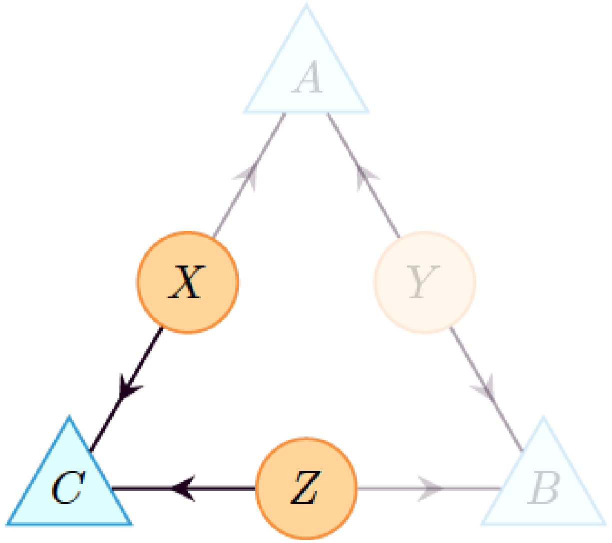

A
X
Y
C
Z
B



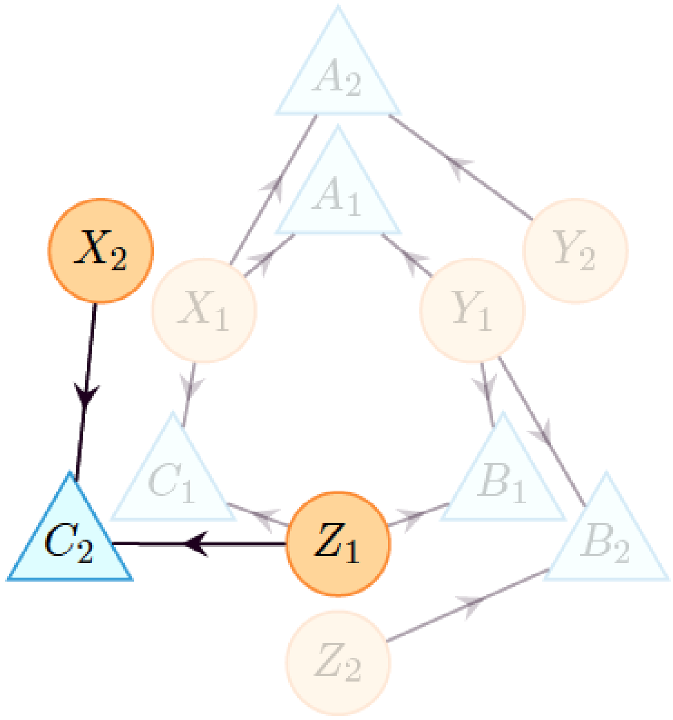
A
X
Y
C
Z
B
A_2
A_1
X_2
X_1
Y_1
Y_2
C_1
B_1
C_2
Z_1
B_2
Z_2



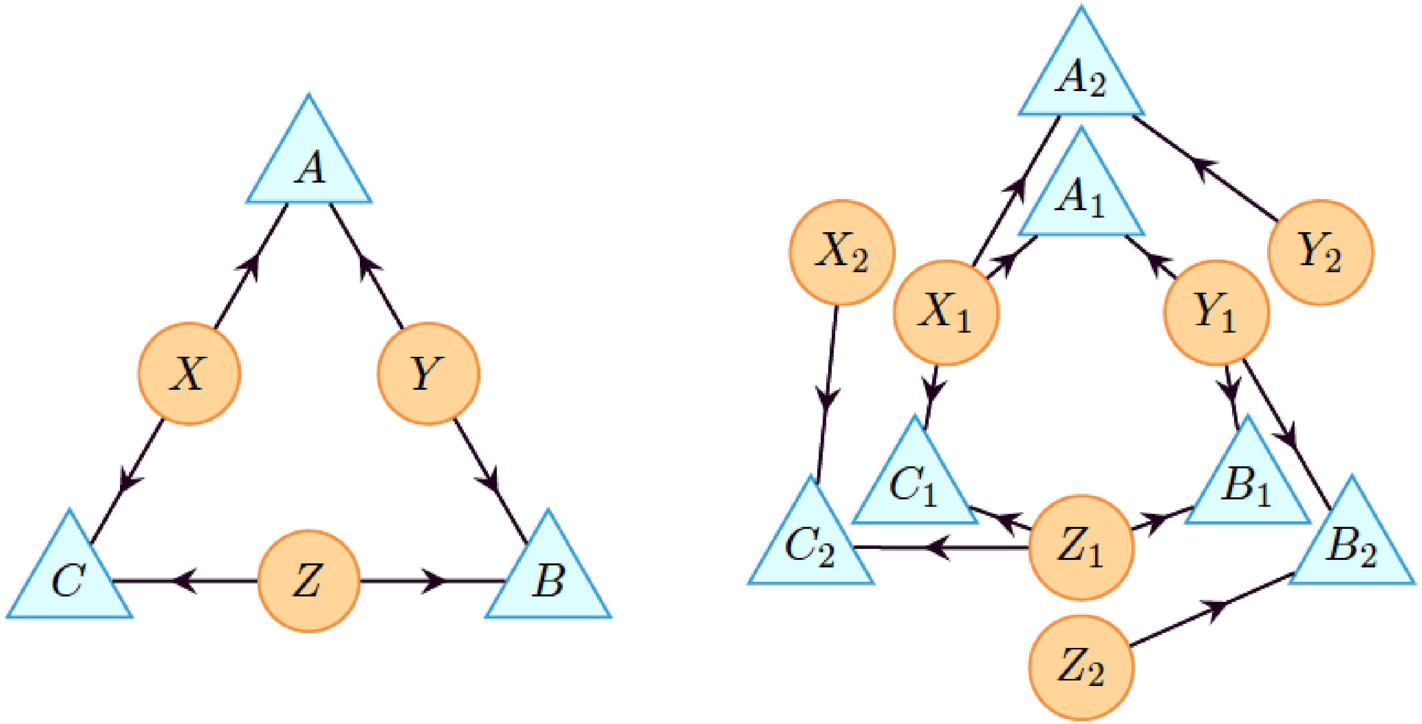
A
X
Y
C
Z
B
A_2
A_1
X_2
Y_2
X_1
Y_1
C_1
B_1
C_2
Z_1
B_2
Z_2



A
X
Y
C
Z
B



A2
A1
X2
X1
Y1
Y2
C1
B1
C2
Z1
B2
Z2



A
X
Y
C
Z
B
A_2
A_1
X_2
X_1
Y_1
Y_2
C_1
B_1
C_2
Z_1
B_2
Z_2

Sameness up to copy-indices

Let **X** ⊆ Nodes(G), **X’** ⊆ Nodes(G’)

X $\sim$ **X’**

denotes that **X’** contains exactly one copy of every node in **X**

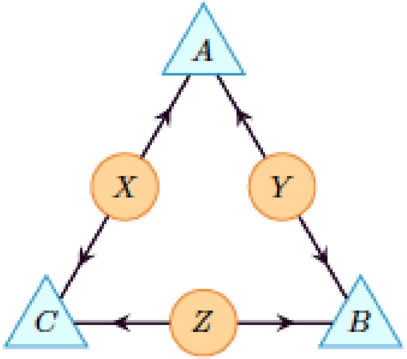
$$\mathrm{ansubgraph}_{G'}(\mathbf{X}') \sim \mathrm{ansubgraph}_{G}(\mathbf{X}).$$

Denotes that **X** $\sim$ **X’** and that an edge is present between two nodes in **X’** iff it is present between the two associated nodes in **X**.

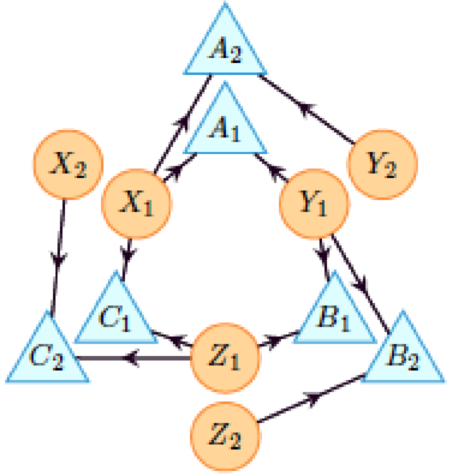
$$G' \in \mathrm{inflations}(G)$$

if and only if

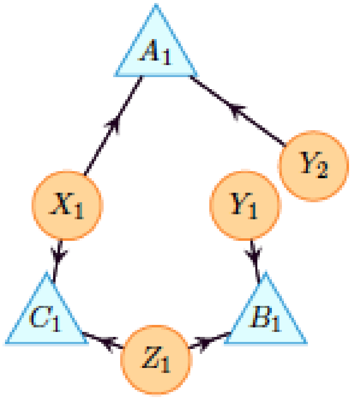
$$\forall A_i \in \mathrm{nodes}(G') : \ \mathrm{ansubgraph}_{G'}(A_i) \sim \mathrm{ansubgraph}_G(A).$$



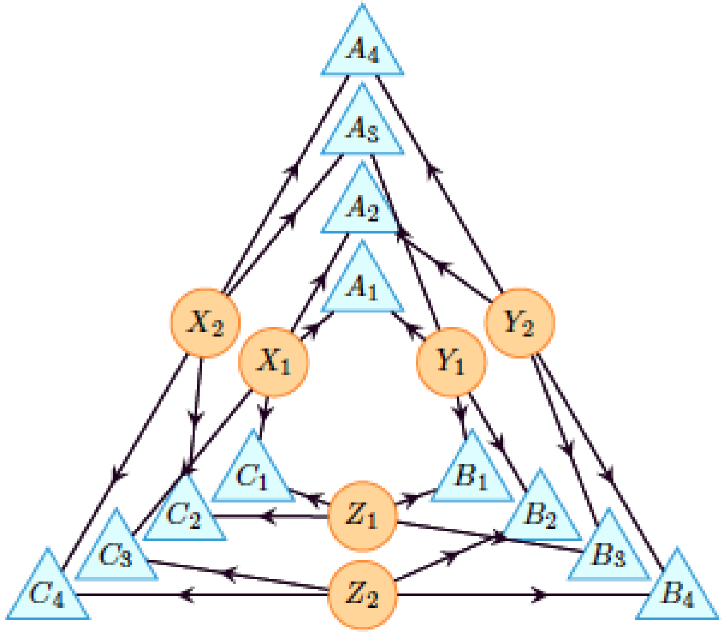
The Triangle Scenario



Spiral Inflation

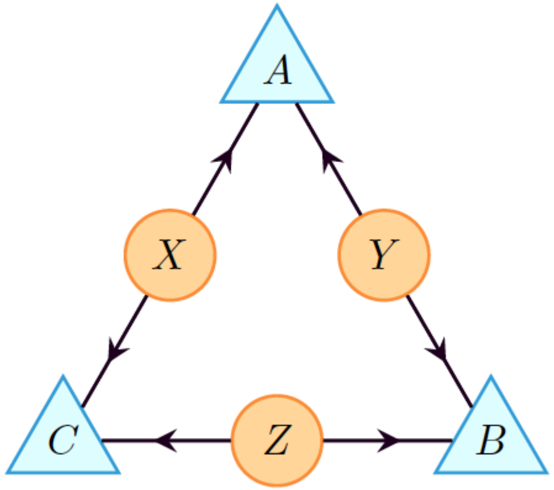


Cut Inflation



Large Inflation

model M on DAG G



$\mathrm{P}_{A|XY}$

$\mathrm{P}_{B|YZ}$

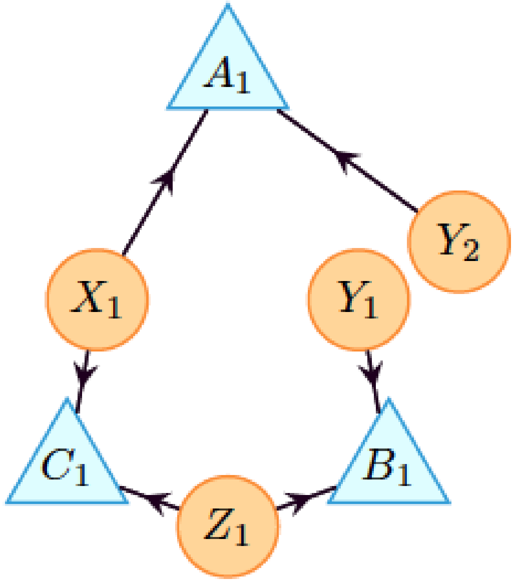
$\mathrm{P}_{C|XZ}$

P_{X}

P_{Y}

P_{Z}

M’ = G $\rightarrow$ G’ Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$

$\mathrm{P}_{B_1|Y_1Z_1}$

$\mathrm{P}_{C_1|X_1Z_1}$

P_{X_1}

P_{Y_1}

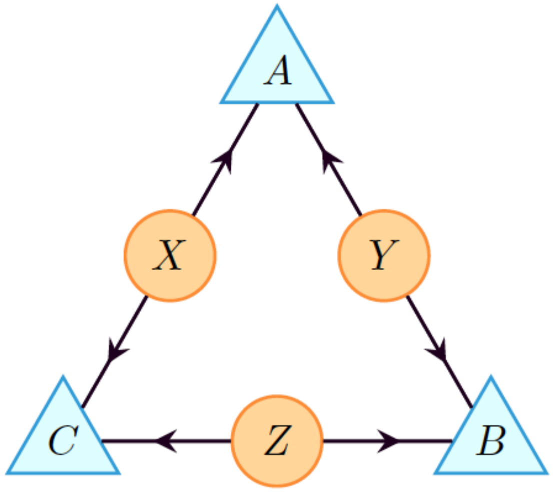
P_{Y_2}

P_{Z_1}

with symmetry constraint:

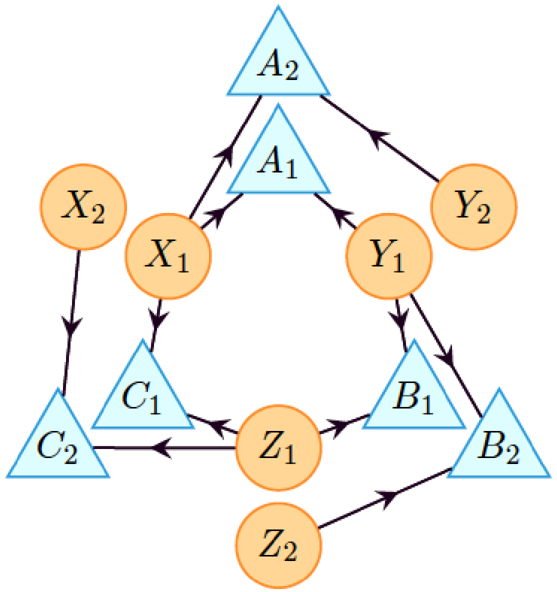
$$\mathrm{P}_{Y_1} = \mathrm{P}_{Y_2}$$

model M on DAG G



$$P_{A|XY}$$
$$P_{B|YZ}$$
$$P_{C|XZ}$$
$$P_X$$
$$P_Y$$
$$P_Z$$

M' = G $\rightarrow$ G' Inflation of M



$$P_{A_1|X_1Y_1} \qquad P_{A_2|X_1Y_2}$$
$$P_{B_1|Y_1Z_1} \qquad P_{B_2|Y_1Z_2}$$
$$P_{C_1|X_1Z_1} \qquad P_{C_2|X_2Z_1}$$
$$P_{X_1} \qquad P_{X_2}$$
$$P_{Y_1} \qquad P_{Y_2}$$
$$P_{Z_1} \qquad P_{Z_2}$$

with symmetry constraints:

$$P_{A_1|X_1Y_1} = P_{A_2|X_1Y_2}$$
$$P_{B_1|Y_1Z_1} = P_{B_2|Y_1Z_2}$$
$$P_{C_1|X_1Z_1} = P_{C_2|X_2Z_1}$$
$$P_{X_1} = P_{X_2}$$
$$P_{Y_1} = P_{Y_2}$$
$$P_{Z_1} = P_{Z_2}$$

Suppose $G' \in \text{inflations}(G)$

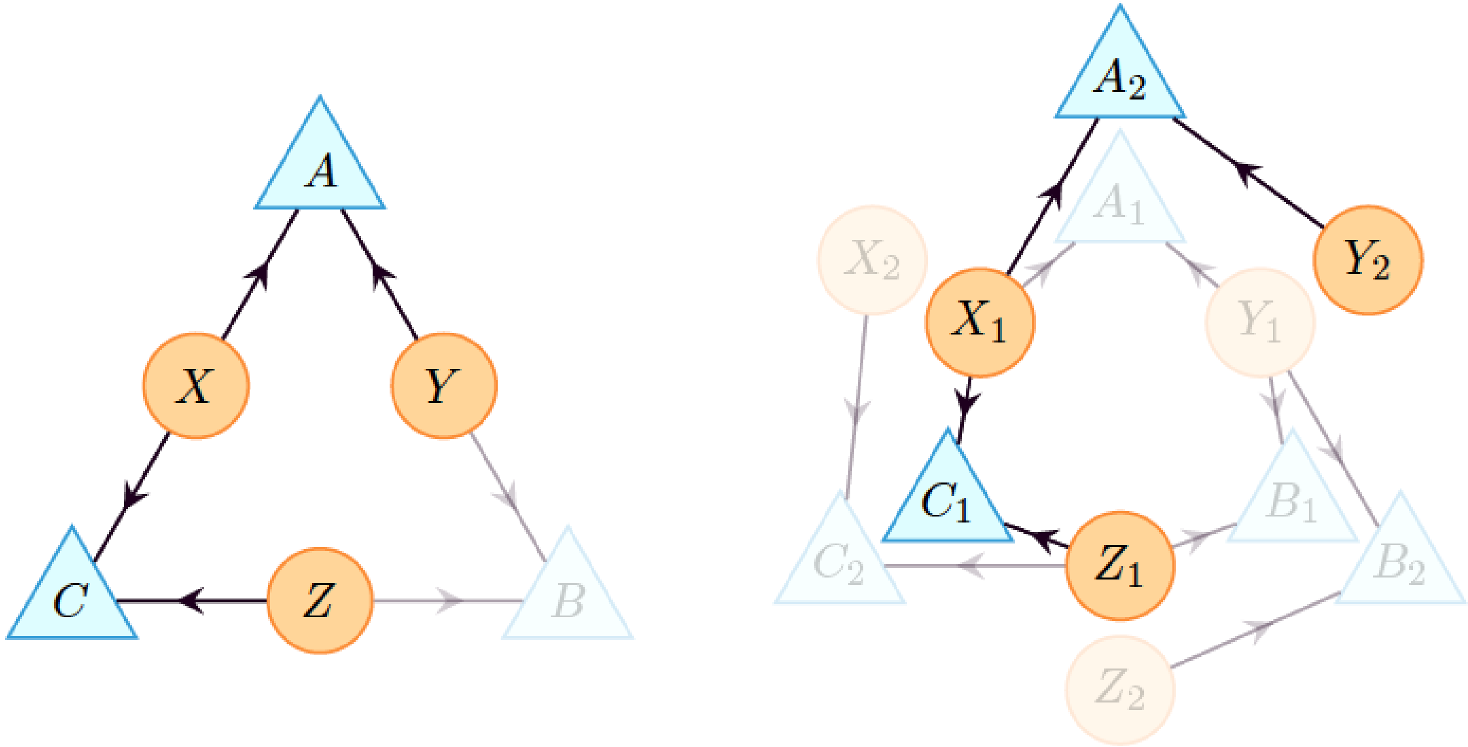
then

$$M' = \text{Inflation}_{G \to G'}(M)$$

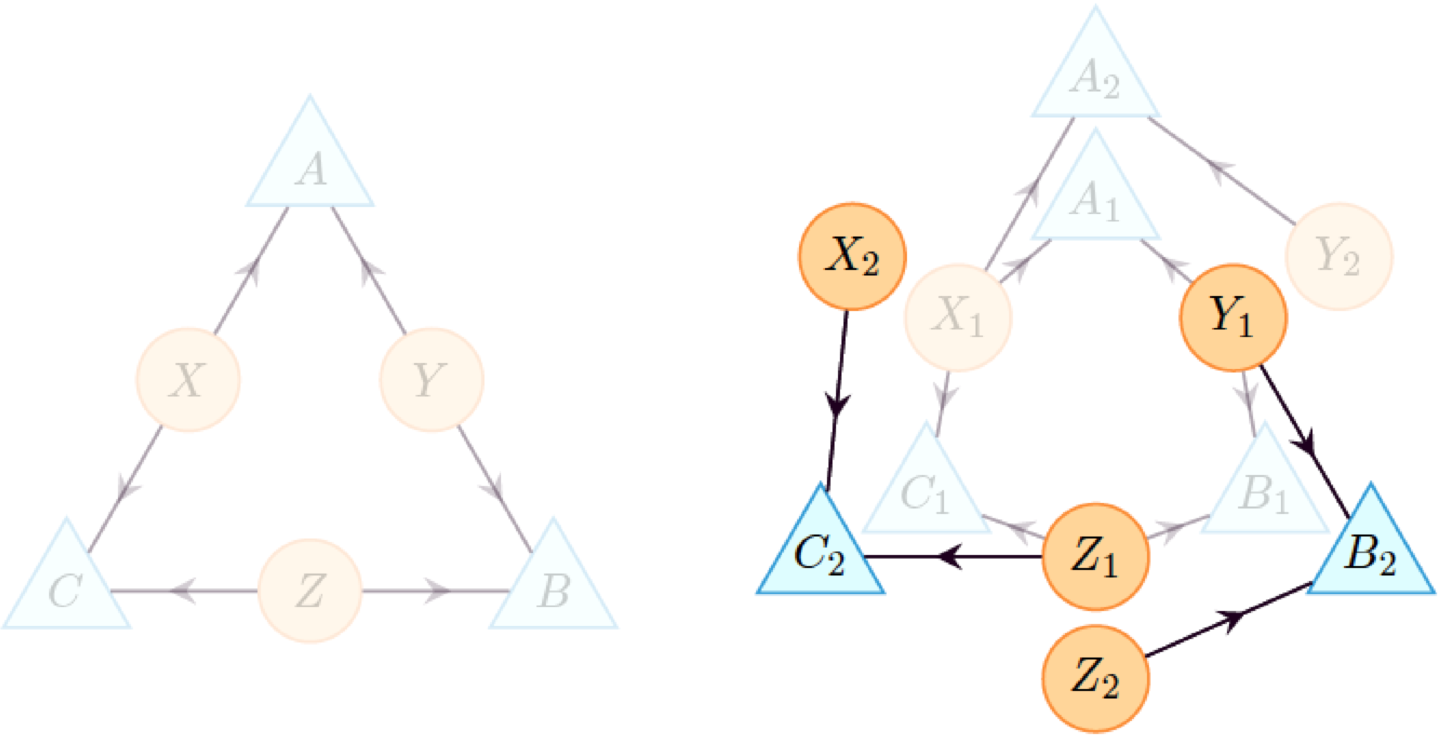
if and only if

$$\forall A_i, A_j \in \text{nodes}(G') : \; \mathrm{P}_{A_i | \mathrm{Pa}_{G'}(A_i)} = \mathrm{P}_{A_j | \mathrm{Pa}_{G'}(A_j)}.$$

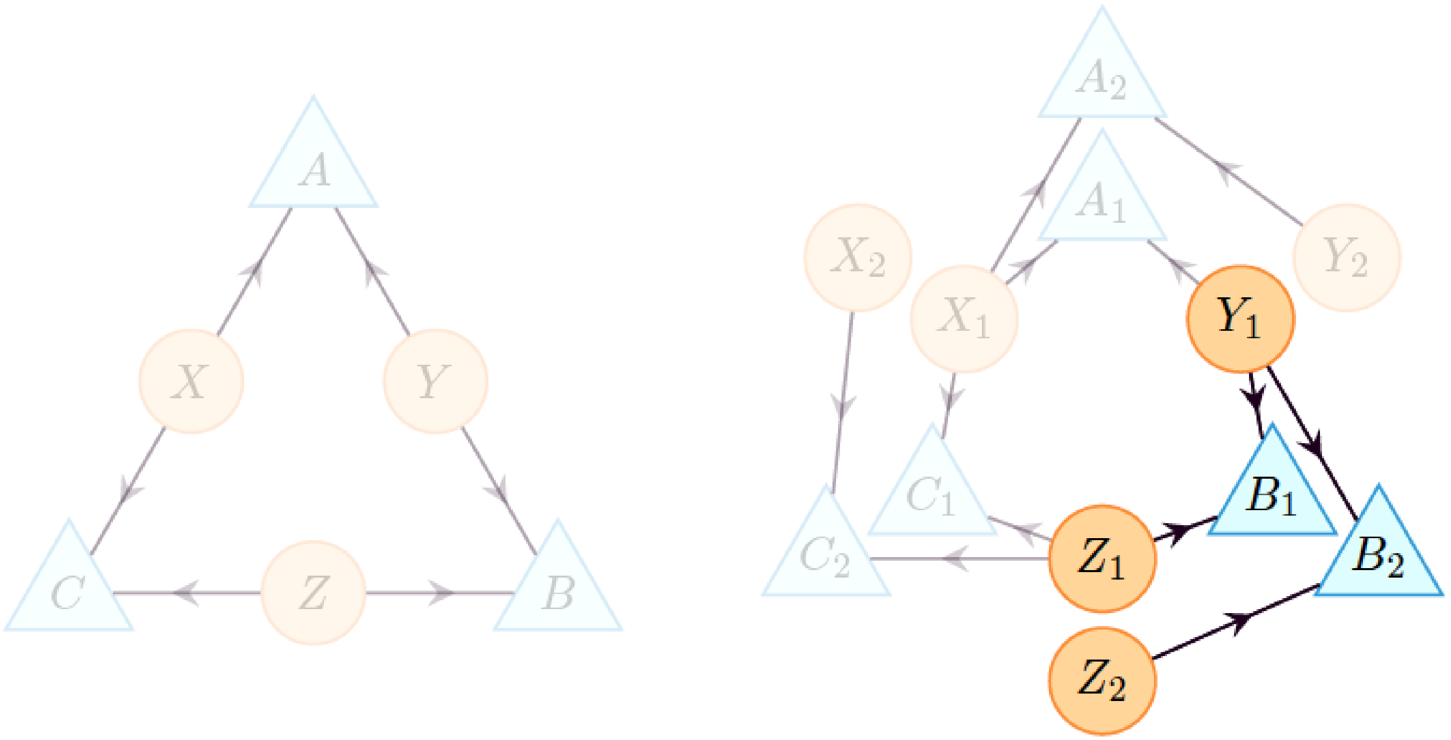
Injectable sets of observed variables
in the inflation DAG



$\{A_2 C_1\}$ is an injectable set



$\{B_2C_2\}$ is *not* an injectable set



$\{B_1 B_2\}$ is *not* an injectable set

Suppose $G' \in \mathrm{inflations}(G)$

Consider

$$\mathbf{V} \subseteq \mathrm{ObservedNodes}(G)$$
$$\mathbf{V}' \subseteq \mathrm{ObservedNodes}(G')$$

such that

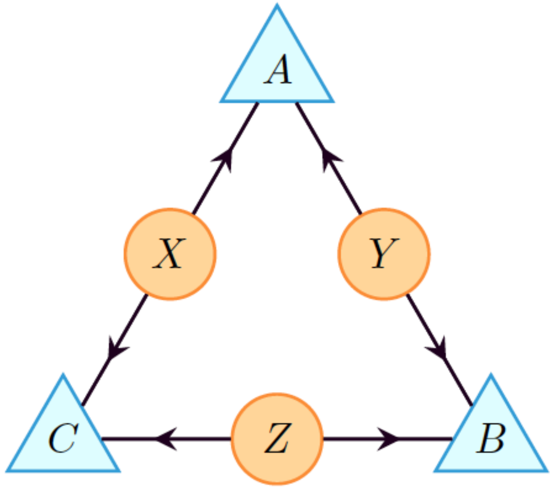
$$\mathbf{V}' \sim \mathbf{V}$$

$\mathbf{V}' \in \mathrm{InjectableSets}(G')$ and $\mathbf{V} \in \mathrm{ImagesInjectableSets}(G)$

if and only if

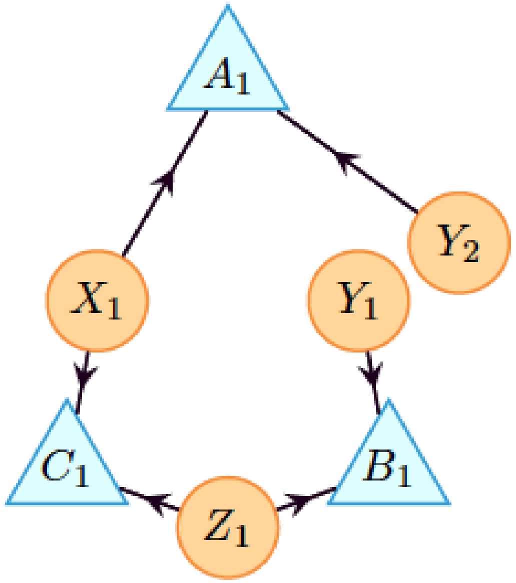
$$\mathrm{ansubgraph}_{G'}(\mathbf{V}') \sim \mathrm{ansubgraph}_G(\mathbf{V}).$$

model M on DAG G



$\mathrm{P}_{A|XY}$
$\mathrm{P}_{B|YZ}$
$\mathrm{P}_{C|XZ}$
P_{X}
P_{Y}
P_{Z}

M’ = G $\rightarrow$ G’ Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$
$\mathrm{P}_{B_1|Y_1Z_1}$
$\mathrm{P}_{C_1|X_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

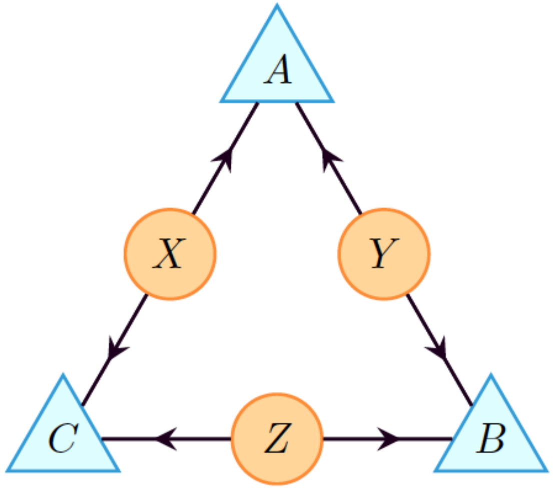
$\{A_1C_1\}$ is an injectable set

$$\mathrm{P}_{A_1C_1} = \sum_{X_1Y_2Z_1} \mathrm{P}_{A_1|X_1Y_2}\mathrm{P}_{C_1|X_1Z_1}\mathrm{P}_{X_1}\mathrm{P}_{Y_2}\mathrm{P}_{Z_1}$$

$$\mathrm{P}_{AC} = \sum_{XYZ} \mathrm{P}_{A|XY}\mathrm{P}_{C|XZ}\mathrm{P}_{X}\mathrm{P}_{Y}\mathrm{P}_{Z}$$

$$P_{AC} \text{ compatible with } M \quad\Longrightarrow\quad P_{A_1C_1} = P_{AC} \text{ compatible with } M'$$

model M on DAG G



$P_{A|XY}$

$P_{B|YZ}$

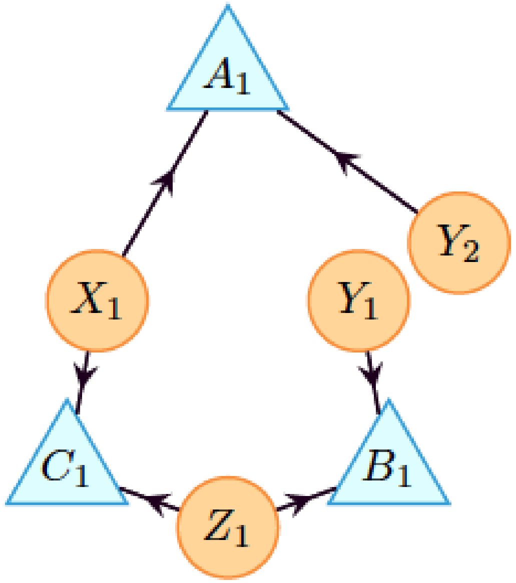
$P_{C|XZ}$

P_X

P_Y

P_Z

M’ = G $\rightarrow$ G’ Inflation of M



$P_{A_1|X_1Y_2}$

$P_{B_1|Y_1Z_1}$

$P_{C_1|X_1Z_1}$

P_{X_1}

P_{Y_1}

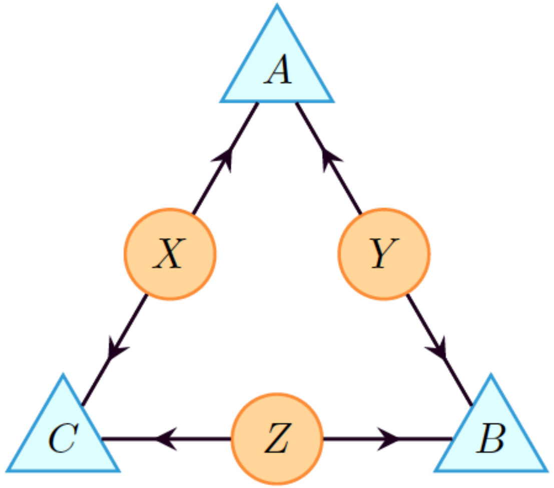
P_{Y_2}

P_{Z_1}

$\{A_1B_1\}$ is *not* an injectable set

$$P_{A_1B_1} = \sum_{X_1Y_1Y_2Z_1} P_{A_1|X_1Y_2} P_{B_1|Y_1Z_1} P_{X_1} P_{Y_1} P_{Y_2} P_{Z_1}$$

model M on DAG G



$\mathrm{P}_{A|XY}$

$\mathrm{P}_{B|YZ}$

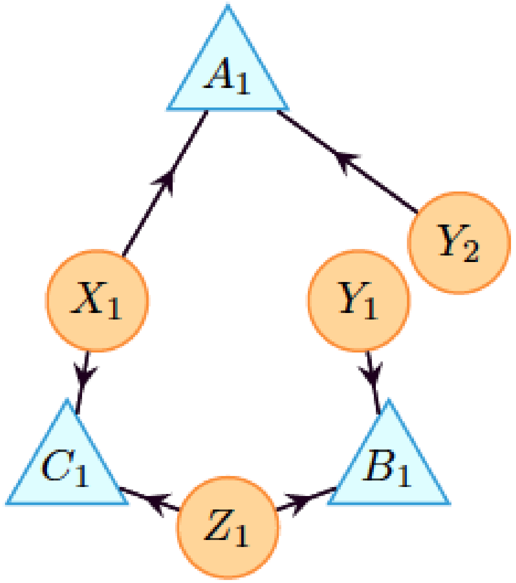
$\mathrm{P}_{C|XZ}$

P_{X}

P_{Y}

P_{Z}

M’ = G $\longrightarrow$ G’ Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$

$\mathrm{P}_{B_1|Y_1Z_1}$

$\mathrm{P}_{C_1|X_1Z_1}$

P_{X_1}

P_{Y_1}

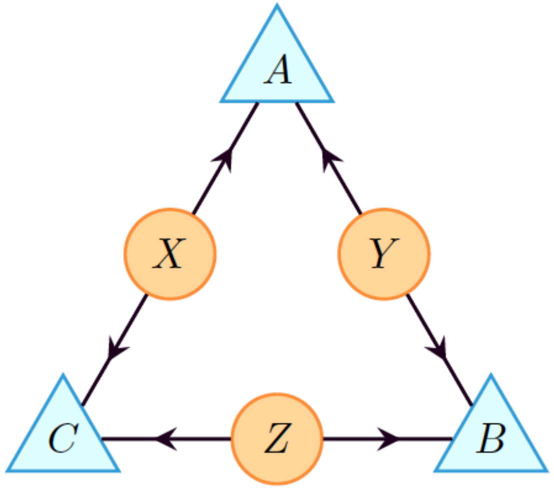
P_{Y_2}

P_{Z_1}

$\{A_1B_1\}$ is *not* an injectable set

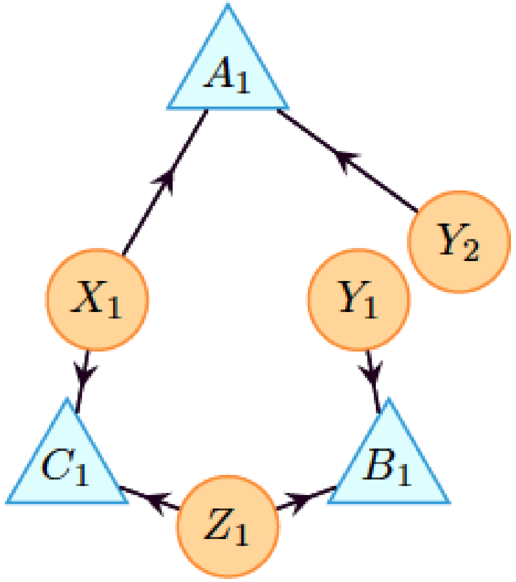
$$\mathrm{P}_{A_1B_1} = \left(\sum_{X_1Y_2} \mathrm{P}_{A_1|X_1Y_2}\mathrm{P}_{Y_2}\mathrm{P}_{X_1}\right)\left(\sum_{Z_1Y_1} \mathrm{P}_{B_1|Y_1Z_1}\mathrm{P}_{Y_1}\mathrm{P}_{Z_1}\right)$$

model M on DAG G



$\mathrm{P}_{A|XY}$
$\mathrm{P}_{B|YZ}$
$\mathrm{P}_{C|XZ}$
P_{X}
P_{Y}
P_{Z}

M’ = G $\rightarrow$ G’ Inflation of M



$\mathrm{P}_{A_1|X_1Y_2}$
$\mathrm{P}_{B_1|Y_1Z_1}$
$\mathrm{P}_{C_1|X_1Z_1}$
P_{X_1}
P_{Y_1}
P_{Y_2}
P_{Z_1}

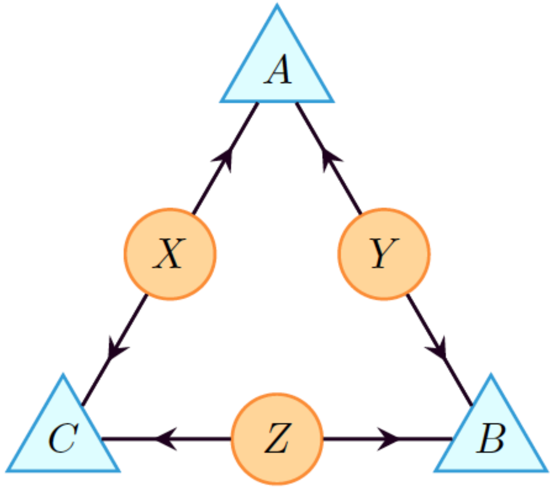
$\{A_1B_1\}$ is *not* an injectable set

$$\mathrm{P}_{A_1B_1} = \left(\sum_{X_1Y_2} \mathrm{P}_{A_1|X_1Y_2}\mathrm{P}_{Y_2}\mathrm{P}_{X_1}\right)\left(\sum_{Z_1Y_1} \mathrm{P}_{B_1|Y_1Z_1}\mathrm{P}_{Y_1}\mathrm{P}_{Z_1}\right)$$

$$\mathrm{P}_{AB} = \sum_{XYZ} \mathrm{P}_{A|XY}\mathrm{P}_{B|YZ}\mathrm{P}_{X}\mathrm{P}_{Y}\mathrm{P}_{Z}$$

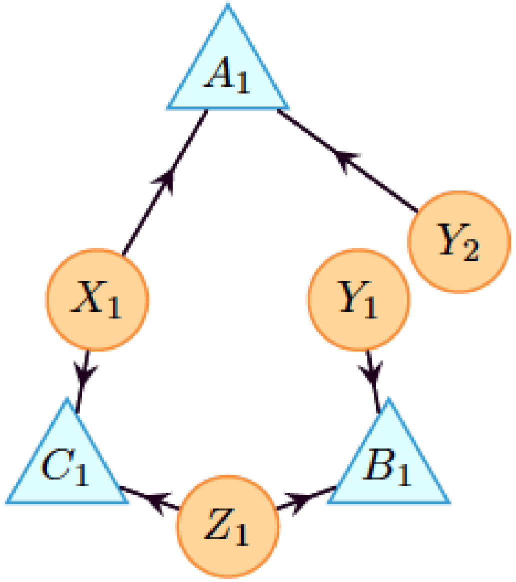
$$P_{AB} \text{ compatible with } M \quad \not\Longrightarrow \quad P_{A_1B_1} = P_{AB} \text{ compatible with } M'$$

model M on DAG G



$$\mathrm{P}_{A|XY} \quad \mathrm{P}_{B|YZ} \quad \mathrm{P}_{C|XZ} \quad \mathrm{P}_{X} \quad \mathrm{P}_{Y} \quad \mathrm{P}_{Z}$$

M' = G → G' Inflation of M



$$\mathrm{P}_{A_1|X_1Y_2} \quad \mathrm{P}_{B_1|Y_1Z_1} \quad \mathrm{P}_{C_1|X_1Z_1} \quad \mathrm{P}_{X_1} \quad \mathrm{P}_{Y_1} \quad \mathrm{P}_{Y_2} \quad \mathrm{P}_{Z_1}$$

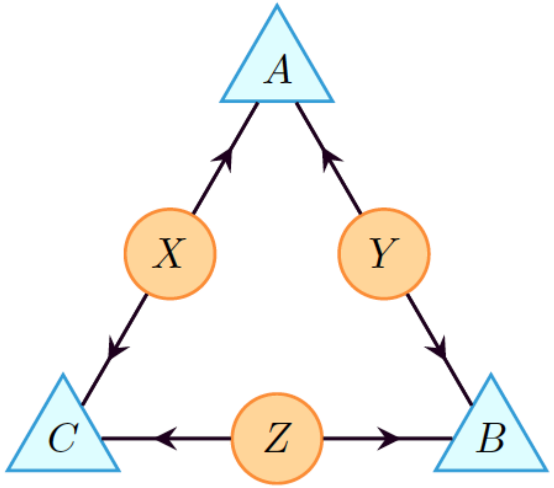
Injectable sets: $\{A_1\}, \{B_1\}, \{C_1\}, \{A_1C_1\}, \{B_1C_1\}$

$$\begin{array}{c}(P_A, P_B, P_C, P_{AC}, P_{BC}) \\ \text{compatible with } M\end{array} \implies \begin{array}{c}(P_{A_1}, P_{B_1}, P_{C_1}, P_{A_1C_1}, P_{B_1C_1}) \\ \text{compatible with } M'\end{array}$$

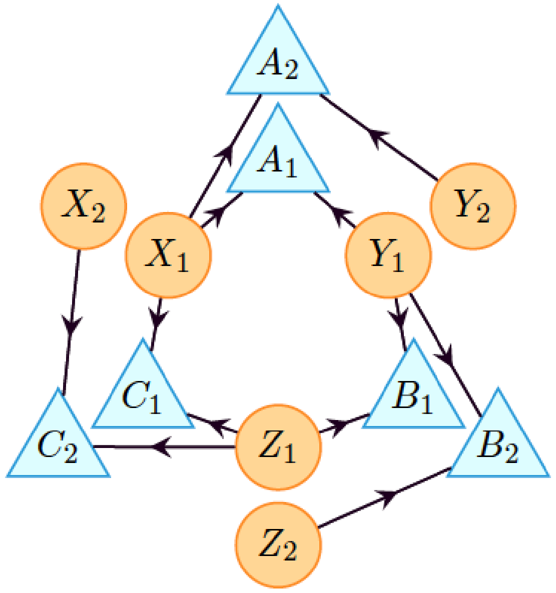
where

$$\mathrm{P}_{A_1} = \mathrm{P}_A \qquad \mathrm{P}_{A_1C_1} = \mathrm{P}_{AC}$$
$$\mathrm{P}_{B_1} = \mathrm{P}_B \qquad \mathrm{P}_{B_1C_1} = \mathrm{P}_{BC}$$
$$\mathrm{P}_{C_1} = \mathrm{P}_C$$

model M on DAG G



M' = G $\rightarrow$ G' Inflation of M



Injectable sets: $\{A_1\}, \{B_1\}, \{C_1\}, \{A_2\}, \{B_2\}, \{C_2\},$
$\{A_1B_1\}, \{A_1, B_2\}, \{B_1C_1\}, \{B_1, C_2\}, \{C_1, A_1\}, \{C_1, A_2\},$
$\{A_1B_1C_1\}$

M' = G $\rightarrow$ G' Inflation of M

$\mathcal{S} \subseteq \mathrm{ImagesInjectableSets}(G)$

$\mathcal{S}' \subseteq \mathrm{InjectableSets}(G')$

$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$
is compatible with M

$\Longrightarrow$

$\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$
where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$
is compatible with M'

M’ = G $\rightarrow$ G’ Inflation of M

$\mathcal{S} \subseteq \mathrm{ImagesInjectableSets}(G)$

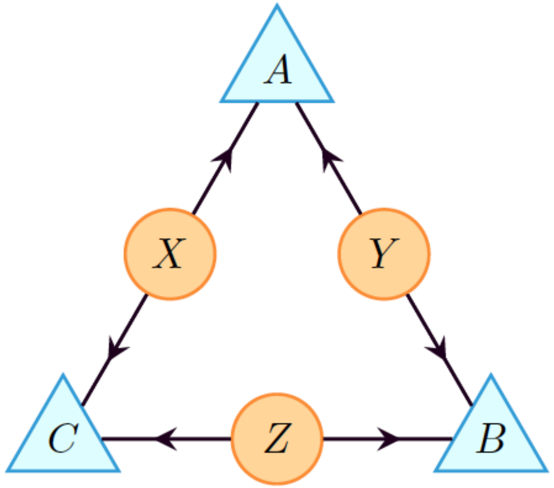
$\mathcal{S}' \subseteq \mathrm{InjectableSets}(G')$

$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$
is not compatible with M

$\Longleftarrow$

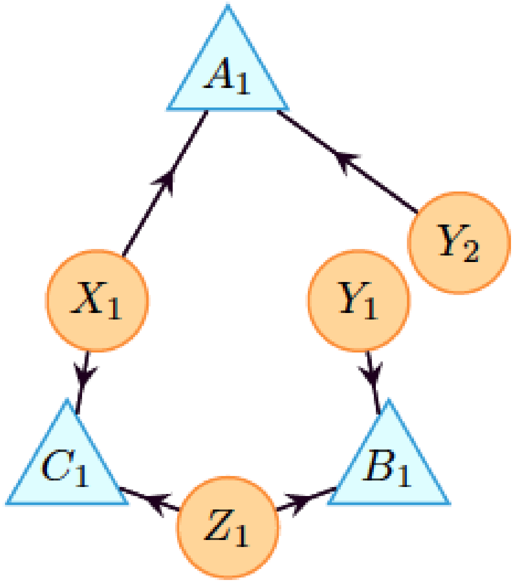
$\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$
where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$
is not compatible with M'

model M on DAG G



$$\mathrm{P}_{A|XY}$$
$$\mathrm{P}_{B|YZ}$$
$$\mathrm{P}_{C|XZ}$$
$$\mathrm{P}_{X}$$
$$\mathrm{P}_{Y}$$
$$\mathrm{P}_{Z}$$

M' = G $\to$ G' Inflation of M



$$\mathrm{P}_{A_1|X_1Y_2}$$
$$\mathrm{P}_{B_1|Y_1Z_1}$$
$$\mathrm{P}_{C_1|X_1Z_1}$$
$$\mathrm{P}_{X_1}$$
$$\mathrm{P}_{Y_1}$$
$$\mathrm{P}_{Y_2}$$
$$\mathrm{P}_{Z_1}$$

Injectable sets: $\{A_1\}, \{B_1\}, \{C_1\}, \{A_1C_1\}, \{B_1C_1\}$

$$(P_A, P_B, P_C, P_{AC}, P_{BC}) \text{ compatible with } M \implies (P_{A_1}, P_{B_1}, P_{C_1}, P_{A_1C_1}, P_{B_1C_1}) \text{ compatible with } M'$$

where
$$\mathrm{P}_{A_1} = \mathrm{P}_A \qquad \mathrm{P}_{A_1C_1} = \mathrm{P}_{AC}$$
$$\mathrm{P}_{B_1} = \mathrm{P}_B \qquad \mathrm{P}_{B_1C_1} = \mathrm{P}_{BC}$$
$$\mathrm{P}_{C_1} = \mathrm{P}_C$$

Deriving
causal compatibility inequalities
by the inflation technique

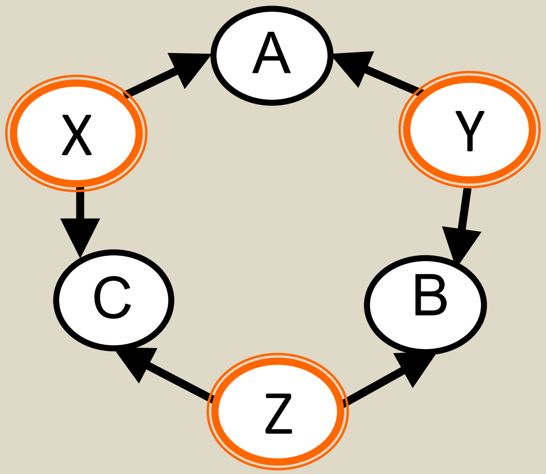
Let $I_{\mathcal{S}}$ be an inequality that acts on the family of distributions $\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

such that whenever

$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$ is compatible with M $\Longrightarrow$ $I_{\mathcal{S}}$ is **satisfied** for $\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

we say that

$I_{\mathcal{S}}$ is a **causal compatibility inequality** for model M



$$\boxed{P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1}$$

causal compatibility inequality

M' = G $\rightarrow$ G' Inflation of M

$\mathcal{S} \subseteq \text{ImagesInjectableSets}(G)$ $\qquad$ $\mathcal{S}' \subseteq \text{InjectableSets}(G')$

$I_{\mathcal{S}'}$ is a **causal compatibility inequality** for model M'

$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$
is compatible with M

$\Longrightarrow$

$\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$
where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$
is compatible with M'

$\Downarrow$

$I_{\mathcal{S}'}$ is **satisfied** for
$\{P_{\mathbf{V}'} : \mathbf{V}' \in \mathcal{S}'\}$
where $P_{\mathbf{V}'} = P_{\mathbf{V}}$ for $\mathbf{V}' \sim \mathbf{V}$

$\Longleftarrow$

$I_{\mathcal{S}}$ is **satisfied** for
$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

M’ = G $\rightarrow$ G’ Inflation of M

$\mathcal{S} \subseteq \text{ImagesInjectableSets}(G)$ $\mathcal{S}' \subseteq \text{InjectableSets}(G')$

$I_{\mathcal{S}'}$ is a **causal compatibility inequality** for model M'

$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

is compatible with M

$$\Downarrow$$

$I_{\mathcal{S}}$ is **satisfied** for

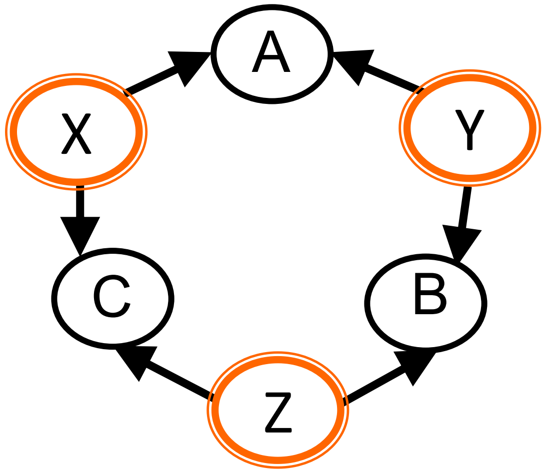
$\{P_{\mathbf{V}} : \mathbf{V} \in \mathcal{S}\}$

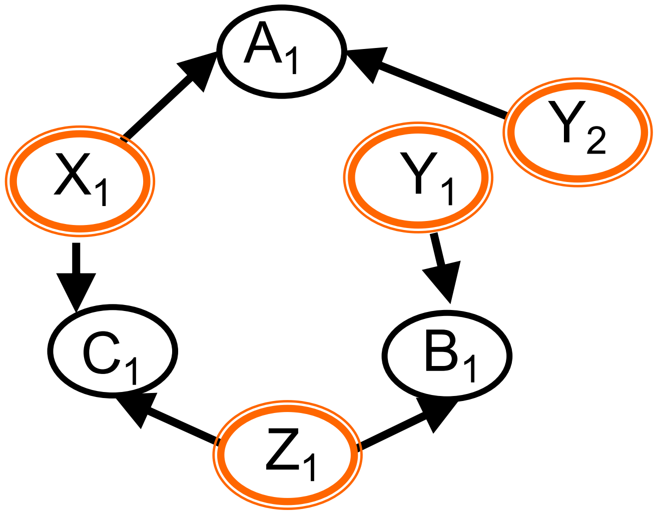
$I_{\mathcal{S}}$ is a **causal compatibility inequality** for model M

M′ = G → G′ Inflation of M

$\mathcal{S} \subseteq \mathrm{ImagesInjectableSets}(G)$ $\mathcal{S}' \subseteq \mathrm{InjectableSets}(G')$

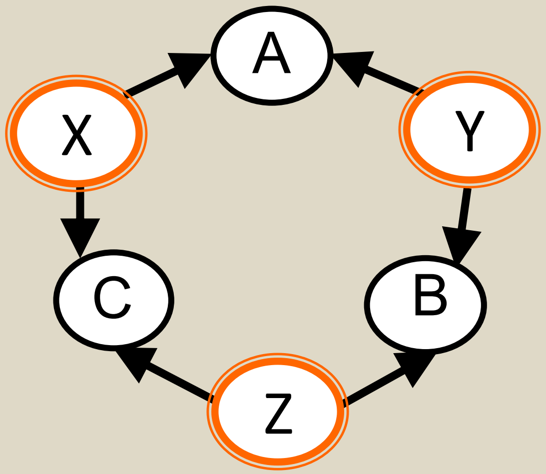
$I_{\mathcal{S}}$ is a **causal compatibility inequality** for model M $\Longleftarrow$ $I_{\mathcal{S}'}$ is a **causal compatibility inequality** for model M'





$P_A+P_B+P_C-P_AP_B-P_{BC}-P_{AC} \leq 1$
is a causal compatibility
inequality for M

$P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1}P_{B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$
is a causal compatibility
inequality for M'



$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$

causal compatibility inequality

$$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$$
is a valid set of marginals

$$\Longrightarrow$$

$$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1}) \text{ satisfy}$$
$$P_{A_1} + P_{B_1} + P_{C_1} - P_{A_1B_1} - P_{B_1C_1} - P_{A_1C_1} \leq 1$$

Linear quantifier elimination

Consider binary X, Y and Z

$$\exists P_{XYZ} \text{ with } \quad P_X, P_Y, P_Z, P_{XY}, P_{YZ}, P_{XZ} \text{ as marginals}$$

$$\Downarrow$$

Marginal compatibility constraints

$$P_X = \textstyle\sum_Y P_{XY} = \sum_Z P_{XZ}$$

$$P_Y = \textstyle\sum_X P_{XY} = \sum_Z P_{YZ}$$

$$P_Z = \textstyle\sum_X P_{XZ} = \sum_Y P_{YZ}$$

$$P_X + P_Y + P_Z - P_{XY} - P_{YZ} - P_{XZ} \leq 1$$

How this is proven:

$$\exists Q_{XYZ} \text{ with } \quad P_X, P_Y, P_Z, P_{XY}, P_{YZ}, P_{XZ} \text{ as marginals}$$

$$\forall x,y : P_{XY}(xy) = \textstyle\sum_z Q_{XYZ}(xyz)$$
$$\forall y,z : P_{YZ}(yz) = \textstyle\sum_x Q_{XYZ}(xyz)$$
$$\forall x,z : P_{XZ}(xz) = \textstyle\sum_y Q_{XYZ}(xyz)$$
$$\forall x : P_X(x) = \textstyle\sum_{y,z} Q_{XYZ}(xyz)$$
$$\forall y : P_Y(y) = \textstyle\sum_{x,z} Q_{XYZ}(xyz)$$
$$\forall z : P_Z(z) = \textstyle\sum_{x,y} Q_{XYZ}(xyz)$$

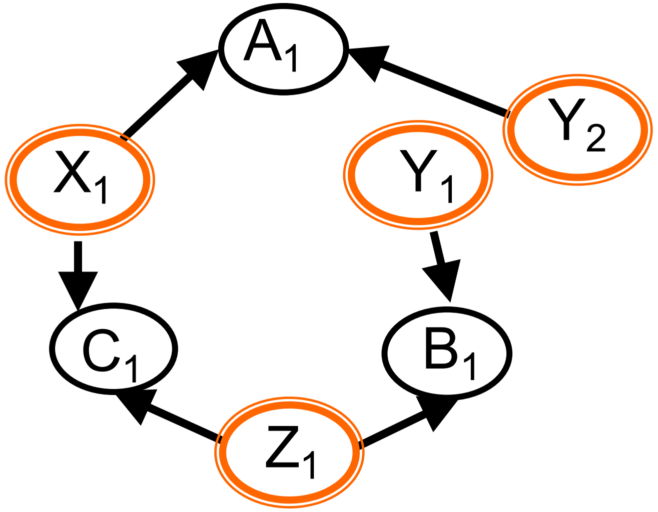
$$\forall x,y,z : 0 \leq Q_{XYZ}(xyz) \leq 1$$

Linear quantifier elimination

$$\forall x : P_X(x) = \textstyle\sum_y P_{XY}(xy) = \sum_z P_{XZ}(xz)$$
$$\forall y : P_Y(y) = \textstyle\sum_x P_{XY}(xy) = \sum_z P_{YZ}(yz)$$
$$\forall z : P_Z(z) = \textstyle\sum_x P_{XZ}(xz) = \sum_y P_{YZ}(yz)$$
$$\forall x,y,z : P_X(x) + P_Y(y) + P_Z(z) - P_{XY}(xy) - P_{YZ}(yz) - P_{XZ}(xz) \leq 1$$

$$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$$
is a valid set of marginals $\implies$ $(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ satisfy
$$P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$$

Linear quantifier elimination

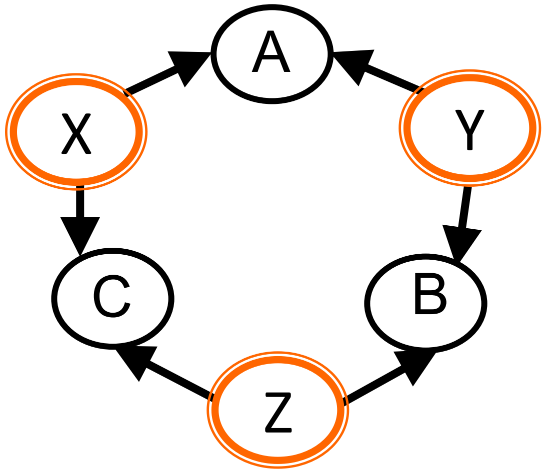


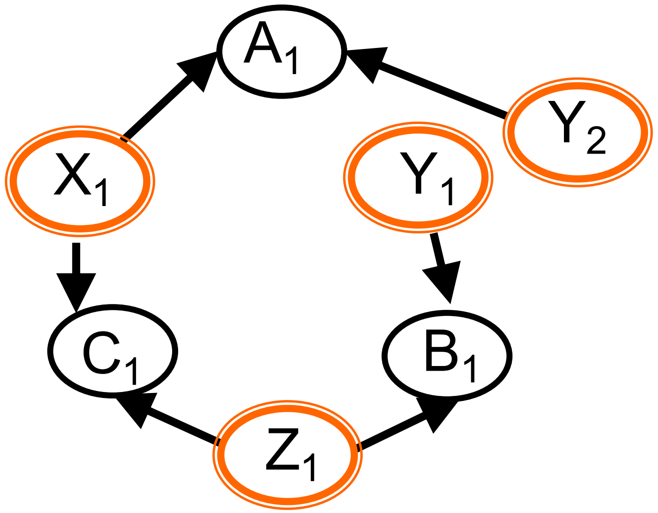
$$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$$
is compatible with M'

$$A_1 \perp_d B_1 \implies P_{A_1B_1} = P_{A_1}P_{B_1}$$

$$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$$
is compatible with M' $\implies$ $P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1}P_{B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$

This is a causal compatibility inequality for M'



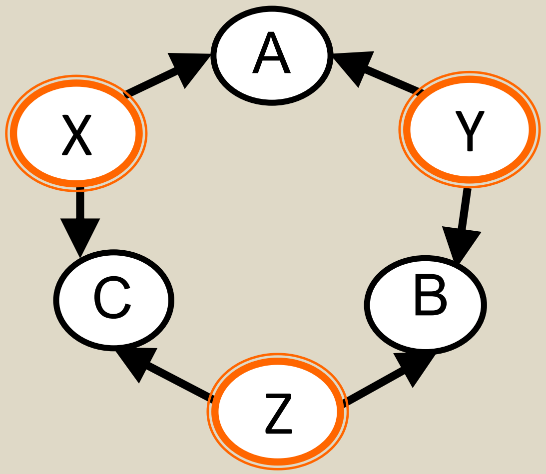


$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$

is a causal compatibility inequality for M

$$P_{A_1} + P_{B_1} + P_{C_1} - P_{A_1} P_{B_1} - P_{B_1 C_1} - P_{A_1 C_1} \leq 1$$

is a causal compatibility inequality for M'

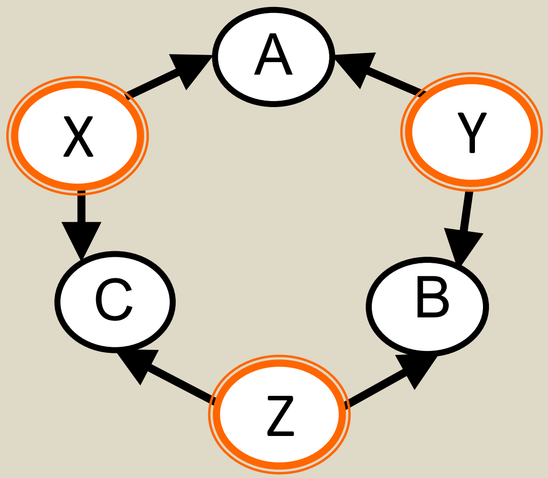


$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$

causal compatibility inequality

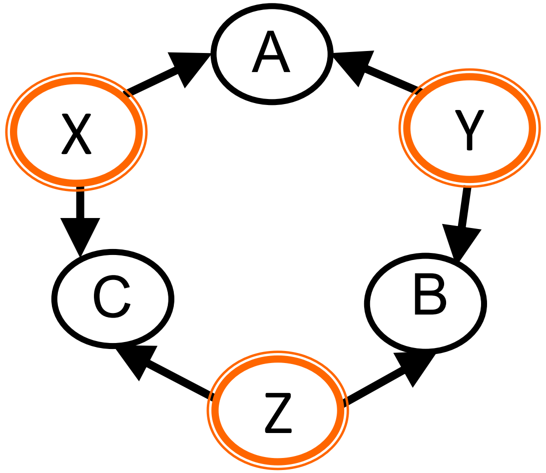
rules out

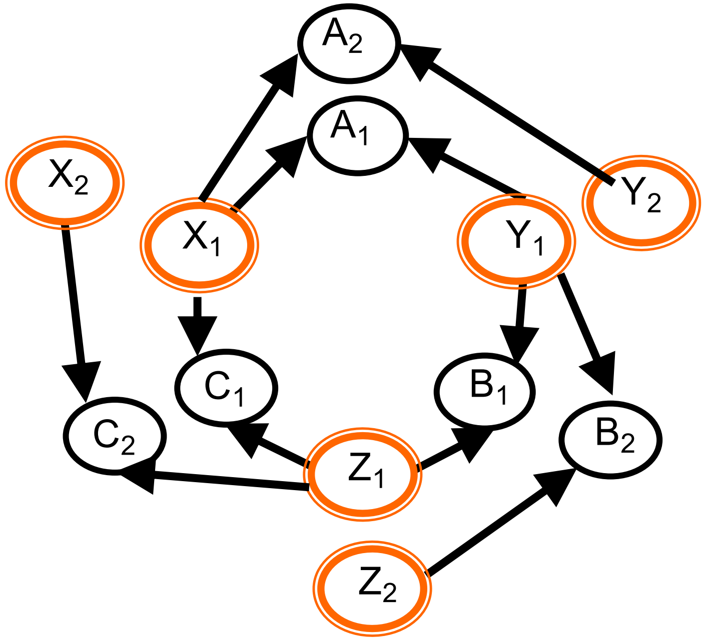
$$P_{ABC} = \tfrac{1}{2}[000] + \tfrac{1}{2}[111]$$



$$
\begin{aligned}
&P_A(1)P_B(1)P_C(1)\\
&\leq P_{AB}(11)P_C(1) + P_{BC}(11)P_A(1)\\
&+ P_{AC}(11)P_B(1) + P_{ABC}(000)
\end{aligned}
$$

causal compatibility inequality





$$P_A(1)P_B(1)P_C(1)$$
$$\leq P_{AB}(11)P_C(1) + P_{BC}(11)P_A(1)$$
$$+ P_{AC}(11)P_B(1) + P_{ABC}(000)$$

is a causal compatibility inequality for M

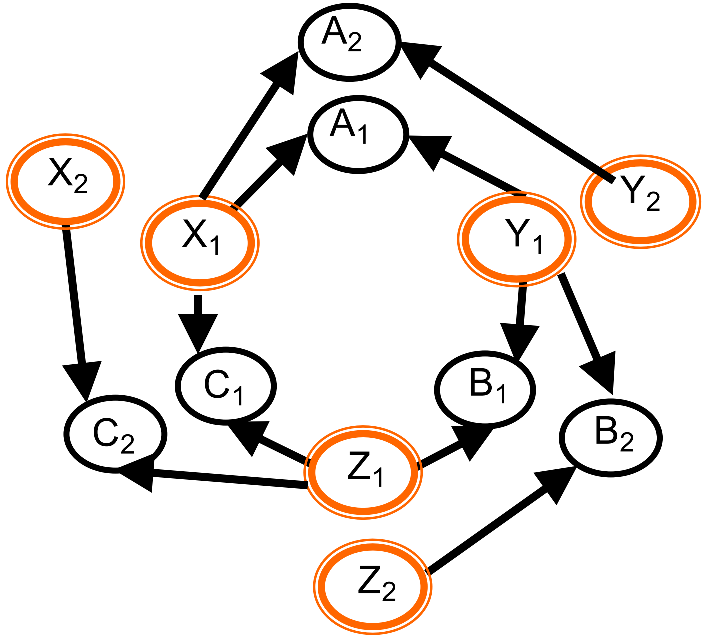
$$P_{A_2}(1)P_{B_2}(1)P_{C_2}(1)$$
$$\leq P_{A_1B_2}(11)P_{C_2}(1) + P_{B_1C_2}(11)P_{A_2}(1)$$
$$+ P_{A_2C_1}(11)P_{B_2}(1) + P_{A_1B_1C_1}(000)$$

is a causal compatibility inequality for M'

$$(P_{A_2B_2C_2}, P_{A_1B_2C_2}, P_{A_2B_1C_2}, P_{A_2B_2C_1}, P_{A_1B_1C_1})$$
is a valid set of marginals

$$\Longrightarrow \quad P_{A_2B_2C_2}(111) \le P_{A_1B_2C_2}(111) + P_{B_1C_2A_2}(111) + P_{A_2C_1B_2}(111) + P_{A_1B_1C_1}(000)$$

Linear quantifier elimination



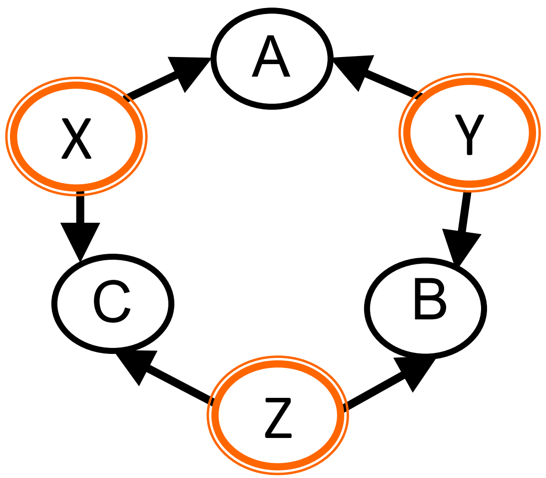
$$(P_{A_2B_2C_2}, P_{A_1B_2C_2}, P_{A_2B_1C_2}, P_{A_2B_2C_1}, P_{A_1B_1C_1})$$
is compatible with M'

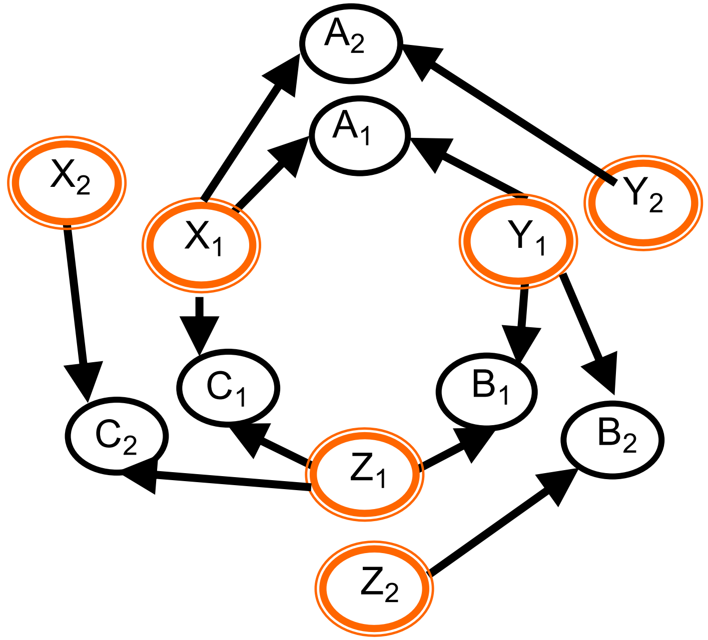
$$\begin{aligned}
A_1B_2 \perp_d C_2 &\implies P_{A_1B_2C_2} = P_{A_1B_2}P_{C_2},\\
B_1C_2 \perp_d A_2 &\implies P_{B_1C_2A_2} = P_{B_1C_2}P_{A_2},\\
A_2C_1 \perp_d B_2 &\implies P_{A_2C_1B_2} = P_{A_2C_1}P_{B_2},\\
A_2 \perp_d B_2 \perp_d C_2 &\implies P_{A_2B_2C_2} = P_{A_2}P_{B_2}P_{C_2}
\end{aligned}$$

$$(P_{A_2B_2C_2}, P_{A_1B_2C_2}, P_{A_2B_1C_2}, P_{A_2B_2C_1}, P_{A_1B_1C_1})$$
is compatible with M'

$$\Longrightarrow \quad P_{A_2}(1)P_{B_2}(1)P_{C_2}(1) \le P_{A_1B_2}(11)P_{C_2}(1) + P_{B_1C_2}(11)P_{A_2}(1) + P_{A_2C_1}(11)P_{B_2}(1) + P_{A_1B_1C_1}(000)$$

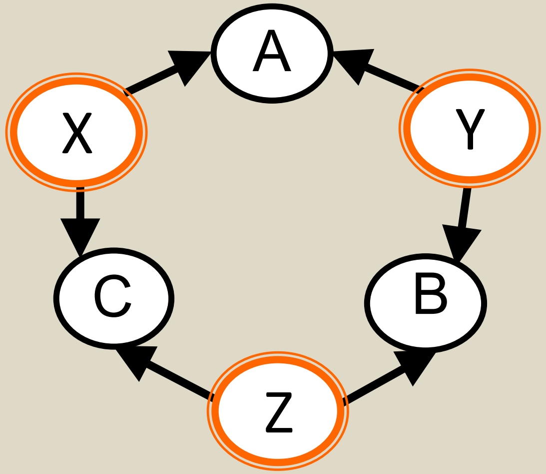
This is a causal compatibility inequality for M'





$$P_A(1)P_B(1)P_C(1) \leq P_{AB}(11)P_C(1) + P_{BC}(11)P_A(1) + P_{AC}(11)P_B(1) + P_{ABC}(000)$$

$$P_{A_2}(1)P_{B_2}(1)P_{C_2}(1) \leq P_{A_1B_2}(11)P_{C_2}(1) + P_{B_1C_2}(11)P_{A_2}(1) + P_{A_2C_1}(11)P_{B_2}(1) + P_{A_1B_1C_1}(000)$$



$$P_A(1)P_B(1)P_C(1) \\ \leq P_{AB}(11)P_C(1) + P_{BC}(11)P_A(1) \\ + P_{AC}(11)P_B(1) + P_{ABC}(000)$$

causal compatibility inequality

rules out

$$P_{ABC} = \tfrac{1}{3}[001] + \tfrac{1}{3}[010] + \tfrac{1}{3}[100]$$

Linear inequality constraints from marginal compatibility
(from linear quantifier elimination)

Polynomial inequality constraints for causal compatibility with the original DAG

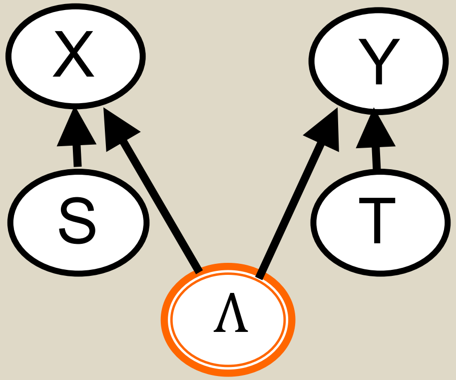
+

Polynomial equality constraints from causal compatibility with the inflated DAG
(e.g., from d-separation relations)

The technique defines an algorithm for deriving causal compatibility inequalities and for testing compatibility

Proof that this provides a convergent heirarchy of tests:
Navascués & Wolfe, J. Causal Inf. 8(1) 70 (2020)

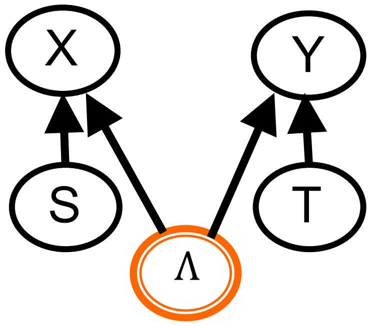
Derivation of Bell inequalities from inflation technique



$$\boxed{\begin{aligned}&\tfrac{1}{4}\textstyle\sum_{x=y} P_{XY|ST}(xy|00) + \tfrac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)\\ &\quad \tfrac{1}{4}\textstyle\sum_{x=y} P_{XY|ST}(xy|10) + \tfrac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \tfrac{3}{4}\end{aligned}}$$

Note that the derivation of the CHSH inequalities from brute-force quantifier elimination did not provide much conceptual insight into their origin

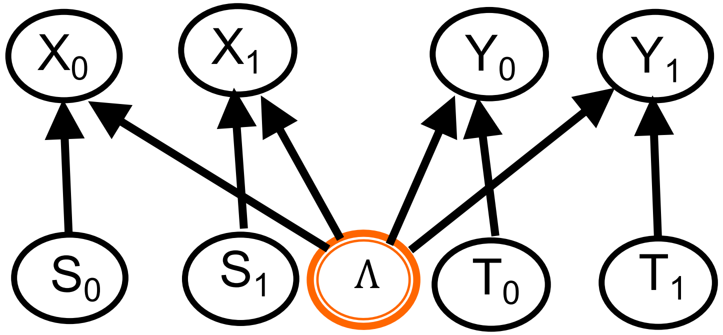
Bell model M



$P_{X|S\Lambda}$
$P_{Y|T\Lambda}$
P_{Λ}

$$P_{XY|ST} = \sum_{\Lambda} P_{X|S\Lambda} P_{Y|T\Lambda} P_{\Lambda}$$

Inflated model M'



$P_{X_0|S_0\Lambda}$
$P_{X_1|S_1\Lambda}$
$P_{Y_0|T_0\Lambda}$
$P_{Y_1|T_1\Lambda}$
P_{Λ}

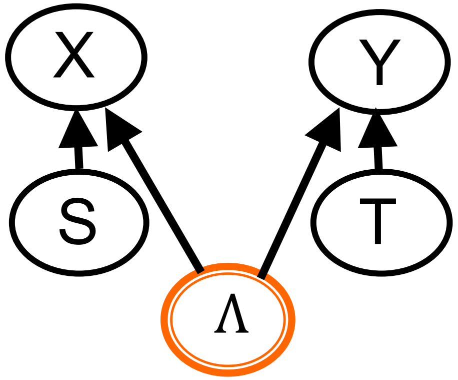
$P_{X_0|S_0\Lambda} = P_{X_1|S_1\Lambda}$
$P_{Y_0|T_0\Lambda} = P_{Y_1|T_1\Lambda}$

$$P_{X_0Y_0|S_0T_0} = \sum_{\Lambda} P_{X_0|S_0\Lambda} P_{Y_0|T_0\Lambda} P_{\Lambda}$$
$$P_{X_0Y_1|S_0T_1} = \sum_{\Lambda} P_{X_0|S_0\Lambda} P_{Y_1|T_1\Lambda} P_{\Lambda}$$
$$P_{X_1Y_0|S_1T_0} = \sum_{\Lambda} P_{X_1|S_1\Lambda} P_{Y_0|T_0\Lambda} P_{\Lambda}$$
$$P_{X_1Y_1|S_1T_1} = \sum_{\Lambda} P_{X_1|S_1\Lambda} P_{Y_1|T_1\Lambda} P_{\Lambda}$$

$P_{XY|ST}$
compatible with M

$P_{X_0Y_0|S_0T_0}, P_{X_0Y_1|S_0T_1}, P_{X_1Y_0|S_1T_0}, P_{X_1Y_1|S_1T_1}$
where $P_{X_iY_j|S_iT_j} = P_{XY|ST}$
compatible with M'

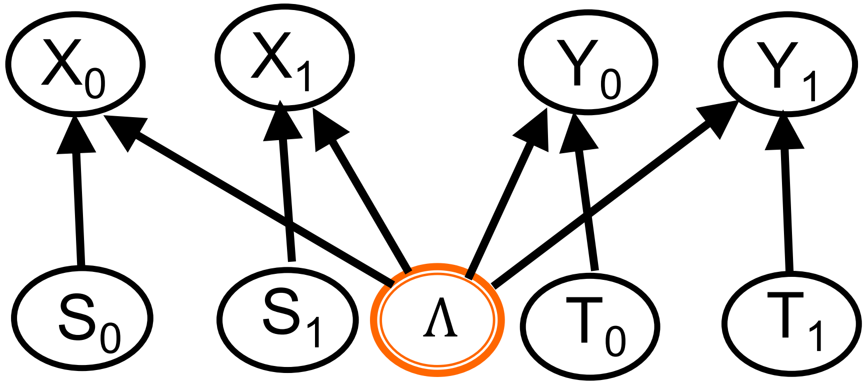
Bell model M



$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \frac{3}{4}$$

Causal compatibility inequality in M

Inflated model M’



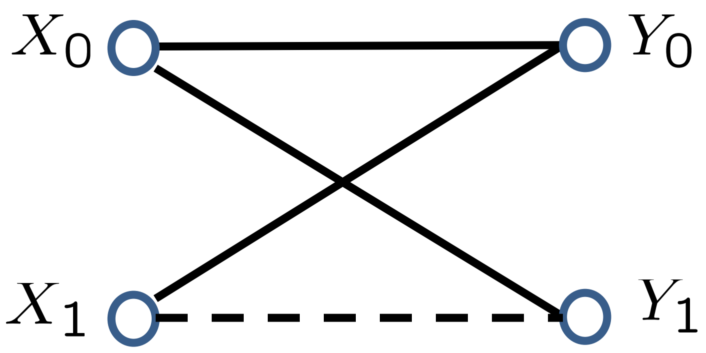
$$\frac{1}{4}\sum_{x=y} P_{X_0Y_0|S_0T_0}(xy|00) + \frac{1}{4}\sum_{x=y} P_{X_0Y_1|S_0T_1}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{X_1Y_0|S_1T_0}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{X_1Y_1|S_1T_1}(xy|11) \leq \frac{3}{4}$$

Causal compatibility inequality in M’

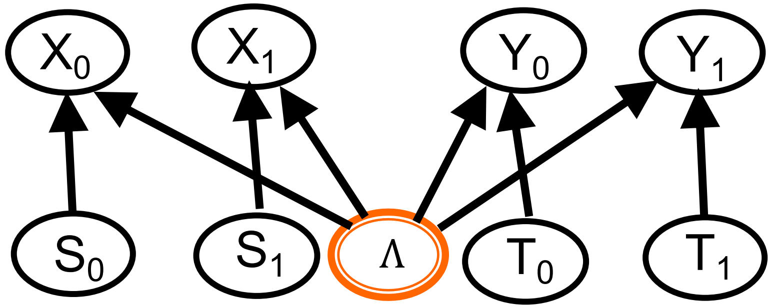
Consider any distribution over 4 variables $Q_{X_0X_1Y_0Y_1}$

The marginals $Q_{X_0Y_0}, Q_{X_0Y_1}, Q_{X_1Y_0}, Q_{X_0Y_1}$ satisfy

$$\frac{1}{4}\sum_{x=y} Q_{X_0Y_0}(xy) + \frac{1}{4}\sum_{x=y} Q_{X_0Y_1}(xy)$$
$$\frac{1}{4}\sum_{x=y} Q_{X_1Y_0}(xy) + \frac{1}{4}\sum_{x\neq y} Q_{X_1Y_1}(xy) \leq \frac{3}{4}$$



Inflated model M’



$$\text{Take } Q_{X_0X_1Y_0Y_1}(\cdot) := P_{X_0X_1Y_0Y_1|S_0S_1T_0T_1}(\cdot|0101)$$

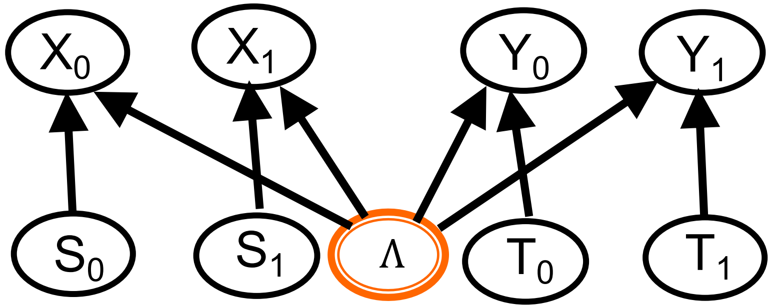
$$\tfrac{1}{4}\textstyle\sum_{x=y} Q_{X_0Y_0}(xy) + \tfrac{1}{4}\sum_{x=y} Q_{X_0Y_1}(xy)$$
$$\tfrac{1}{4}\textstyle\sum_{x=y} Q_{X_1Y_0}(xy) + \tfrac{1}{4}\sum_{x\neq y} Q_{X_1Y_1}(xy) \leq \tfrac{3}{4}$$

$$\Rightarrow$$

$$\tfrac{1}{4}\textstyle\sum_{x=y} P_{X_0Y_0|S_0S_1T_0T_1}(xy|0101) + \tfrac{1}{4}\sum_{x=y} P_{X_0Y_1|S_0S_1T_0T_1}(xy|0101)$$
$$\tfrac{1}{4}\textstyle\sum_{x=y} P_{X_1Y_0|S_0S_1T_0T_1}(xy|0101) + \tfrac{1}{4}\sum_{x\neq y} P_{X_1Y_1|S_0S_1T_0T_1}(xy|0101) \leq \tfrac{3}{4}$$

This is a marginal compatibility constraint

Inflated model M’



$$X_0Y_0 \perp S_1T_1|S_0T_0 \implies P_{X_0Y_0|S_0T_0S_1T_1} = P_{X_0Y_0|S_0T_0}$$

$$X_0Y_1 \perp S_1T_0|S_0T_1 \implies P_{X_0Y_1|S_0T_0S_1T_1} = P_{X_0Y_1|S_0T_1}$$

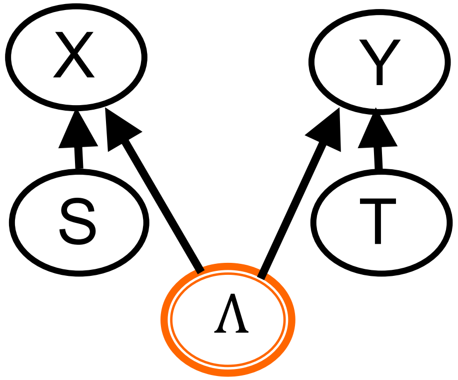
$$X_1Y_0 \perp S_0T_1|S_1T_0 \implies P_{X_1Y_0|S_0T_0S_1T_1} = P_{X_1Y_0|S_1T_0}$$

$$X_1Y_1 \perp S_0T_0|S_1T_1 \implies P_{X_1Y_1|S_0T_0S_1T_1} = P_{X_1Y_1|S_1T_1}$$

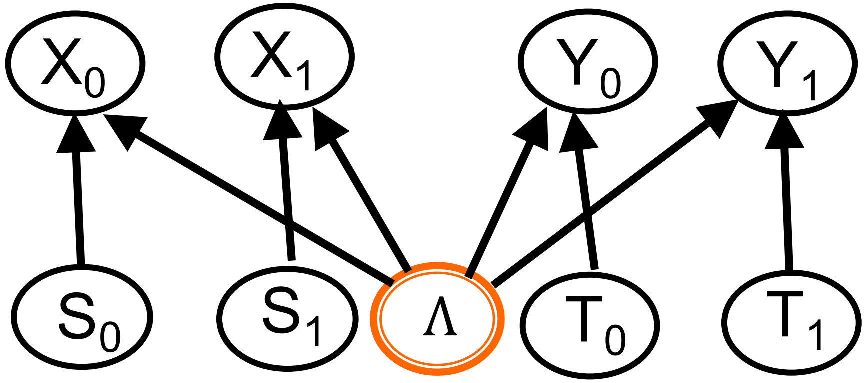
$$\tfrac{1}{4}\textstyle\sum_{x=y} P_{X_0Y_0|S_0S_1T_0T_1}(xy|0101) + \tfrac{1}{4}\sum_{x=y} P_{X_0Y_1|S_0S_1T_0T_1}(xy|0101)$$
$$\tfrac{1}{4}\textstyle\sum_{x=y} P_{X_1Y_0|S_0S_1T_0T_1}(xy|0101) + \tfrac{1}{4}\sum_{x\neq y} P_{X_1Y_1|S_0S_1T_0T_1}(xy|0101) \leq \tfrac{3}{4}$$

$$\tfrac{1}{4}\textstyle\sum_{x=y} P_{X_0Y_0|S_0T_0}(xy|00) + \tfrac{1}{4}\sum_{x=y} P_{X_0Y_1|S_0T_1}(xy|01)$$
$$\tfrac{1}{4}\textstyle\sum_{x=y} P_{X_1Y_0|S_1T_0}(xy|10) + \tfrac{1}{4}\sum_{x\neq y} P_{X_1Y_1|S_1T_1}(xy|11) \leq \tfrac{3}{4}$$

Bell model M



Inflated model M’



$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \frac{3}{4}$$

Causal compatibility inequality in M

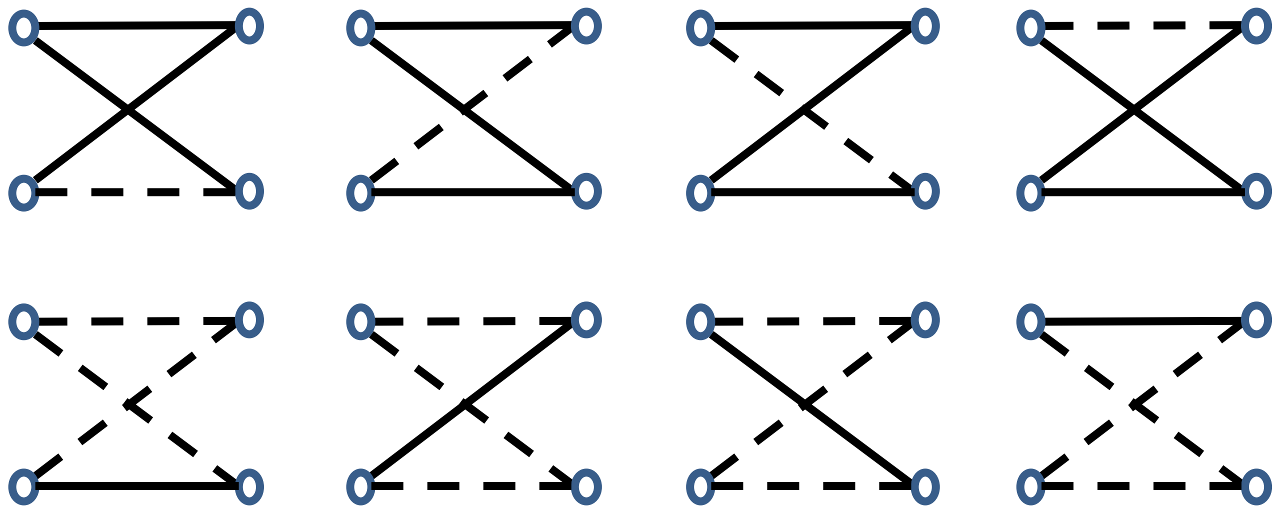
$$\frac{1}{4}\sum_{x=y} P_{X_0Y_0|S_0T_0}(xy|00) + \frac{1}{4}\sum_{x=y} P_{X_0Y_1|S_0T_1}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{X_1Y_0|S_1T_0}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{X_1Y_1|S_1T_1}(xy|11) \leq \frac{3}{4}$$

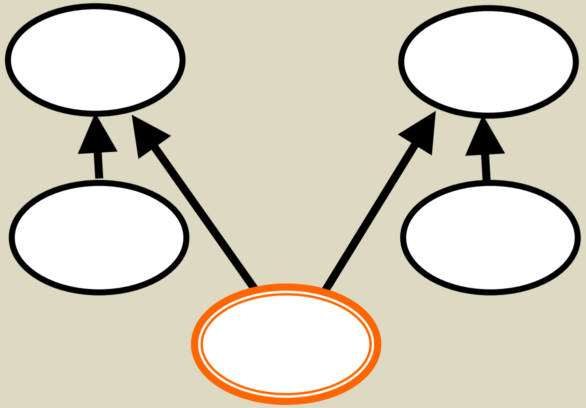
Causal compatibility inequality in M’

All the Bell inequalities for binary settings and outcomes

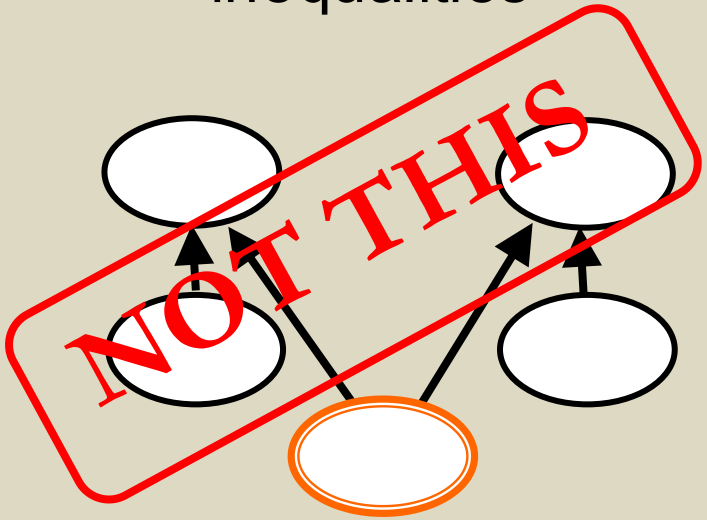
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01)$$
$$\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \frac{3}{4}$$

+ permutations corresponding to the 8 frustrated four-node networks

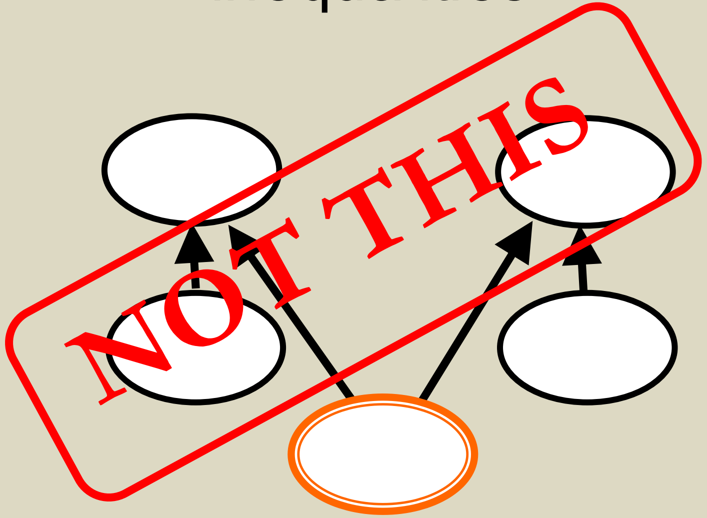




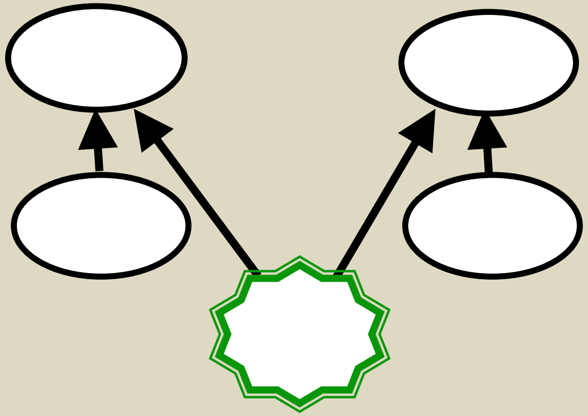
Violation of Bell inequalities



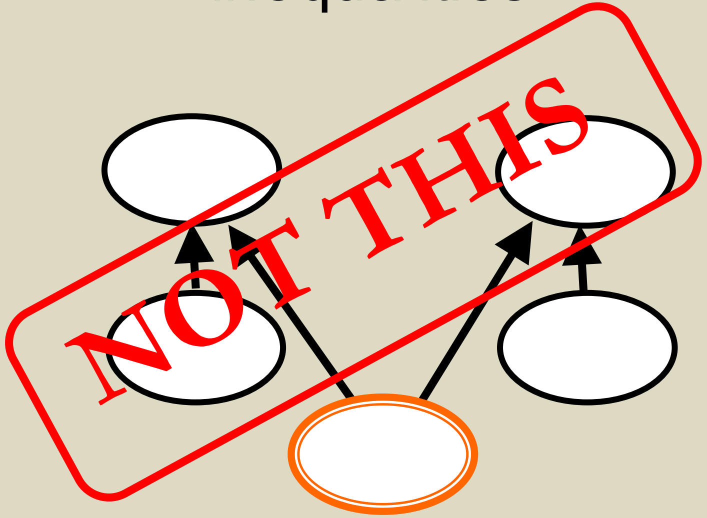
Violation of Bell inequalities



Witnessing quantumness

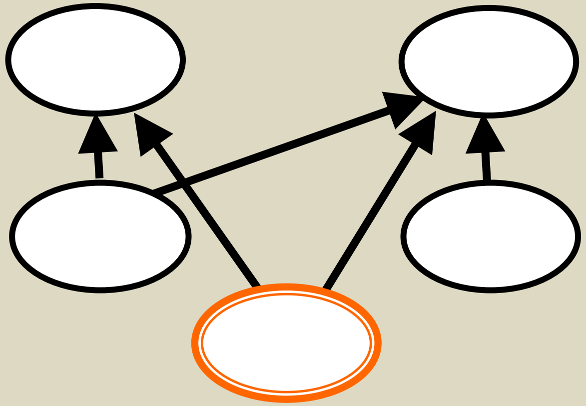


Violation of Bell inequalities

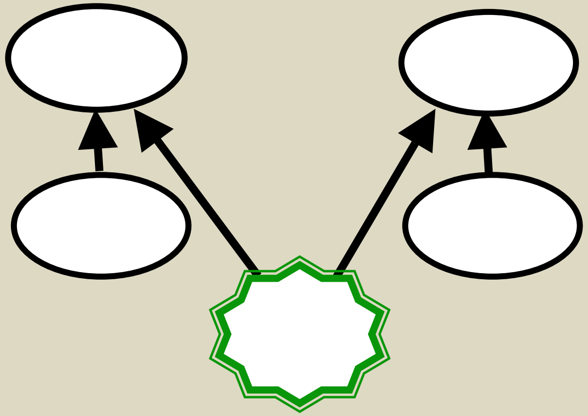


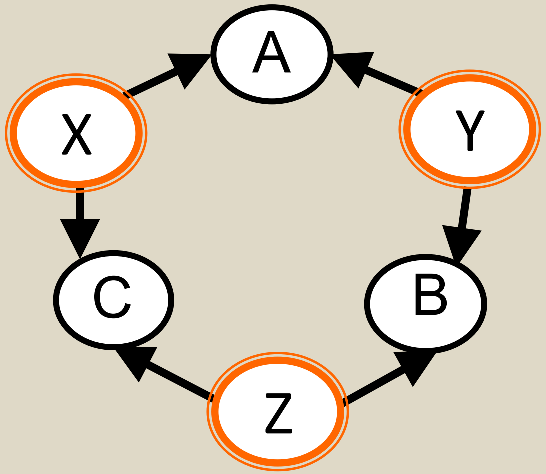
or

Witnessing need for different structure

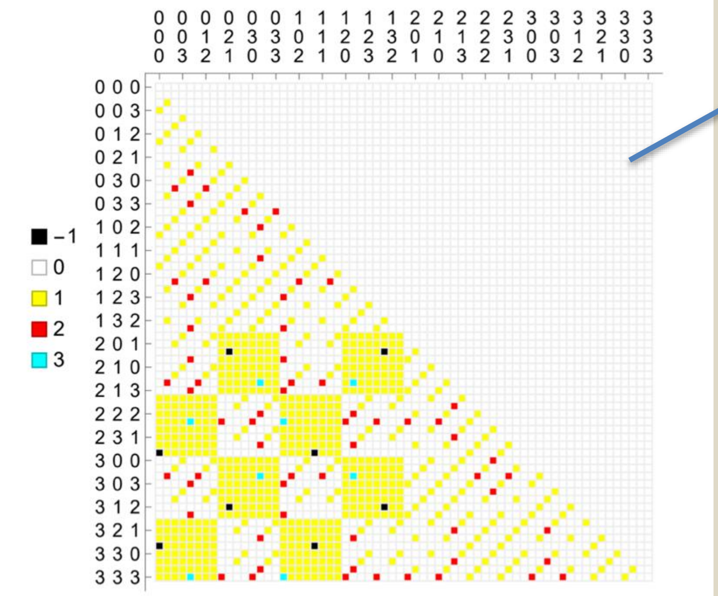


Witnessing quantumness





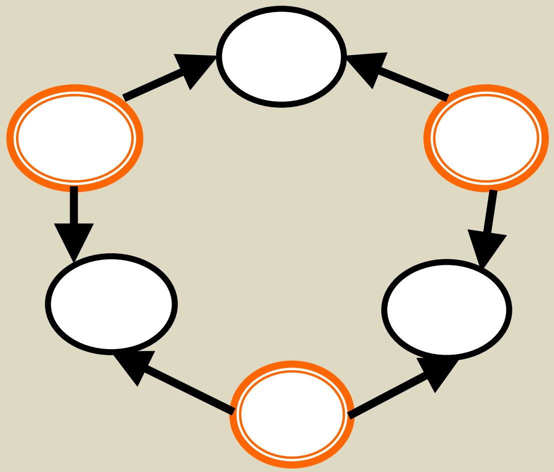
$$V := \sum_{a,b,c,a',b',c'} y_{abca'b'c'} P_{ABC}(abc) P_{ABC}(a'b'c') \geq 0$$



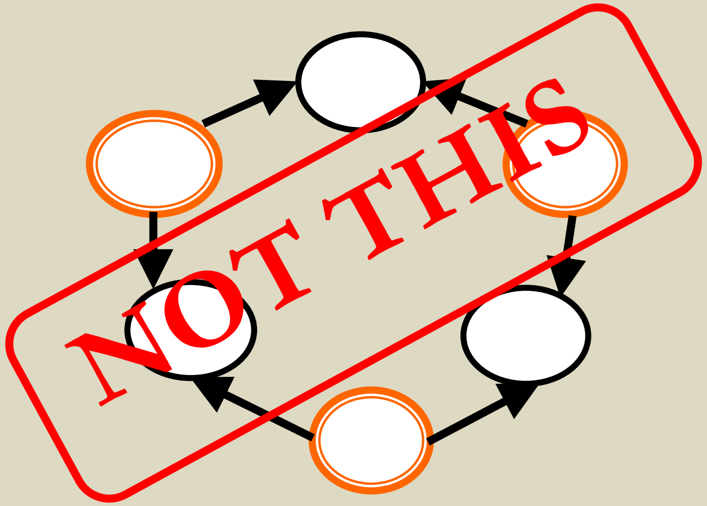
(b) **The coefficients of a quadratic inequality**

This causal compatibility inequality is violated by the data

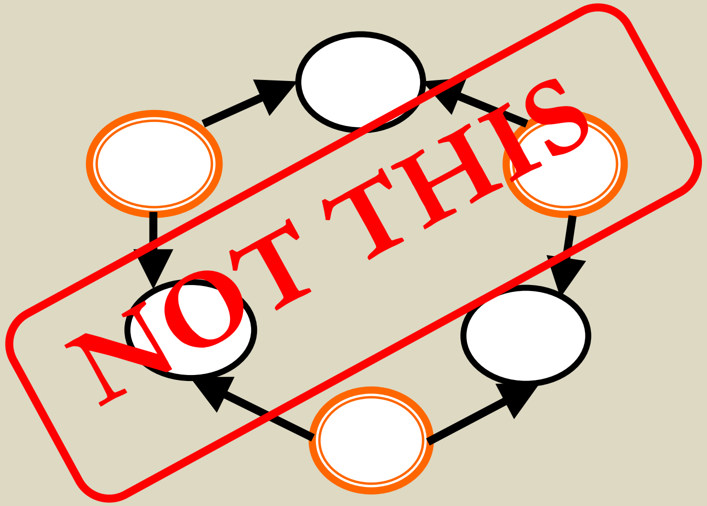
$$V_{\text{exp}} = -0.02436 \pm 0.00016$$



Violation of certain causal compatibility inequalities

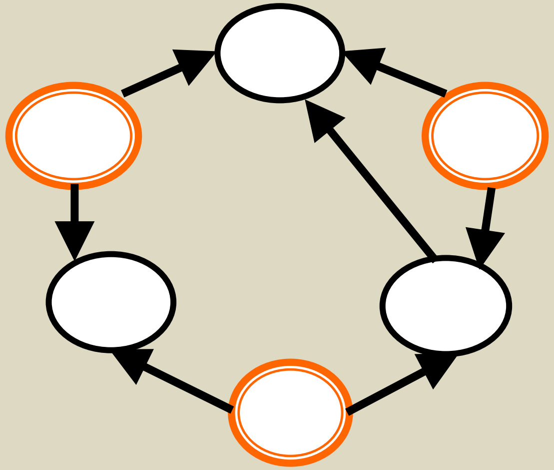


Violation of certain causal compatibility inequalities

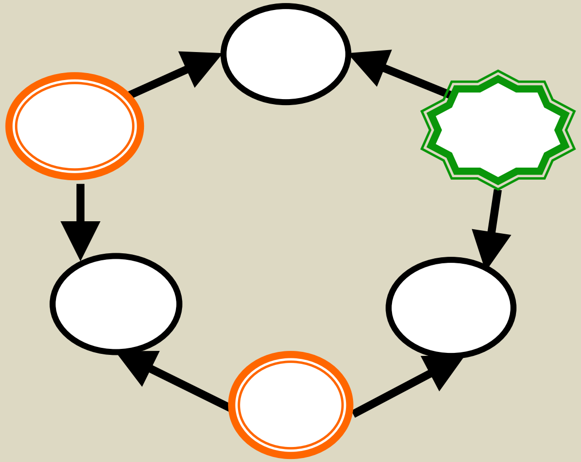


or

Witnessing need for different structure

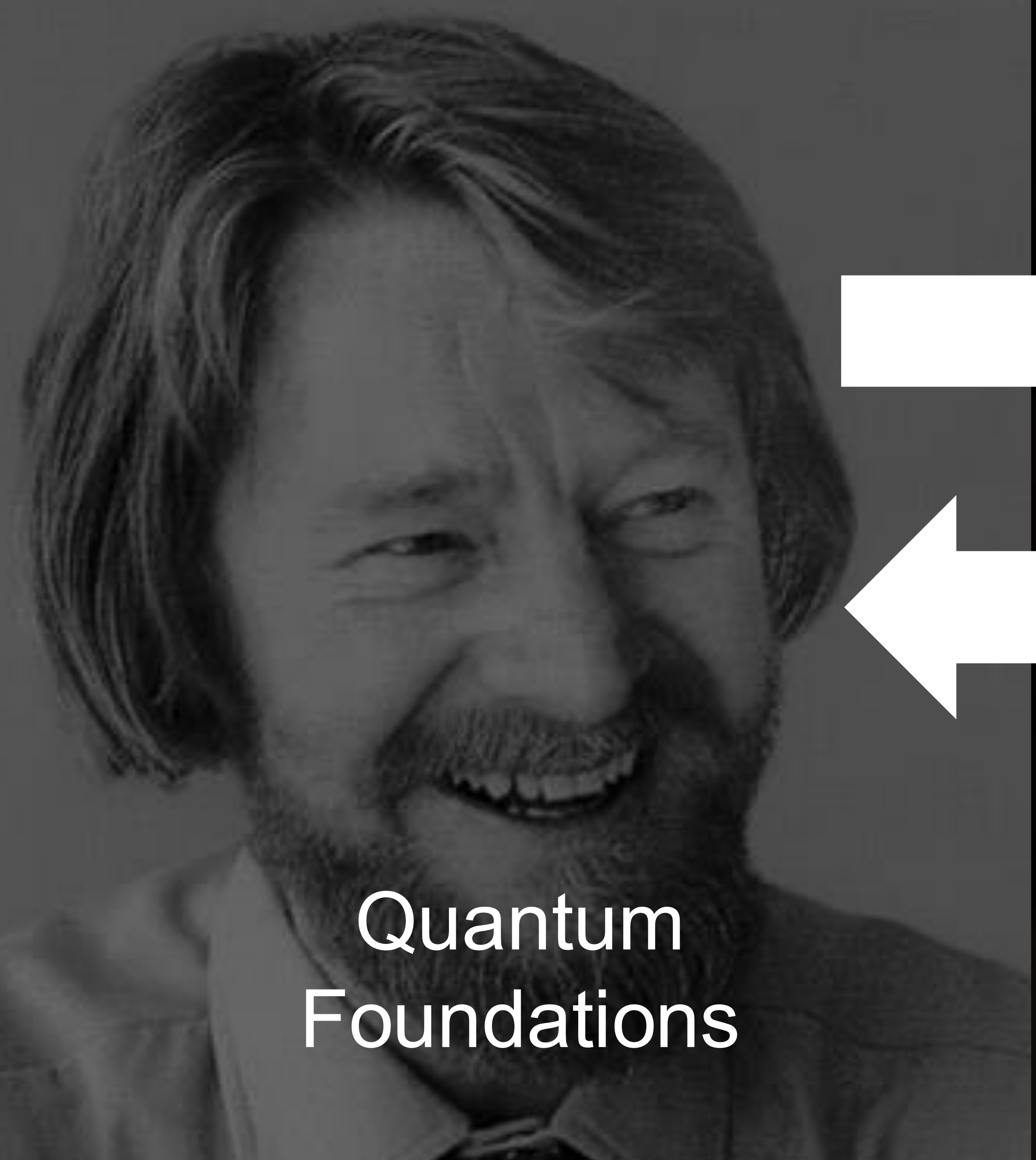


Witnessing quantumness



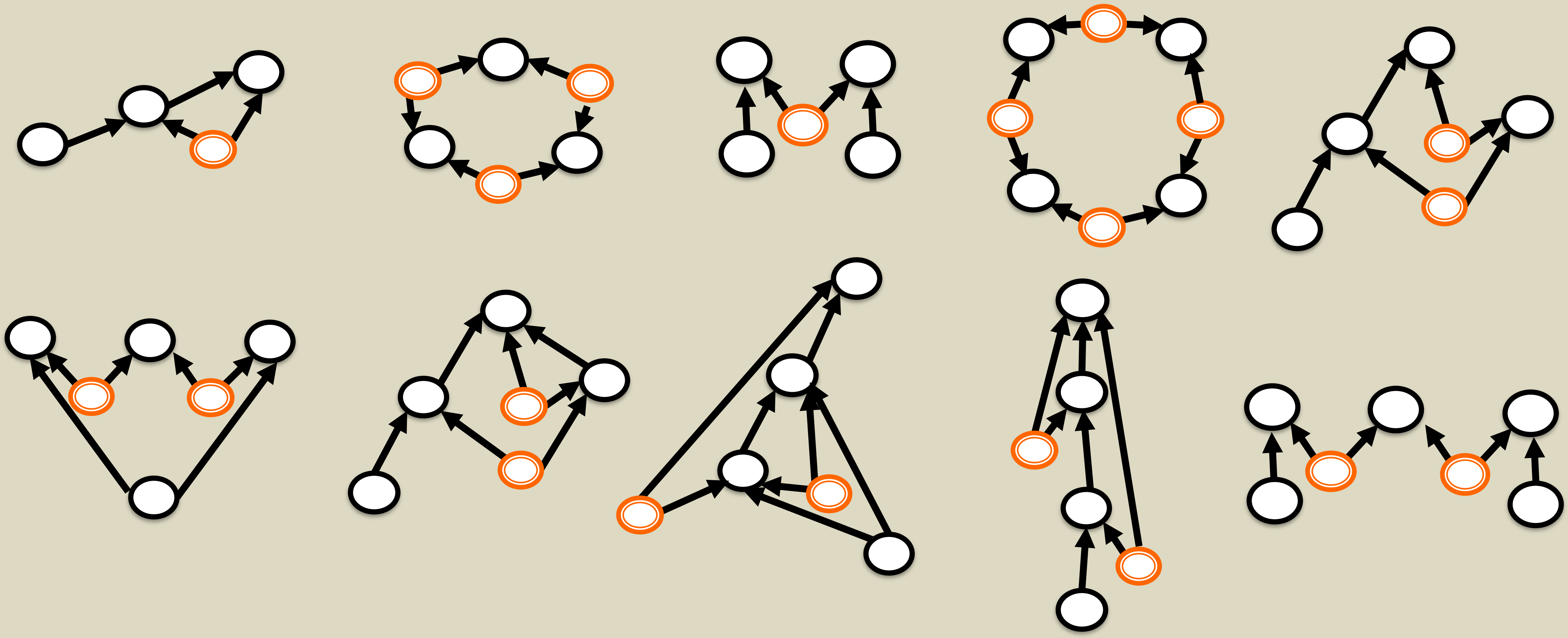
Approaches to Bell arguments that follow essentially
the logic of the inflation technique:

Fine’s proof of CHSH inequalities
Hardy’s proof of Bell’s theorem
The Greenberger-Horne-Zeilinger proof of Bell’s theorem
Bell’s theorem by way of Kochen-Specker noncontextuality
Braunstein-Caves entropic inequalities
Symmetric extensions / shareability of local correlations
…


Quantum
Foundations


Causal
Inference

Some causal structures that admit of inequality constraints



Causal inference techniques