Causal Inference: Classical and Quantum

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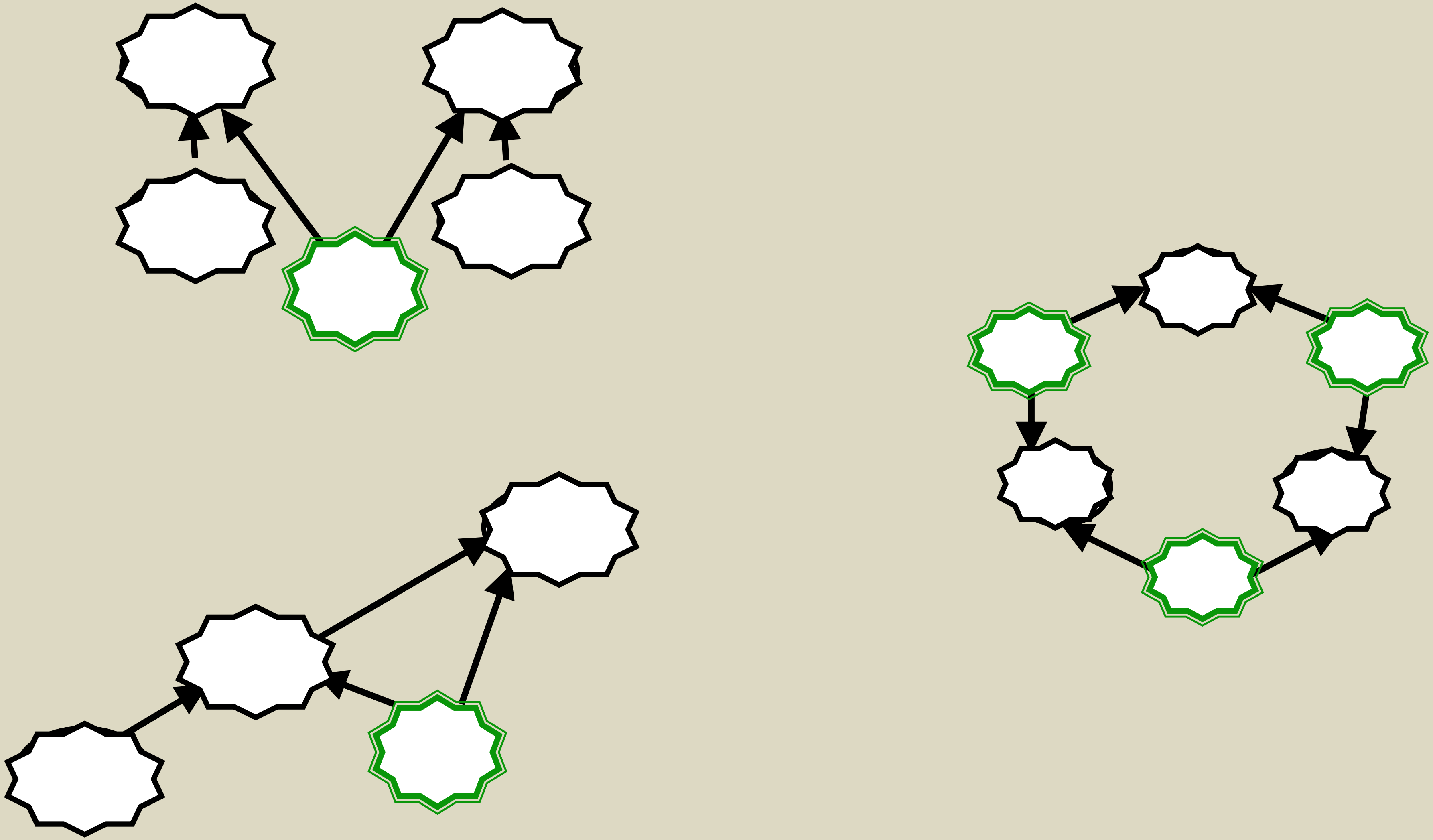
Causarum Investigatio
“Investigate the causes”

Fully quantum causal models

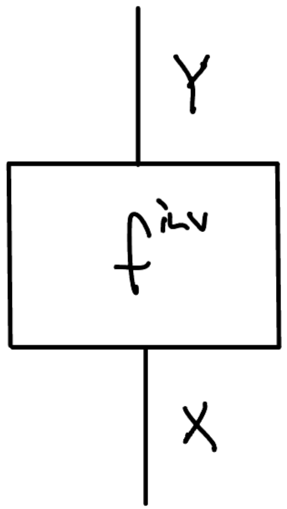


The causal theory

Focus on deterministic and reversible evolution in closed systems:

Unitary evolution

invertible function



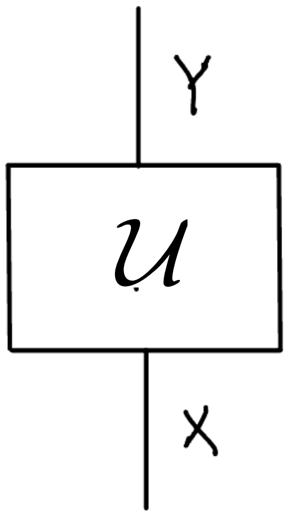
For $|X| = |Y|$

$$f^{\mathsf{inv}} : X \to Y$$

$$:: x \mapsto f^{\mathsf{inv}}(x)$$

where f^{inv} is an invertible function

Unitary channel



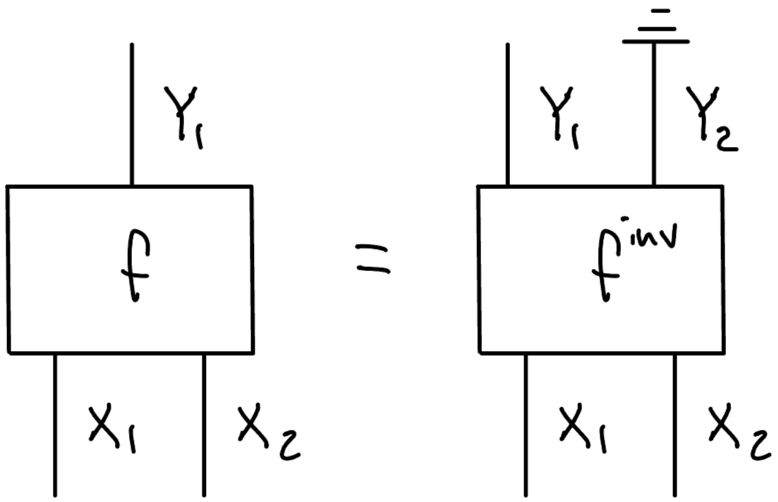
For $\dim(\mathcal{H}_X) = \dim(\mathcal{H}_Y)$

$$\mathcal{U} : \mathcal{L}(\mathcal{H}_X) \to \mathcal{L}(\mathcal{H}_Y)$$

$$:: A \mapsto \mathcal{U}(A) := UAU^\dagger$$

where U is a unitary operator

general function



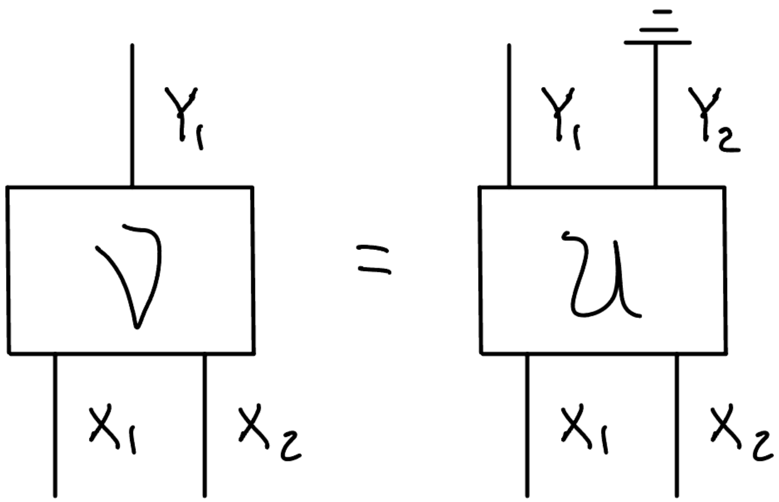
For $|X_1||X_2| = |Y_1||Y_2|$

$$f : X_1 \times X_2 \to Y_1$$

$$f := f^{\mathsf{inv}}|_{Y_1}$$

where f^{inv} is an invertible function

Reduced unitary



For $\mathsf{dim}(\mathcal{H}_{X_1} \otimes \mathcal{H}_{X_2}) = \mathsf{dim}(\mathcal{H}_{Y_1} \otimes \mathcal{H}_{Y_2})$

$$\mathcal{V} : \mathcal{L}(\mathcal{H}_{X_1} \otimes \mathcal{H}_{X_2}) \to \mathcal{L}(\mathcal{H}_{Y_1})$$

$$\mathcal{V} := \mathrm{Tr}_{Y_2} \circ \mathcal{U}$$

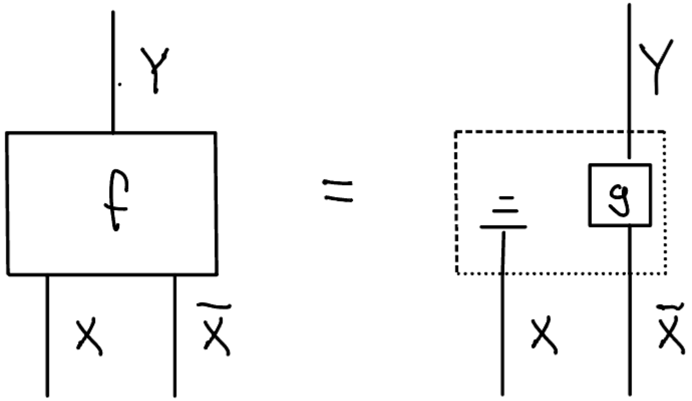
where $\mathcal{U}$ is a unitary channel

What is the definition of causal influence quantumly?

Classical

variable X has no influence on variable Y if Y has a trivial dependence on X

for a general function f

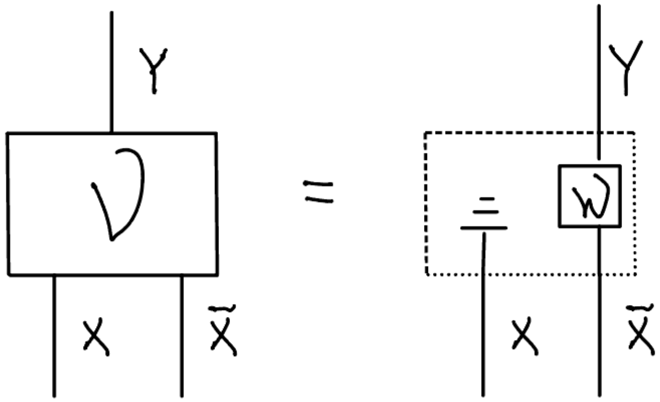


$$f(X,\bar{X}) = g(\bar{X})$$

Quantum

system X has no influence on system Y if Y has a trivial dependence on X

for a reduced unitary channel V



$$\mathcal{V}_{Y|X\bar{X}} = \mathrm{Tr}_X \otimes \mathcal{W}_{Y|\bar{X}}$$

The inferential theory

Joint states for systems that are common-cause connected

	Classical	Quantum
State of knowledge	P_A	ρ_A
Normalization	$\sum_A P_A = 1$	$\mathrm{Tr}_A(\rho_A) = 1$
Joint state	P_{AB}	ρ_{AB}
Marginalization	$P_B = \sum_A P_{AB}$	$\rho_B = \mathrm{Tr}_A(\rho_{AB})$

Marginal independence of A and B

$$\rho_{AB} = \rho_A \otimes \rho_B$$

Denote this

$$(A \perp B)$$

Conditional states for systems that are cause-effect connected

Classical belief propagation

$$P_B = \Gamma_{B|A}[P_A]$$

Stochastic map preserving positivity and normalization

In terms of a conditional

$$P_B = \sum_A P_{B|A} P_A$$

$$\sum_B P_{B|A} = 1$$

$$P_{B|A} \geq 0$$

Quantum belief propagation

$$\rho_B = \mathcal{E}_{B|A}(\rho_A)$$

Completely positive trace-preserving map

In terms of a conditional

$$\rho_B = \mathrm{Tr}_A(\rho_{B|A}\rho_A)$$

$$\mathrm{Tr}_B(\rho_{B|A}) = I_A$$

$$\rho_{B|A}^{T_A} \geq 0$$

The Choi-Jamiolkowski isomorphism

$$\rho_{B|A} := (\Phi_{B|A'} \otimes \mathrm{id}_A)(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |k\rangle\langle j|_A)$$

$$\Phi_{B|A}(\cdot) = \mathrm{Tr}_A(\rho_{B|A} \cdot)$$

Proof: $$\mathrm{Tr}_A(\rho_{B|A}\rho_A) = \sum_{j,k} \Phi_{B|A'}(|j\rangle\langle k|_{A'})\langle j|\rho_A|k\rangle$$

$$= \Phi_{B|A}(\sum_{j,k} |j\rangle\langle j|_A \rho_A |k\rangle\langle k|_A)$$

$$= \Phi_{B|A}(\rho_A)$$ QED

The Choi-Jamiolkowski isomorphism

$$\rho_{B|A} := (\Phi_{B|A'} \otimes \mathrm{id}_A)(\textstyle\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |k\rangle\langle j|_A)$$

$$\Phi_{B|A}(\cdot) = \mathrm{Tr}_A(\rho_{B|A} \cdot)$$

$$\Phi_{B|A'} \text{ is trace-preserving} \quad \leftrightarrow \quad \mathrm{Tr}_B(\rho_{B|A}) = I_A$$

Proof:

$$\begin{aligned}
\mathrm{Tr}_B(\rho_{B|A}) &= \mathrm{Tr}_B\left[(\Phi_{B|A'} \otimes \mathrm{id}_A)(\textstyle\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |k\rangle\langle j|_A)\right] \\
&= \textstyle\sum_{j,k} \mathrm{Tr}_B \circ \Phi_{B|A'}\left(|j\rangle\langle k|_{A'}\right) |k\rangle\langle j|_A \\
&= \textstyle\sum_{j,k} \mathrm{Tr}_{A'}\left(|j\rangle\langle k|_{A'}\right) |k\rangle\langle j|_A \\
&= \textstyle\sum_{j,k} \delta_{j,k} |k\rangle\langle j|_A \\
&= I_A \qquad \text{QED}
\end{aligned}$$

The Choi-Jamiolkowski isomorphism

$$\rho_{B|A} := (\Phi_{B|A'} \otimes \mathrm{id}_A)(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |k\rangle\langle j|_A)$$

$$\Phi_{B|A}(\cdot) = \mathrm{Tr}_A(\rho_{B|A}\cdot)$$

$$\Phi_{B|A} \;\text{ is completely positive } \;\leftrightarrow\; \rho_{B|A}^{T_A} \geq 0$$

$$\rho_{B|A} \text{ is PPT}$$

Proof: $\rho_{B|A} = (\Phi_{B|A'} \otimes \mathrm{id}_A)\left[(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |j\rangle\langle k|_A)^{T_A}\right]$

$= \left[(\Phi_{B|A'} \otimes \mathrm{id}_A)(d_A|\Psi^+\rangle_{A'A}\langle\Psi^+|)\right]^{T_A}$ QED

Classical belief propagation

$$P_B = \Gamma_{B|A}[P_A]$$

Stochastic map preserving positivity and normalization

In terms of a conditional

$$P_B = \sum_A P_{B|A} P_A$$

$$\sum_B P_{B|A} = 1$$

$$P_{B|A} \geq 0$$

Quantum belief propagation

$$\rho_B = \mathcal{E}_{B|A}(\rho_A)$$

Completely positive trace-preserving map

In terms of a conditional

$$\rho_B = \mathrm{Tr}_A(\rho_{B|A}\rho_A)$$

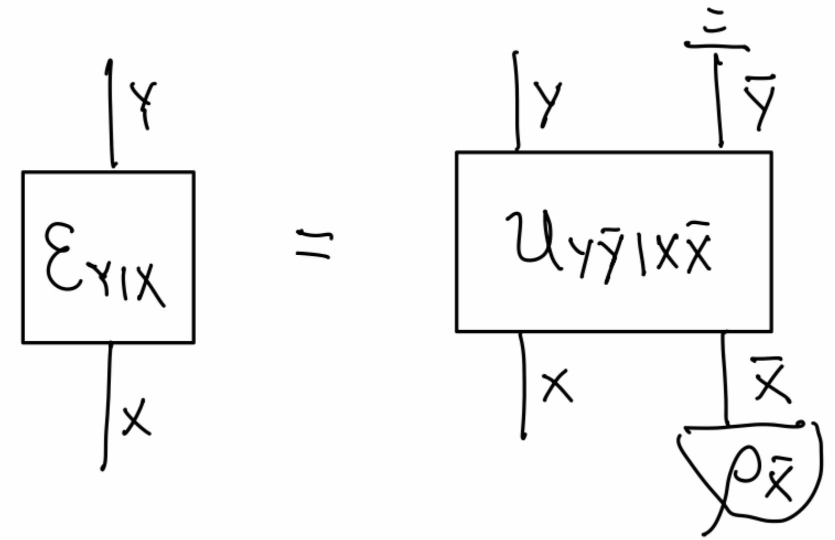
$$\mathrm{Tr}_B(\rho_{B|A}) = I_A$$

$$\rho_{B|A}^{T_A} \geq 0$$

The quantum analogue of the difference between deterministic dependence and probabilistic dependence

The Stinespring dilation theorem

Completely positive trace-preserving map



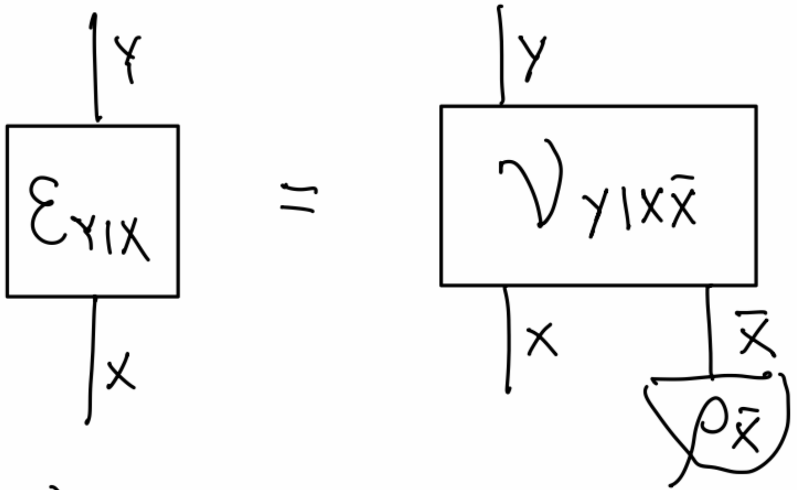
Unitary map

$$\mathcal{U}(\cdot) = U \cdot U^\dagger$$

$\exists\, \mathcal{U}_{Y\bar{Y}|X\bar{X}},\ \rho_{\bar{X}}$:

$$\mathcal{E}_{Y|X}(\cdot) = \mathrm{Tr}_{\bar{X}}\left(\mathcal{U}_{Y\bar{Y}|X\bar{X}}(\cdot \otimes \rho_{\bar{X}})\right)$$

Completely positive trace-preserving map



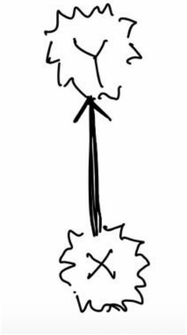
Reduced unitary map

$\exists\ \mathcal{V}_{Y|X\bar{X}},\ \rho_{\bar{X}}:$

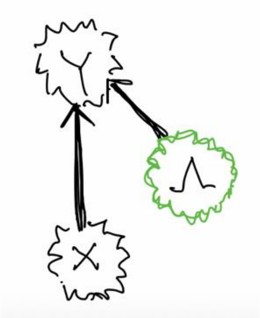
$$\mathcal{E}_{Y|X}(\cdot) = \mathrm{Tr}_{\bar{X}}\left(\mathcal{V}_{Y|X\bar{X}}(\cdot \otimes \rho_{\bar{X}})\right)$$

$$\rho_{Y|X} = \mathrm{Tr}_{\bar{X}}\left(\rho^{det}_{Y|X\bar{X}}\, \rho_{\bar{X}}\right)$$

where $\rho^{det}_{Y|X\bar{X}}$ is CJ op' for $\mathcal{V}_{Y|X\bar{X}}$



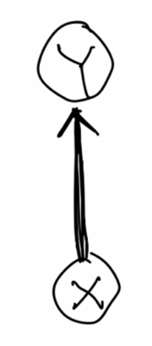
$\rho_{Y|X}$



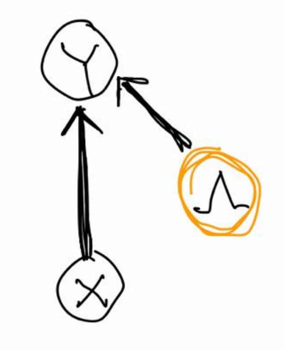
$\mathcal{V}_{Y|X\Lambda}$
ρ_Λ

$$\rho_{Y|X} = \mathrm{Tr}_\Lambda \left(\rho^{\mathrm{det}}_{Y|X\Lambda} \rho_\Lambda \right)$$

where $\rho^{\mathrm{det}}_{Y|X\Lambda}$ is CJ-isomorphic to $\mathcal{V}_{Y|X\Lambda}$



$P_{Y|X}$

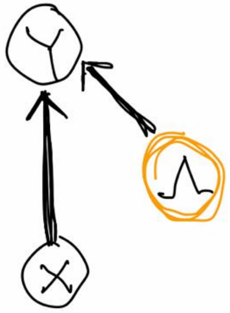


$Y = f(X, \Lambda)$
P_{Λ}

$$P_{Y|X} = \sum_{\Lambda} P^{\mathrm{det}}_{Y|X\Lambda} P_{\Lambda}$$

where $P^{\mathrm{det}}_{Y|X\Lambda} = \delta_{Y,f(X,\Lambda)}$

$P_{Y|X}$

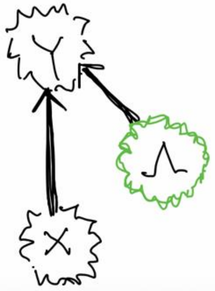


$Y = f(X,\Lambda)$
P_Λ

$$P_{Y|X} = \sum_\Lambda P^{\text{det}}_{Y|X\Lambda} P_\Lambda$$

where $P^{\text{det}}_{Y|X\Lambda} = \delta_{Y,f(X,\Lambda)}$

$\rho_{Y|X}$

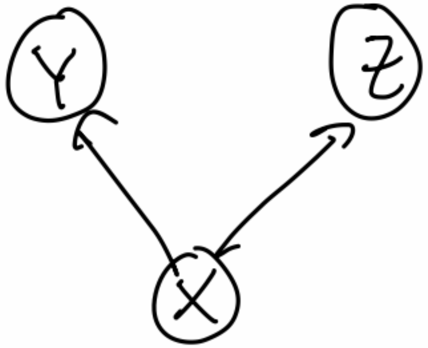


$\mathcal{V}_{Y|X\Lambda}$
ρ_Λ

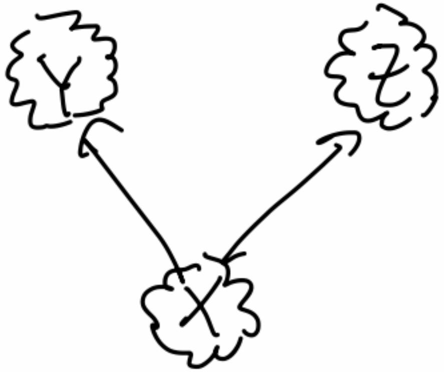
$$\rho_{Y|X} = \mathrm{Tr}_\Lambda\left(\rho^{\text{det}}_{Y|X\Lambda}\rho_\Lambda\right)$$

where $\rho^{\text{det}}_{Y|X\Lambda}$ is CJ-isomorphic to $\mathcal{V}_{Y|X\Lambda}$

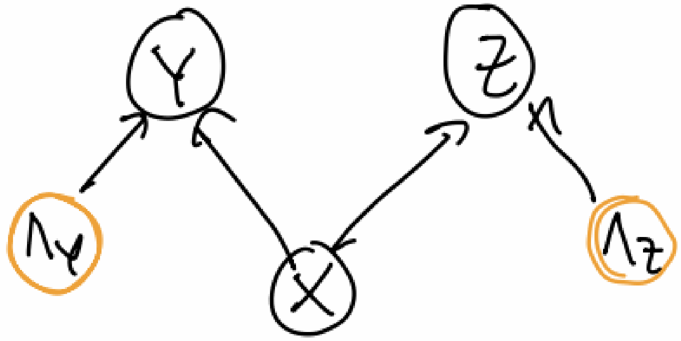
One system a complete common cause of two others



$$P_{YZ|X} = P_{Y|X} P_{Z|X}$$



$$\rho_{YZ|X} = \rho_{Y|X} \rho_{Z|X}$$

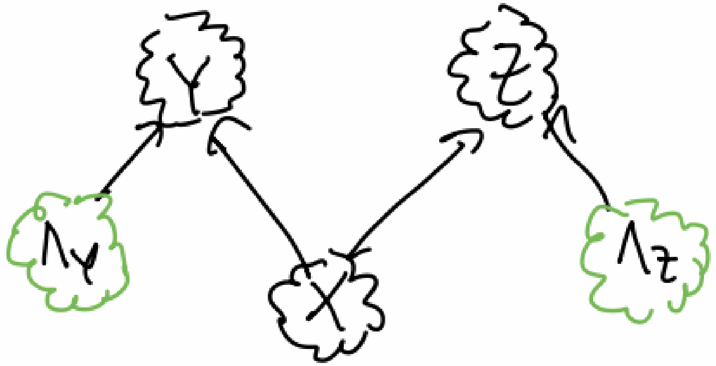


$$P^{det}_{YZ|\Lambda_Y X \Lambda_Z} = P^{det}_{Y|\Lambda_Y X}\, P^{det}_{Z|\Lambda_Z X}$$

$$P_{YZ|X} = \sum_{\Lambda_Y \Lambda_Z} P^{det}_{YZ|\Lambda_Y X \Lambda_Z} P_{\Lambda_Y} P_{\Lambda_Z}$$

$$= \Big(\sum_{\Lambda_Y} P^{det}_{Y|\Lambda_Y X} P_{\Lambda_Y}\Big)\Big(\sum_{\Lambda_Z} P^{det}_{Z|\Lambda_Z X} P_{\Lambda_Z}\Big)$$

$$= P_{Y|X}\, P_{Z|X}$$

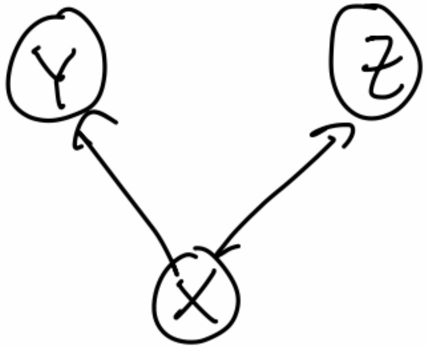


$$\rho^{det}_{YZ|\Lambda_Y X \Lambda_Z} = \rho^{det}_{Y|\Lambda_Y X}\, \rho^{det}_{Z|\Lambda_Z X}$$

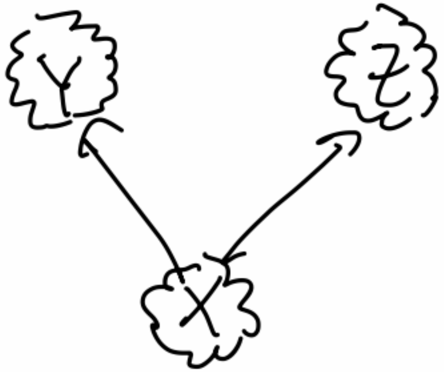
$$\rho_{YZ|X} = \mathrm{Tr}_{\Lambda_Y \Lambda_Z}\big(\rho^{det}_{YZ|\Lambda_Y X \Lambda_Z}\, \rho_{\Lambda_Y} \otimes \rho_{\Lambda_Z}\big)$$

$$= \mathrm{Tr}_{\Lambda_Y}\big(\rho^{det}_{Y|\Lambda_Y X}\, \rho_{\Lambda_Y}\big)\, \mathrm{Tr}_{\Lambda_Z}\big(\rho^{det}_{Z|\Lambda_Z X}\, \rho_{\Lambda_Z}\big)$$

$$= \rho_{Y|X}\, \rho_{Z|X}$$



$$p_{YZ|X} = p_{Y|X}\, p_{Z|X}$$



$$\rho_{YZ|X} = \rho_{Y|X}\, \rho_{Z|X}$$

B and C are conditionally independent given A

$$\rho_{BC|A} = \rho_{B|A}\rho_{C|A}$$

Denote this
$(B \perp C|A)$

Note: Hermiticity of $\rho_{BC|A}, \rho_{B|A}, \rho_{C|A}$

implies $[\rho_{B|A}, \rho_{C|A}] = 0$

There is also a quantum analogue of the
causal Markov condition

If X_1, X_2,. . . , X_n have causal relations described by a DAG G, then the set of compatible quantum processes are those of the form:

$$\rho_{X_1,X_2,\ldots,X_n} = \prod_i \rho_{X_i|\mathrm{Parents}_G(X_i)}$$

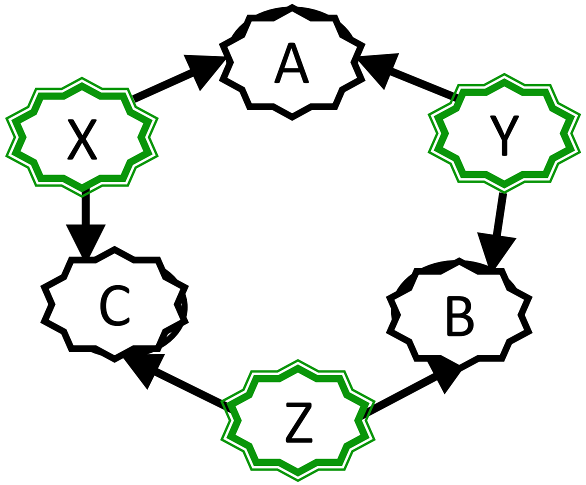
(but I won't be able to explain how it is different)

The fully quantum causal compatibility problem:

What quantum states/processes are compatible with a given causal structure?

Conventional assumption:
dimension of latent quantum systems are arbitrary

Triangle scenario

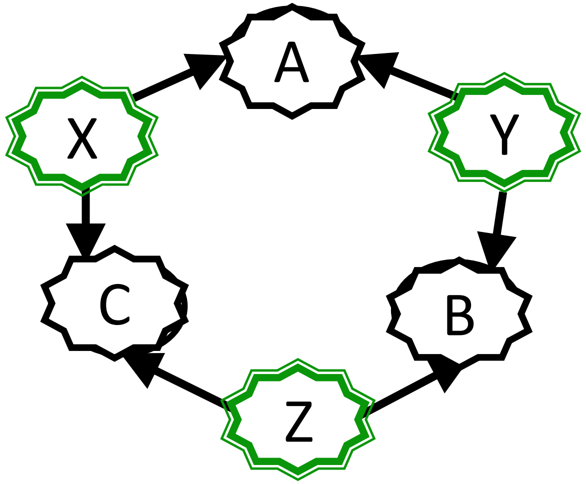


$$\rho_{A|XY}$$
$$\rho_{B|YZ}$$
$$\rho_{C|XZ}$$
$$\rho_X$$
$$\rho_Y$$
$$\rho_Z$$

$$[\rho_{A|XY}, \rho_{B|YZ}] = 0$$
$$[\rho_{B|YZ}, \rho_{C|XZ}] = 0$$
$$[\rho_{A|XY}, \rho_{C|XZ}] = 0$$

$$\rho_{ABC} = \mathrm{Tr}_{XYZ}\left(\rho_{A|XY}\rho_{B|YZ}\rho_{C|XZ}\rho_X\rho_Y\rho_Z\right)$$

Triangle scenario



$\rho_{A|X_A Y_A}$

$\rho_{B|Y_B Z_B}$

$\rho_{C|X_C Z_C}$

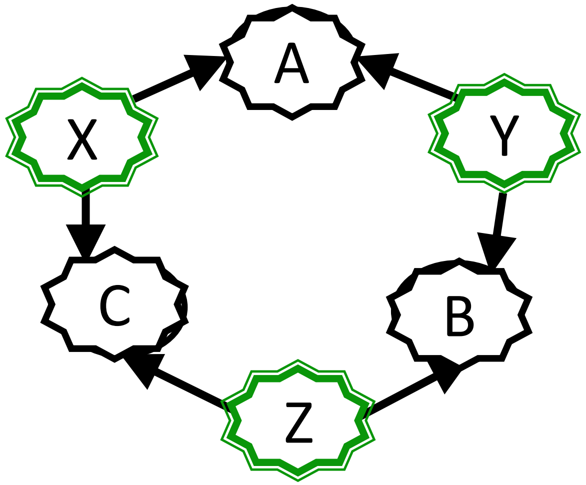
$\rho_{X_A X_C}$

$\rho_{Y_A Y_B}$

$\rho_{Z_B Z_C}$

$$\rho_{ABC} = \mathrm{Tr}_{X_A X_C Y_A Y_B Z_C Z_B}\left(\rho_{A|X_A Y_A}\rho_{B|Y_B Z_B}\rho_{C|X_C Z_C}\rho_{X_A X_C}\rho_{Y_A Y_B}\rho_{Z_B Z_C}\right)$$

Triangle scenario



$\mathcal{E}_{A|X_AY_A}$

$\mathcal{E}_{B|Y_BZ_B}$

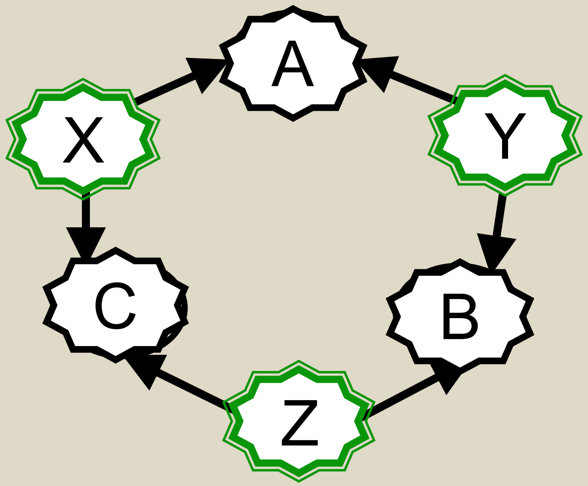
$\mathcal{E}_{C|X_CZ_C}$

$\rho_{X_AX_C}$

$\rho_{Y_AY_B}$

$\rho_{Z_BZ_C}$

$$\rho_{ABC} = \mathcal{E}_{A|X_AY_A} \otimes \mathcal{E}_{B|Y_BZ_B} \otimes \mathcal{E}_{C|X_CZ_C} \left(\rho_{X_AX_C}\rho_{Y_AY_B}\rho_{Z_BZ_C}\right)$$

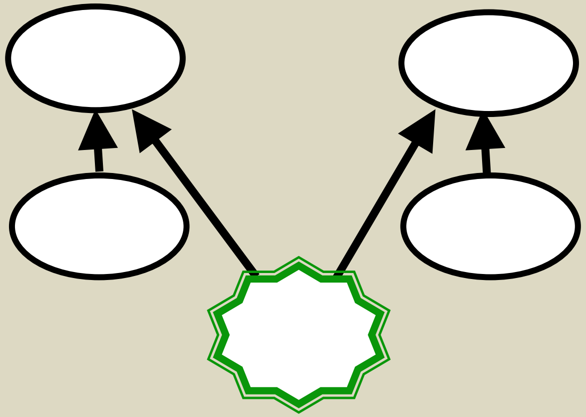


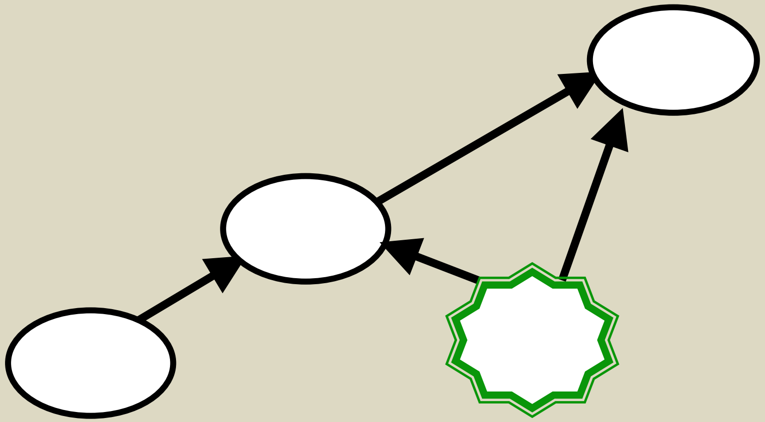
A, B and C
qubits

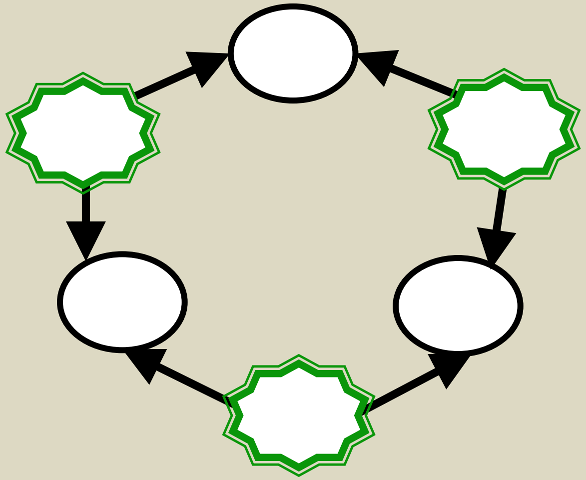
$$I - \rho_A - \rho_B - \rho_C + \rho_A \otimes \rho_B + \rho_{BC} + \rho_{AC} \geq 0$$

where identity operators are implicit

Causal models where only the latents can be quantum





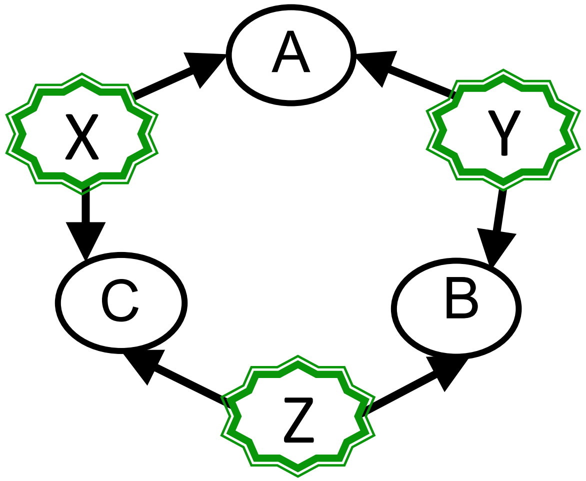


The causal compatibility problem in this context:

What probability distributions over classical variables are compatible with a given causal structure when the latent systems can be quantum?

Conventional assumption:

dimension of latent quantum system is arbitrary



$\rho_{A|XY}$

$\rho_{B|YZ}$

$\rho_{C|XZ}$

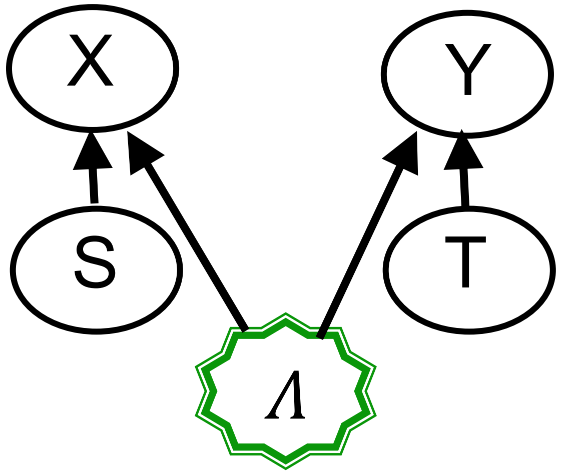
ρ_X

ρ_Y

ρ_Z

$$[\rho_{A|XY}, \rho_{B|YZ}] = 0$$
$$[\rho_{B|YZ}, \rho_{C|XZ}] = 0$$
$$[\rho_{A|XY}, \rho_{C|XZ}] = 0$$

$$P_{ABC} = \mathrm{Tr}_{XYZ}\left(\rho_{A|XY}\rho_{B|YZ}\rho_{C|XZ}\rho_X\rho_Y\rho_Z\right)$$



$\rho_{X|S\Lambda}$

$\rho_{Y|T\Lambda}$

ρ_{Λ}

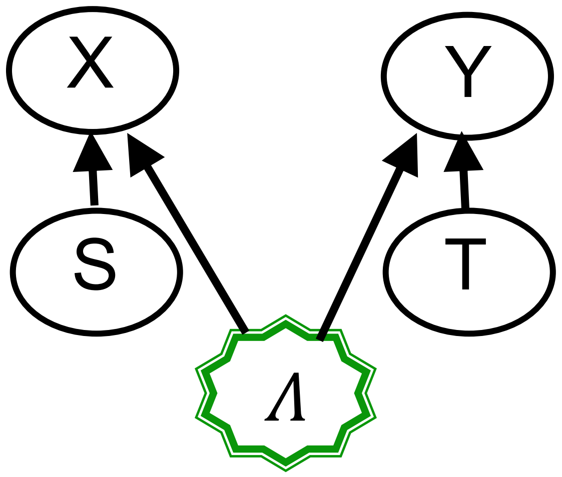
$$[\rho_{X|S\Lambda}, \rho_{Y|T\Lambda}] = 0$$

$$P_{XY|ST} = \mathrm{Tr}_{\Lambda}(\rho_{X|S\Lambda}\rho_{Y|T\Lambda}\rho_{\Lambda})$$

Quantum causal models
include
classical causal models
as a special case

(when all conditional states are diagonal in some product basis)

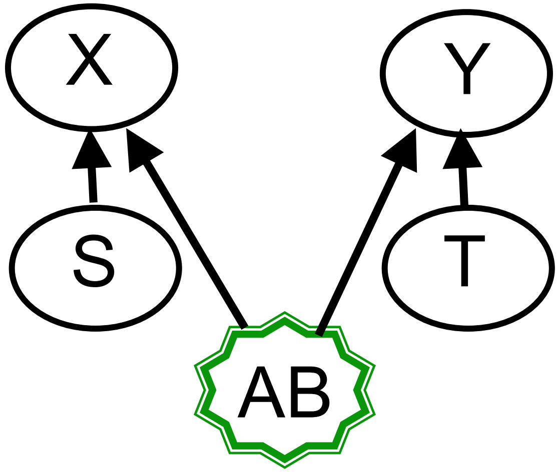
Bell scenario



$$\rho_{X|S\Lambda}$$
$$\rho_{Y|T\Lambda}$$
$$\rho_{\Lambda}$$

$$[\rho_{X|S\Lambda}, \rho_{Y|T\Lambda}] = 0$$

$$P_{XY|ST} = \mathrm{Tr}_{\Lambda}(\rho_{X|S\Lambda}\rho_{Y|T\Lambda}\rho_{\Lambda})$$

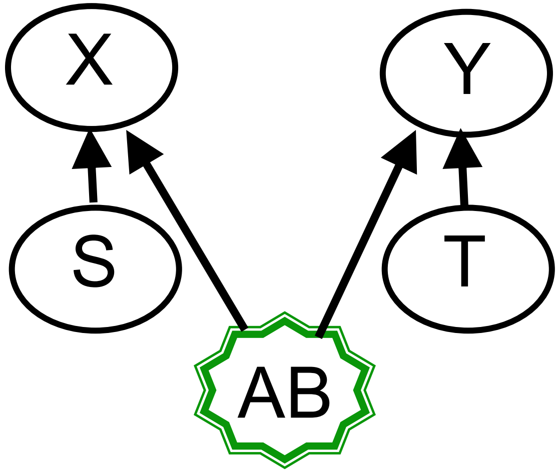


$\rho_{X|SA}$

$\rho_{Y|TB}$

ρ_{AB}

$$P_{XY|ST} = \mathrm{Tr}_{AB}(\rho_{X|SA}\rho_{Y|TB}\rho_{AB})$$

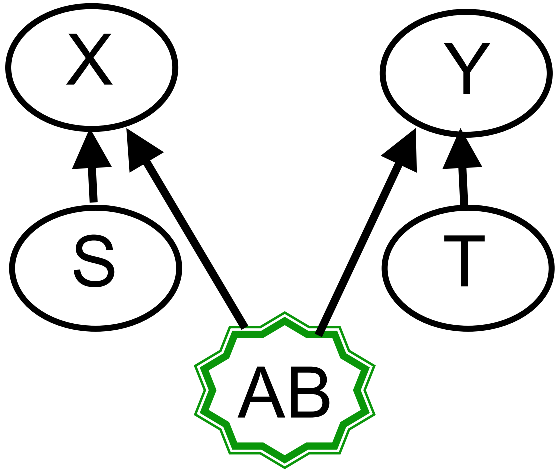


$\{E^A_{x|s}\}_x$ for each s

$\{E^B_{y|t}\}_y$ for each t

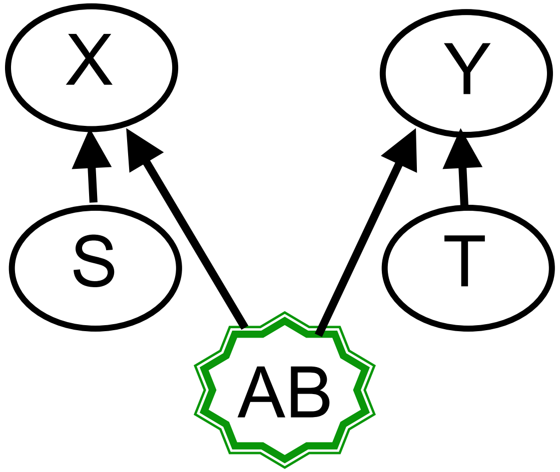
ρ_{AB}

$$P_{XY|ST}(xy|st) = \mathrm{Tr}_{AB}((E^A_{x|s} \otimes E^B_{y|t})\rho_{AB})$$



$$\rho_{X|SA}$$
$$\rho_{Y|TB}$$
$$\rho_{AB}$$

$$P_{XY|ST} = \mathrm{Tr}_{AB}(\rho_{X|SA}\rho_{Y|TB}\rho_{AB})$$



$$\rho_{X|SA}$$
$$\rho_{Y|TB}$$
$$\rho_{AB}$$

$$P_{XY|ST} = \mathrm{Tr}_{AB}(\rho_{X|SA}\rho_{Y|TB}\rho_{AB})$$

Causal compatibility constraints:

$$P_{X|ST} = P_{X|S}$$
$$P_{Y|ST} = P_{Y|T}$$

The d-separation theorem:

Consider a causal structure G and three disjoint subsets of variables **X**, **Y** and **Z**.

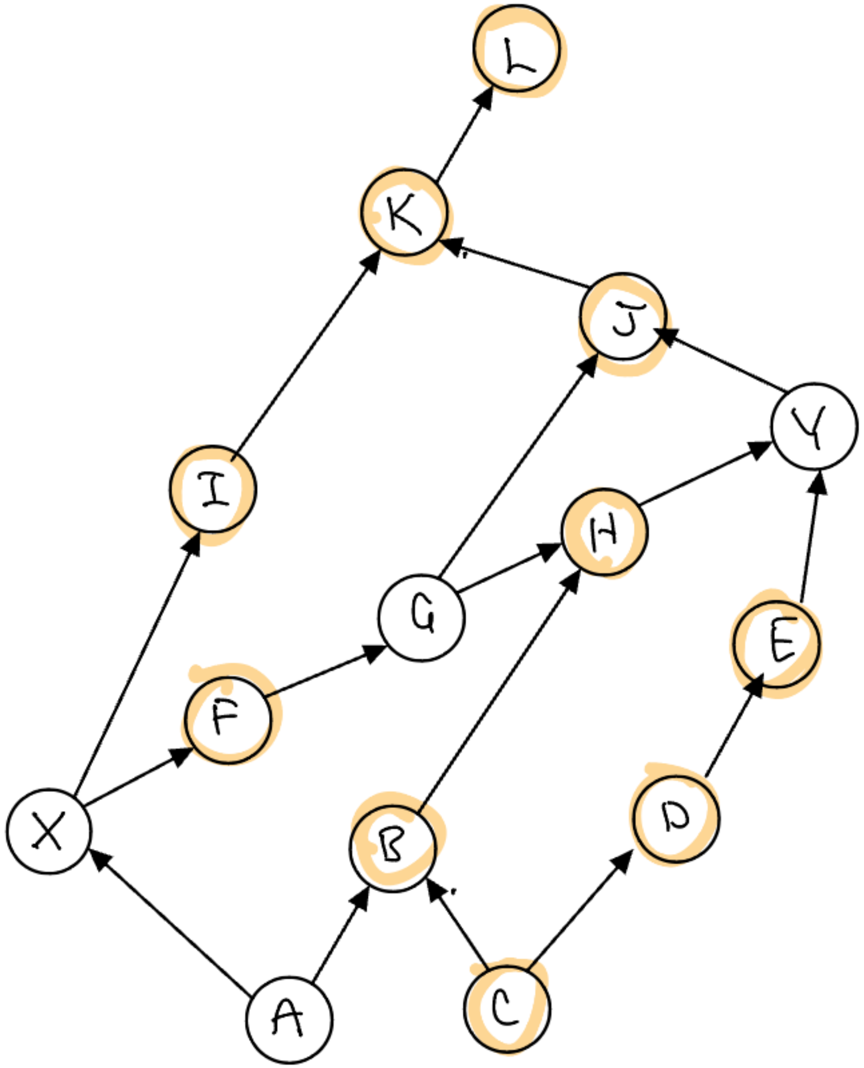
Soundness

d-separation between **X** and **Y** given **Z** in G
implies $\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \mid \mathbf{Z}$ in all distributions P compatible with G

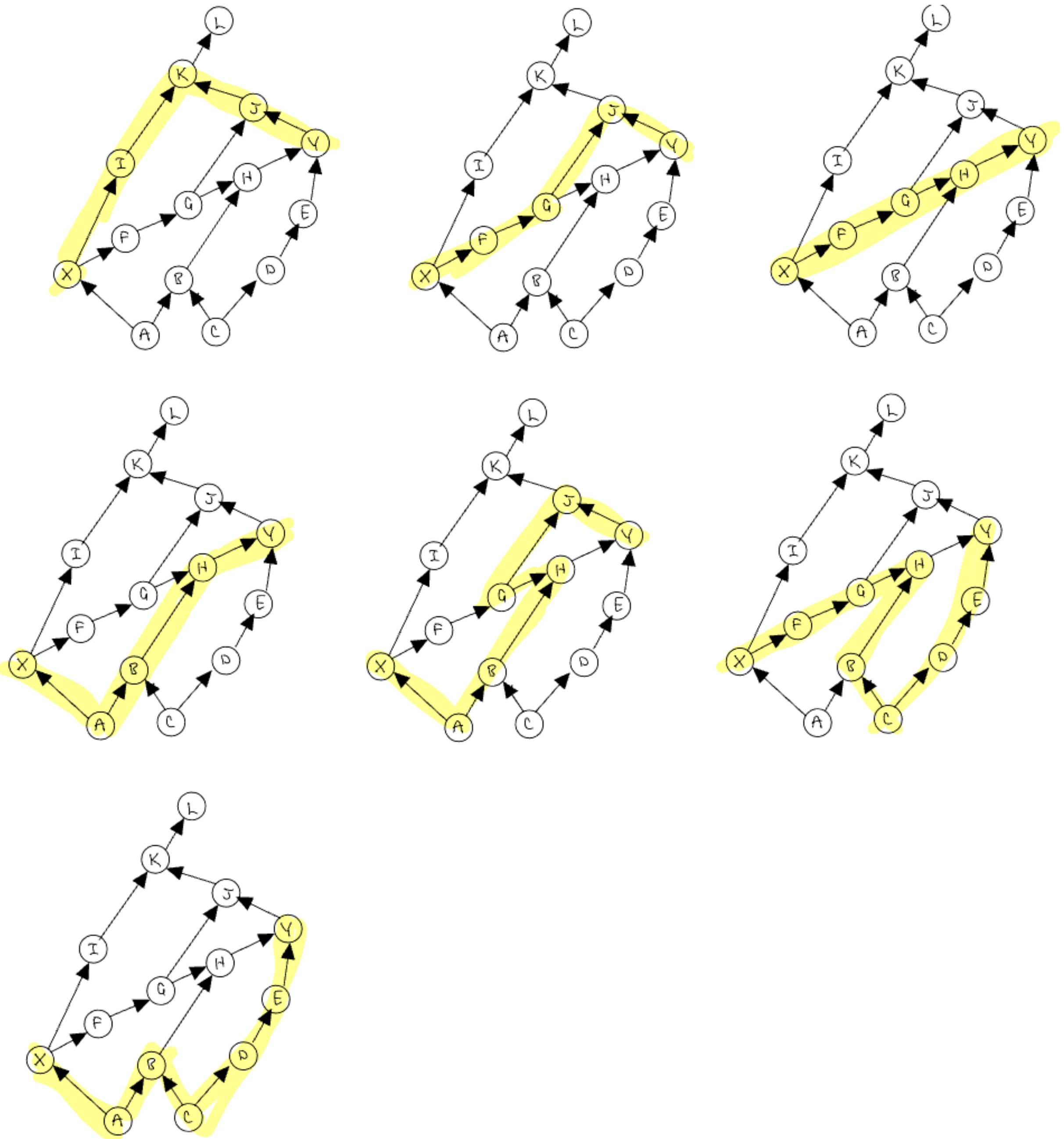
Completeness

$\mathbf{X} \perp\!\!\!\perp \mathbf{Y} \mid \mathbf{Z}$ in all distributions P compatible with G
implies d-separation between **X** and **Y** given **Z** in G

For classical sets of variables **X**, **Y** and **Z** in the DAG, this still holds, even if some or all of the latent nodes are quantum

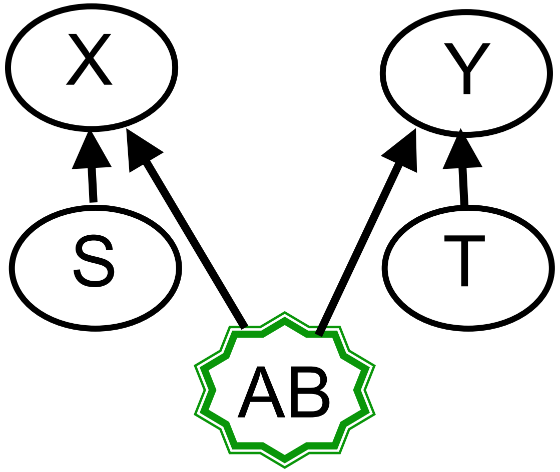


$$X \perp Y | GA$$



Intuition for proof: path blocking removes correlations, regardless of whether the latent variables are classical or quantum

Conditioning on a classical noncollider in the path from X to Y still makes X and Y independent
Marginalizing on a classical or quantum collider in the path from X to Y, still makes X and Y independent



$\rho_{X|SA}$

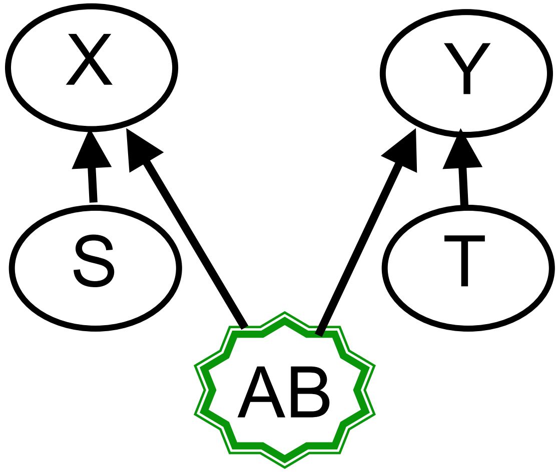
$\rho_{Y|TB}$

ρ_{AB}

$$P_{XY|ST} = \mathrm{Tr}_{AB}(\rho_{X|SA}\rho_{Y|TB}\rho_{AB})$$

Causal compatibility constraints:

$$P_{X|ST} = P_{X|S}$$
$$P_{Y|ST} = P_{Y|T}$$



$\rho_{X|SA}$

$\rho_{Y|TB}$

ρ_{AB}

$$P_{XY|ST} = \mathrm{Tr}_{AB}(\rho_{X|SA}\rho_{Y|TB}\rho_{AB})$$

Causal compatibility constraints:

$$P_{X|ST} = P_{X|S}$$
$$P_{Y|ST} = P_{Y|T}$$

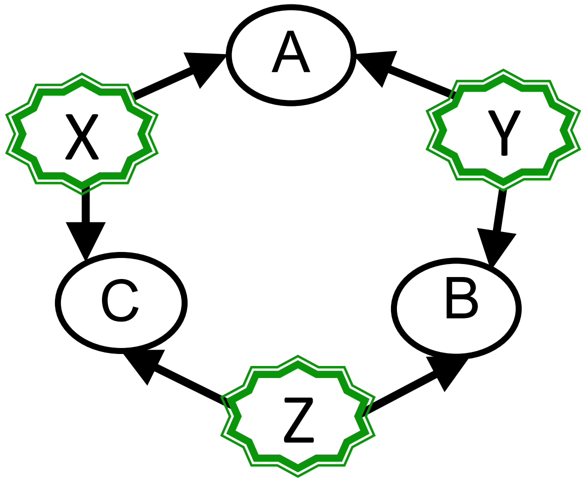
$$\boxed{\begin{array}{l}\frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|00) + \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|01) \\ \quad \frac{1}{4}\sum_{x=y} P_{XY|ST}(xy|10) + \frac{1}{4}\sum_{x\neq y} P_{XY|ST}(xy|11) \leq \frac{1}{2} + \frac{1}{2\sqrt{2}}\end{array}}$$

Tsirelson, Lett. Math. Phys. 4, 93 (1980)

In Bell scenario, quantum bounds can be derived from
The NPA semidefinite programming hierarchy
Navascués, Pironio, and Acín, New J. Phys. 10, 073013 (2008)

For other causal structures, one needs a new tool:
inflation technique with quantum latent nodes

Triangle scenario



$\rho_{A|XY}$

$\rho_{B|YZ}$

$\rho_{C|XZ}$

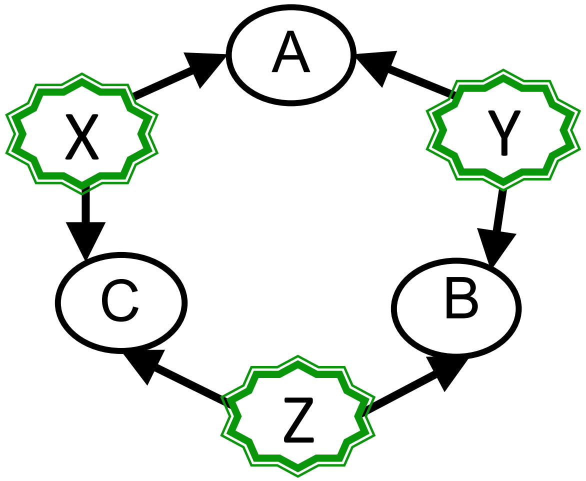
ρ_X

ρ_Y

ρ_Z

$$[\rho_{A|XY}, \rho_{B|YZ}] = 0$$
$$[\rho_{B|YZ}, \rho_{C|XZ}] = 0$$
$$[\rho_{A|XY}, \rho_{C|XZ}] = 0$$

$$P_{ABC} = \mathrm{Tr}_{XYZ}\left(\rho_{A|XY}\rho_{B|YZ}\rho_{C|XZ}\rho_X\rho_Y\rho_Z\right)$$



$$\rho_{A|X_A Y_A}$$

$$\rho_{B|Y_B Z_B}$$

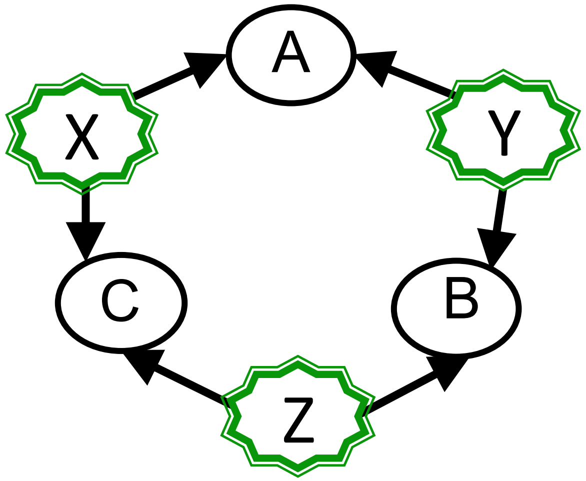
$$\rho_{C|X_C Z_C}$$

$$\rho_{X_A X_C}$$

$$\rho_{Y_A Y_B}$$

$$\rho_{Z_B Z_C}$$

$$P_{ABC} = \mathrm{Tr}_{X_A X_C Y_A Y_B Z_C Z_B}\left(\rho_{A|X_A Y_A}\rho_{B|Y_B Z_B}\rho_{C|X_C Z_C}\rho_{X_A X_C}\rho_{Y_A Y_B}\rho_{Z_B Z_C}\right)$$



$\{E_a^{X_A Y_A}\}_a$

$\{E_b^{Y_B Z_B}\}_b$

$\{E_c^{X_C Z_C}\}_c$

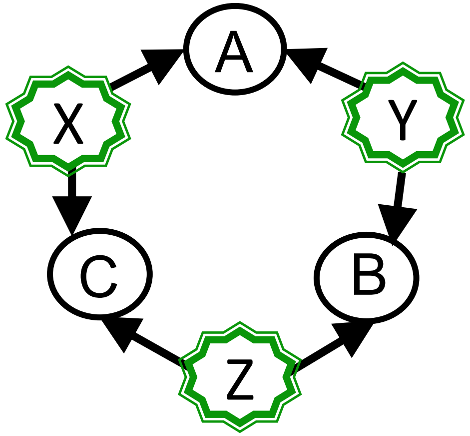
$\rho_{X_A X_C}$

$\rho_{Y_A Y_B}$

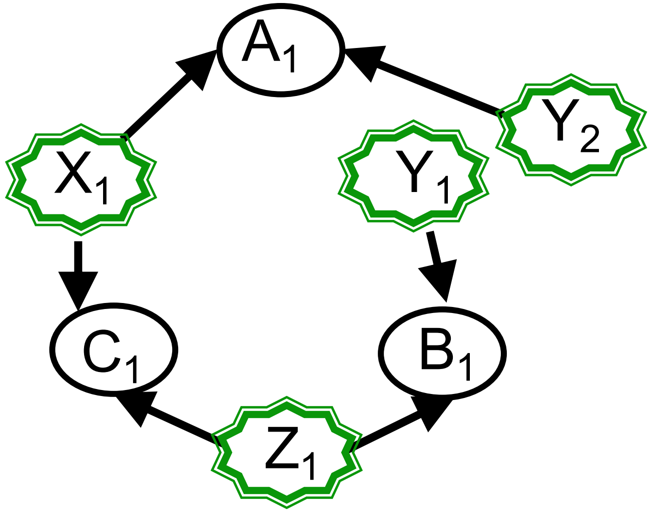
$\rho_{Z_B Z_C}$

$$P_{ABC}(abc) = \mathrm{Tr}_{X_A X_C Y_A Y_B Z_A Z_C}\left(E_a^{X_A Y_A} E_b^{Y_B Z_B} E_c^{X_C Z_C} \rho_{X_A X_C} \rho_{Y_A Y_B} \rho_{Z_B Z_C}\right)$$

No symmetry of dependence (e.g., A_1 and A_2 on X_1) because of **no-cloning**

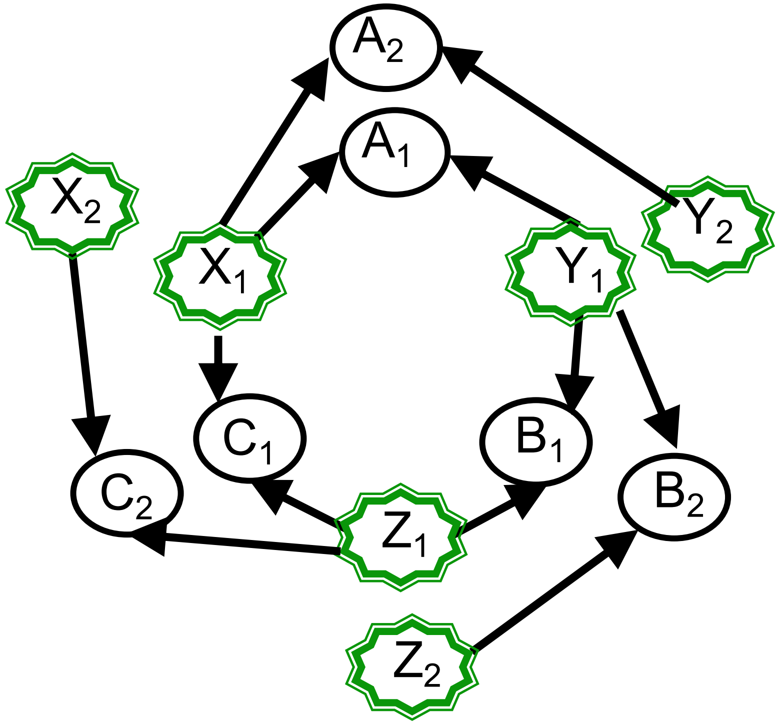


Triangle



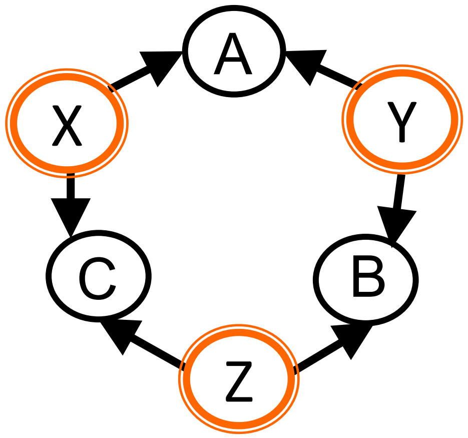
Cut inflation of Triangle

Nonfanout

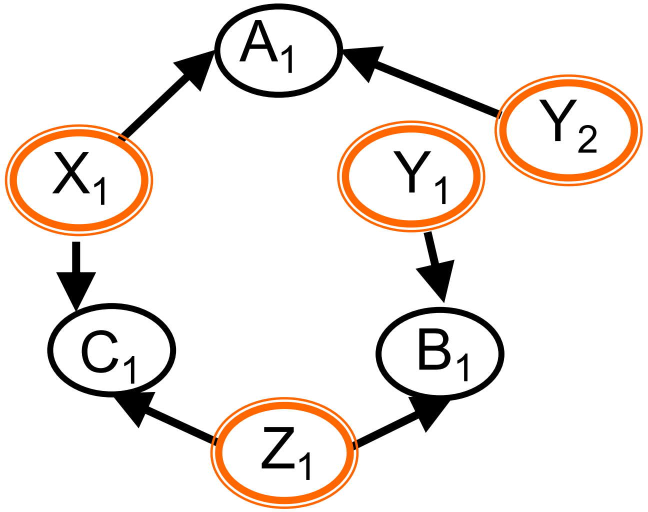


Spiral inflation of Triangle

Fanout

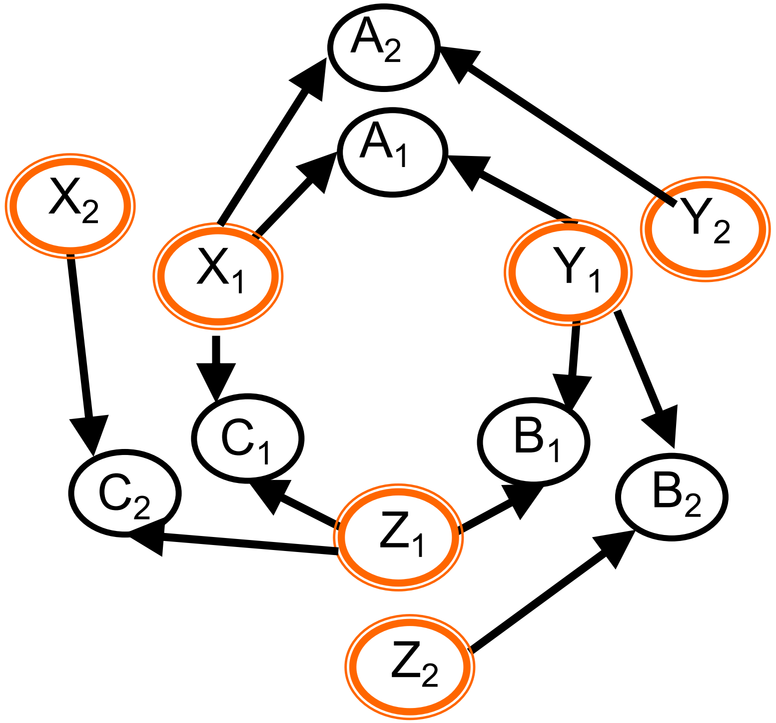


Triangle



Cut inflation of Triangle

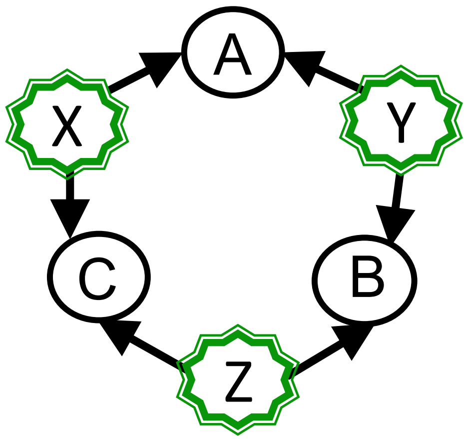
Nonfanout



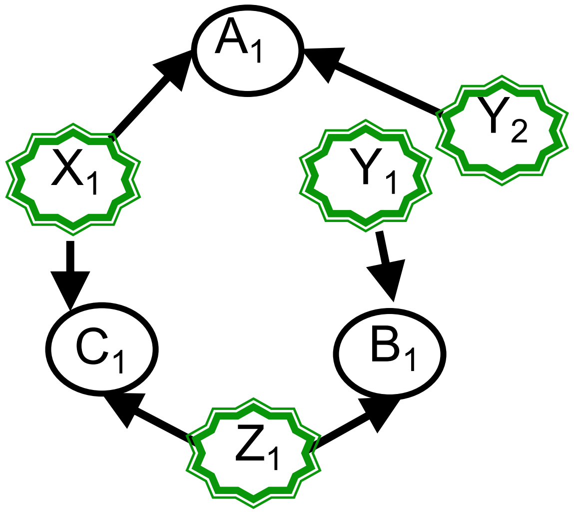
Spiral inflation of Triangle

Fanout

Nonfanout inflations



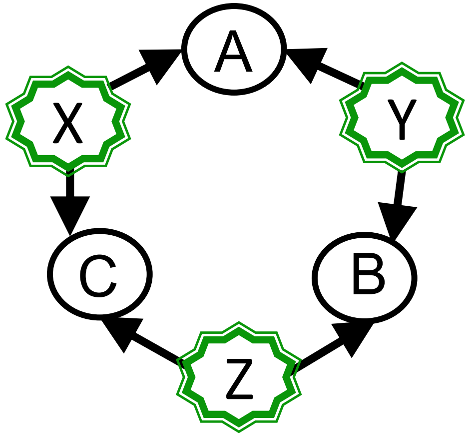
Triangle



Cut inflation of Triangle

Nonfanout

Quantum model M on DAG G



$\rho_{A|XY}$

$\rho_{B|YZ}$

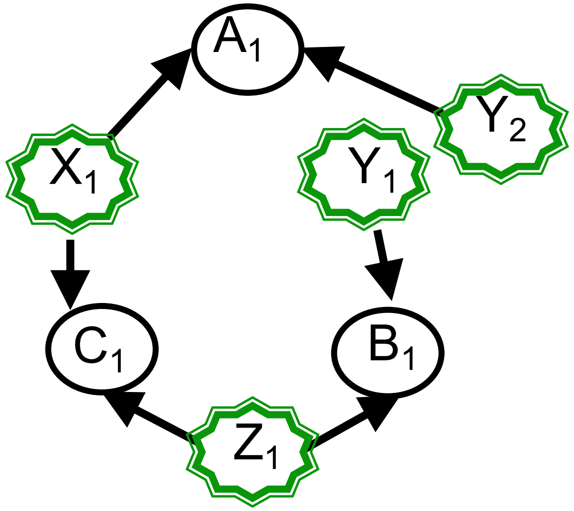
$\rho_{C|XZ}$

ρ_X

ρ_Y

ρ_Z

Quantum model M' = G $\rightarrow$ G' Inflation of M



$\rho_{A_1|X_1Y_2}$

$\rho_{B_1|Y_1Z_1}$

$\rho_{C_1|X_1Z_1}$

ρ_{X_1}

ρ_{Y_1}

ρ_{Y_2}

ρ_{Z_1}

with symmetry constraint:

$\rho_{Y_1} = \rho_{Y_2}$

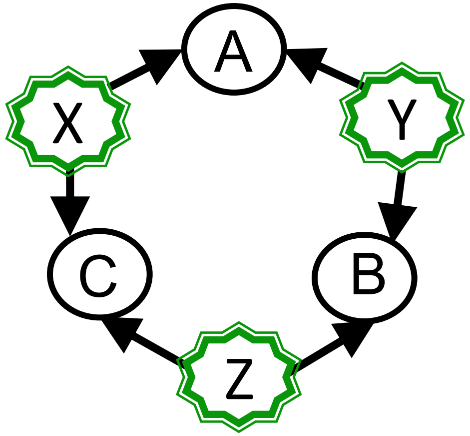
$\{A_1C_1\}$ is an injectable set

$$P_{A_1C_1} = \mathrm{Tr}_{X_1Y_2Z_1}\left(\rho_{A_1|X_1Y_2}\rho_{C_1|X_1Z_1}\rho_{X_1}\rho_{Y_2}\rho_{Z_1}\right)$$

$$P_{AC} = \mathrm{Tr}_{XYZ}\left(\rho_{A|XY}\rho_{C|XZ}\rho_X\rho_Y\rho_Z\right)$$

$$P_{AC} \text{ compatible with } M \quad \Longrightarrow \quad P_{A_1C_1} = P_{AC} \text{ compatible with } M'$$

Quantum model M on DAG G



$\rho_{A|XY}$

$\rho_{B|YZ}$

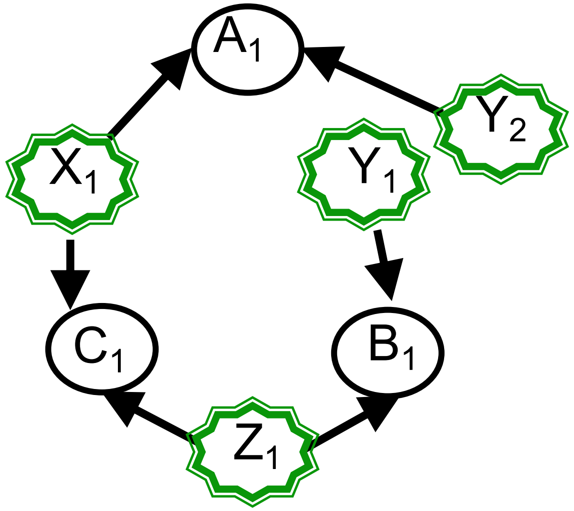
$\rho_{C|XZ}$

ρ_X

ρ_Y

ρ_Z

Quantum model M' = G → G' Inflation of M



$\rho_{A_1|X_1Y_2}$

$\rho_{B_1|Y_1Z_1}$

$\rho_{C_1|X_1Z_1}$

ρ_{X_1}

ρ_{Y_1}

ρ_{Y_2}

ρ_{Z_1}

with symmetry constraint:

$\rho_{Y_1} = \rho_{Y_2}$

Injectable sets: $\{A_1\}, \{B_1\}, \{C_1\}, \{A_1C_1\}, \{B_1C_1\}$

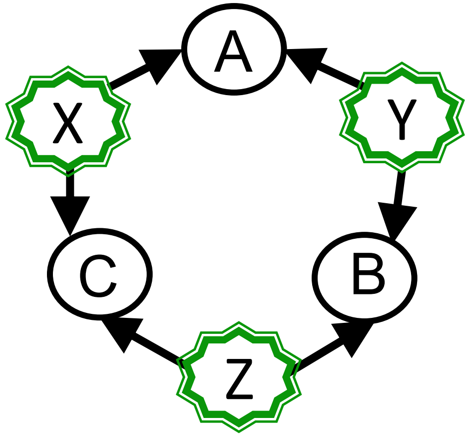
$(P_A, P_B, P_C, P_{AC}, P_{BC})$ is compatible with M $\Longrightarrow$ $(P_{A_1}, P_{B_1}, P_{C_1}, P_{A_1C_1}, P_{B_1C_1})$

where

$$\mathrm{P}_{A_1} = \mathrm{P}_A \qquad \mathrm{P}_{A_1C_1} = \mathrm{P}_{AC}$$
$$\mathrm{P}_{B_1} = \mathrm{P}_B \qquad \mathrm{P}_{B_1C_1} = \mathrm{P}_{BC}$$
$$\mathrm{P}_{C_1} = \mathrm{P}_C$$

is compatible with M'

Quantum model M on DAG G



$\rho_{A|XY}$

$\rho_{B|YZ}$

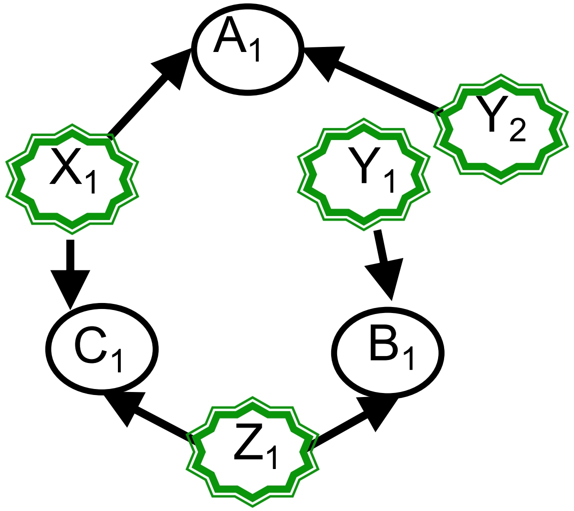
$\rho_{C|XZ}$

ρ_X

ρ_Y

ρ_Z

Quantum model M' = G → G' Inflation of M



$\rho_{A_1|X_1Y_2}$

$\rho_{B_1|Y_1Z_1}$

$\rho_{C_1|X_1Z_1}$

ρ_{X_1}

ρ_{Y_1}

ρ_{Y_2}

ρ_{Z_1}

with symmetry constraint:

$\rho_{Y_1} = \rho_{Y_2}$

Injectable sets: $\{A_1\}, \{B_1\}, \{C_1\}, \{A_1C_1\}, \{B_1C_1\}$

$(P_A, P_B, P_C, P_{AC}, P_{BC})$

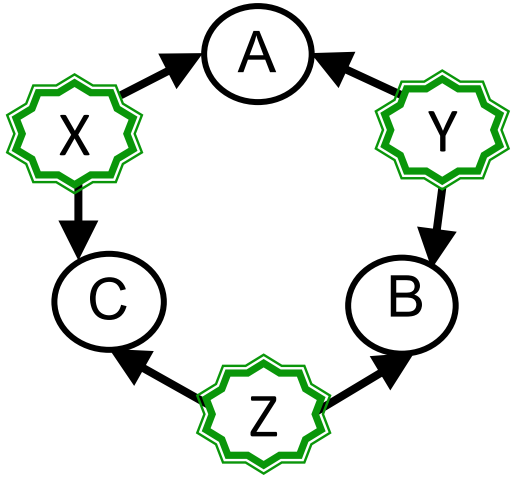
is not compatible with M

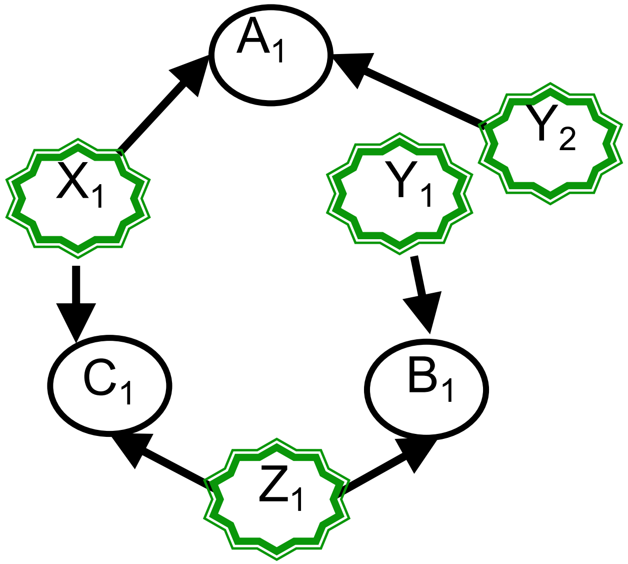
$\Longleftarrow$

$(P_{A_1}, P_{B_1}, P_{C_1}, P_{A_1C_1}, P_{B_1C_1})$

where
$\mathrm{P}_{A_1} = \mathrm{P}_A$ $\quad$ $\mathrm{P}_{A_1C_1} = \mathrm{P}_{AC}$
$\mathrm{P}_{B_1} = \mathrm{P}_B$ $\quad$ $\mathrm{P}_{B_1C_1} = \mathrm{P}_{BC}$
$\mathrm{P}_{C_1} = \mathrm{P}_C$

is not compatible with M'





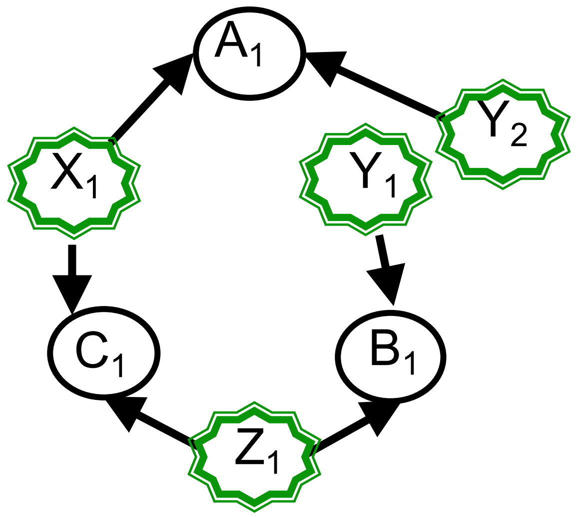
$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$
is a causal compatibility
inequality for M

$\Longleftarrow$

$P_{A_1} + P_{B_1} + P_{C_1} - P_{A_1} P_{B_1} - P_{B_1 C_1} - P_{A_1 C_1} \leq 1$
is a causal compatibility
inequality for M'

$$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$$
is a valid set of marginals $\implies$ $(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ satisfy

$$P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$$



$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is compatible with M'

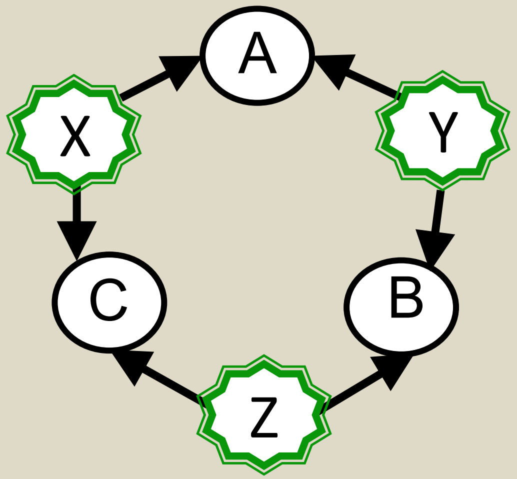
$$A_1 \perp_d B_1 \implies P_{A_1B_1} = P_{A_1}P_{B_1}$$

Implications from d-separation to conditional independences hold in quantum causal models

$(P_{A_1C_1}, P_{B_1C_1}, P_{A_1B_1})$ is compatible with M' $\implies$

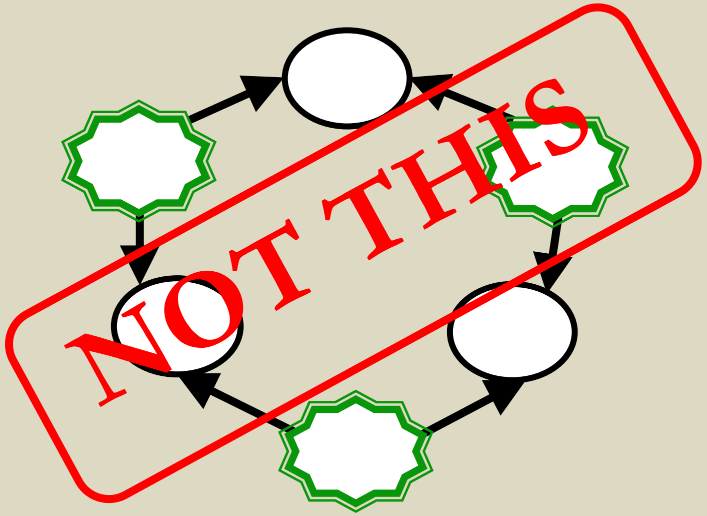
$$P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1}P_{B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$$

$$P_{A_1}+P_{B_1}+P_{C_1}-P_{A_1}P_{B_1}-P_{B_1C_1}-P_{A_1C_1} \leq 1$$
is a causal compatibility inequality for M'



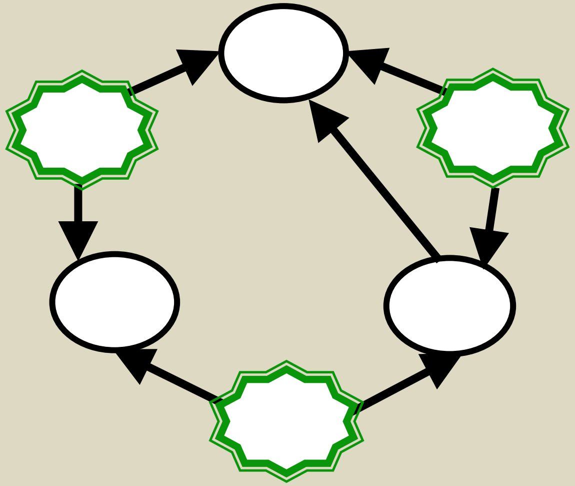
$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$

Violation of causal compatibility inequalities from **nonfanout** inflations

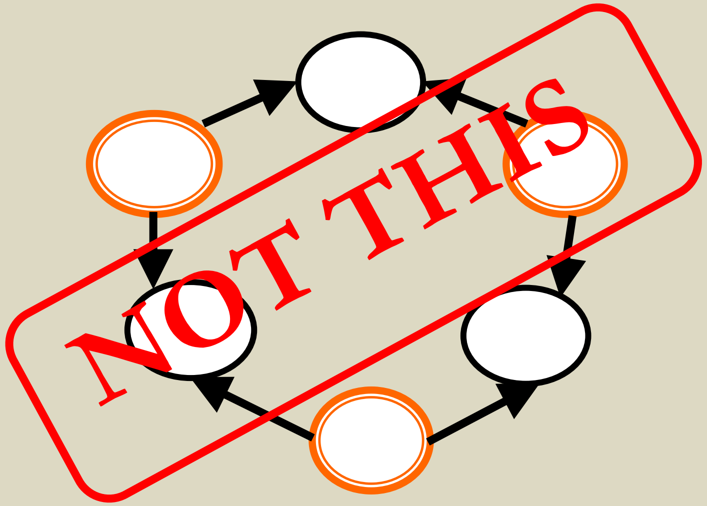


$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \leq 1$$

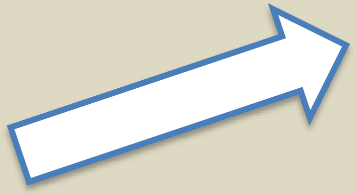
Witnessing need for different structure



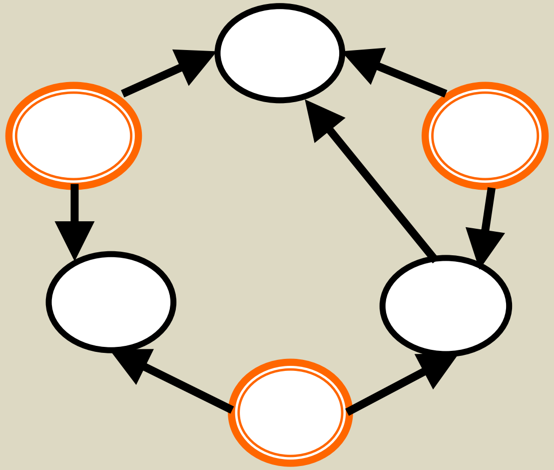
Violation of causal compatibility inequalities from **nonfanout** inflations



$$P_A + P_B + P_C - P_A P_B - P_{BC} - P_{AC} \le 1$$



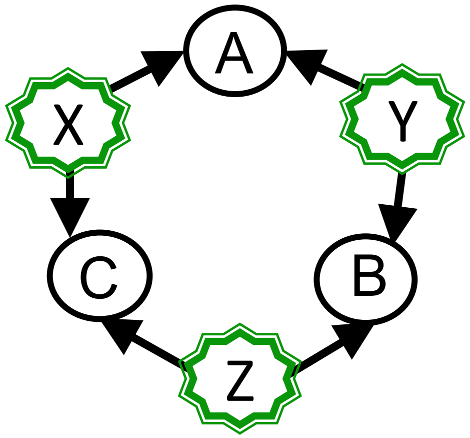
Witnessing need for different structure



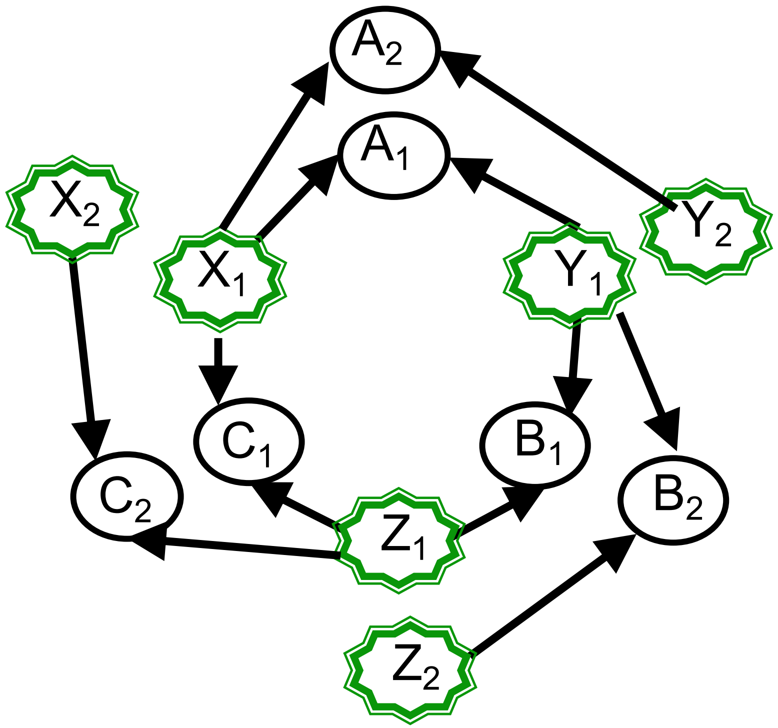
Inequalities derived from nonfanout inflations are causal compatibility constraints that hold *for any theory*

Consequently, the only inequality constraints that can serve to prove gaps between theories (e.g., quantum-classical gaps or postquantum-quantum gaps) are those derived from fanout inflations

Fanout inflations



Triangle



Spiral inflation of Triangle

Fan-out

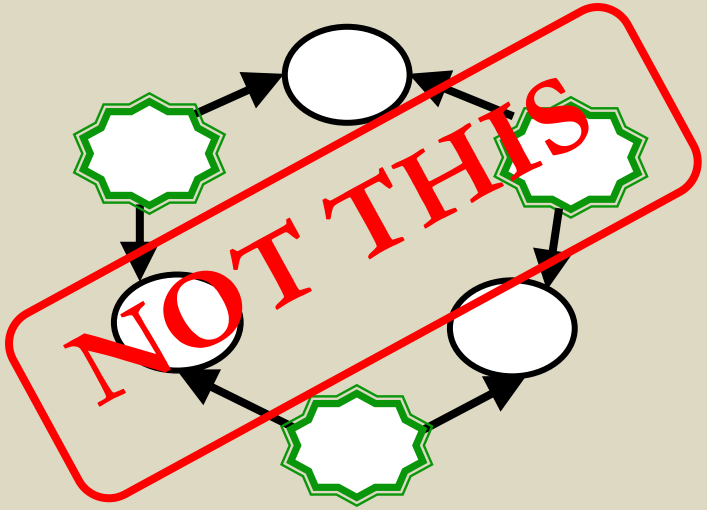
Quantum inflation technique

Wolfe, Pozas-Kerstjens, Grinberg, Rosset, Acín, Navascués, Phys. Rev. X **11**, 021043 (2021)

Generalization of NPA semidefinite programming hierarchy from the Bell scenario

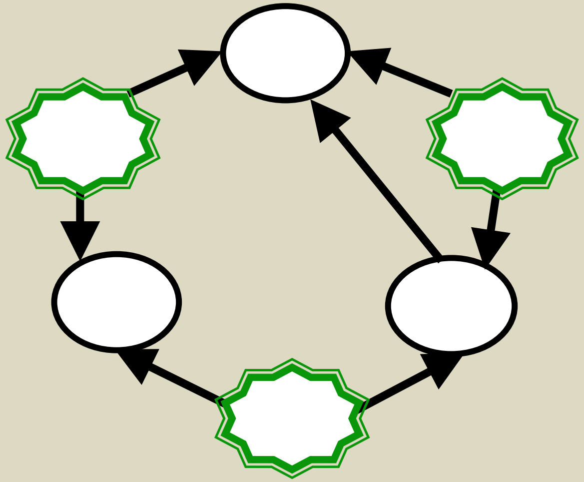
Navascués, Pironio, and Acín, New J. Phys. 10, 073013 (2008)

Violation of causal compatibility inequalities from **fanout** inflations

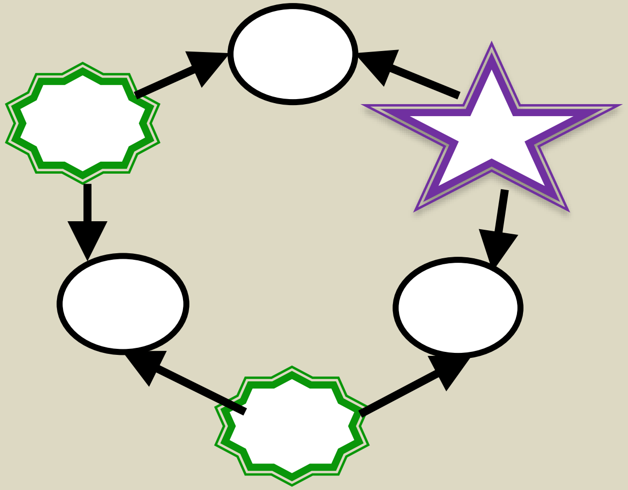


or

Witnessing need for different structure



Witnessing **post-quantumness**

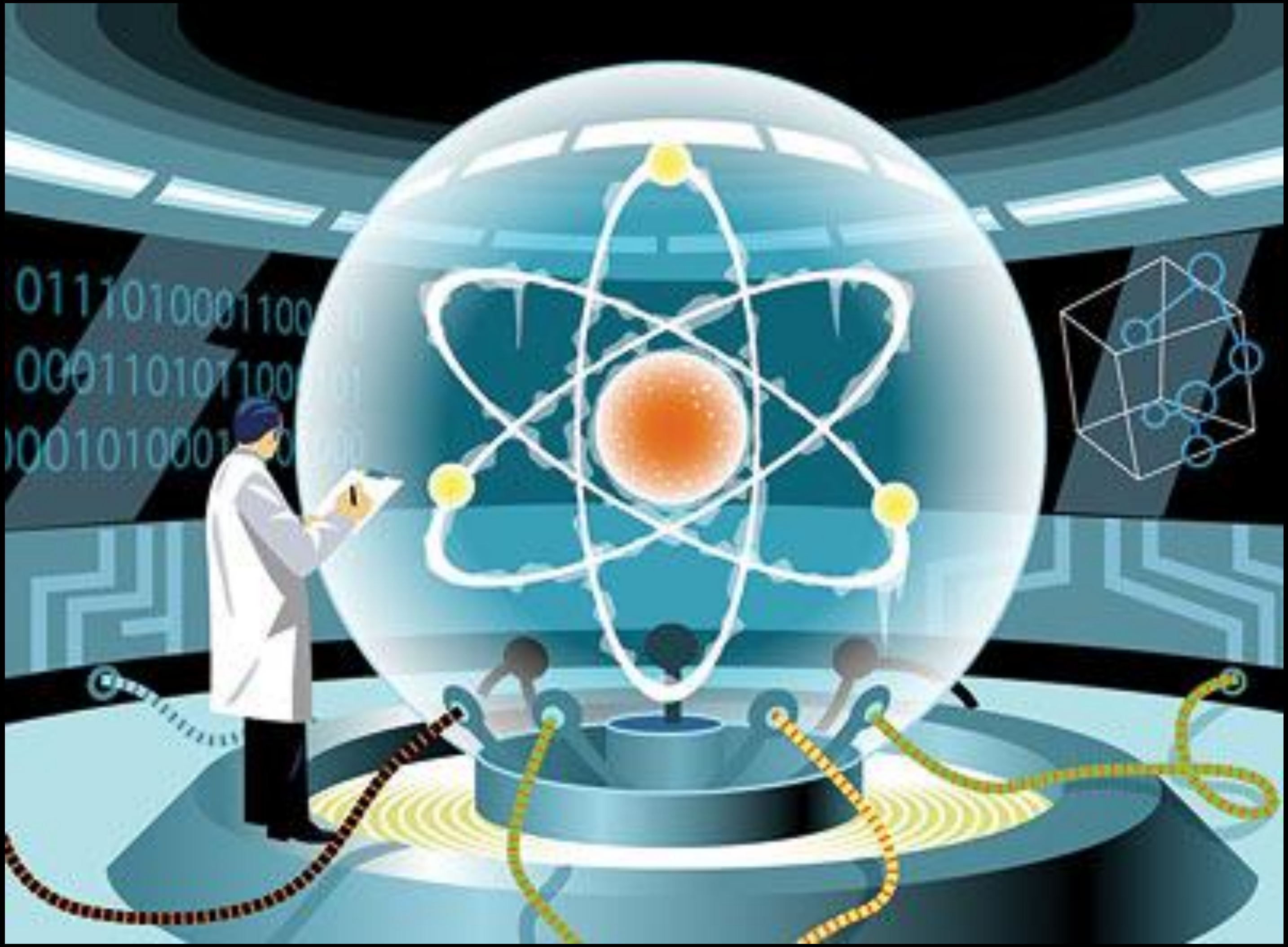


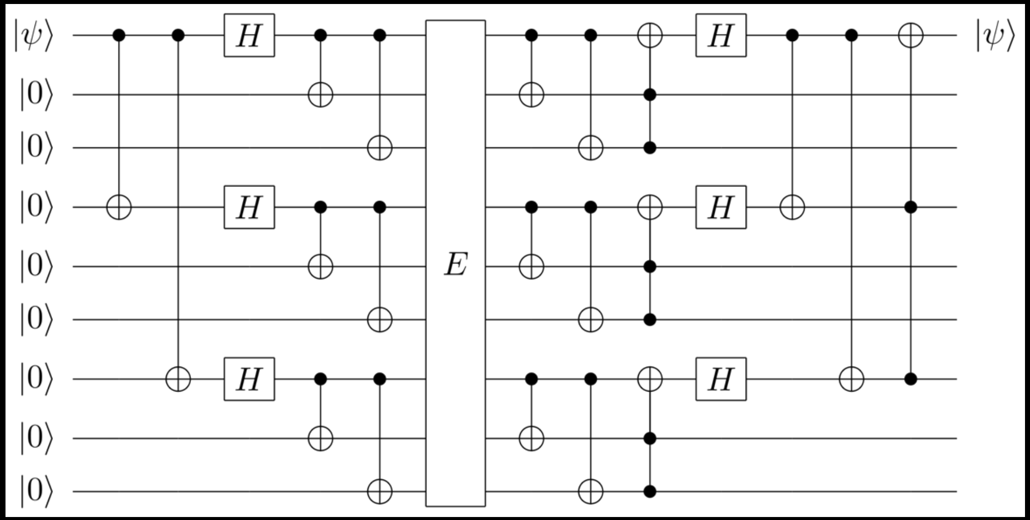
Potential Applications in Modern Physics

Quantum Foundations

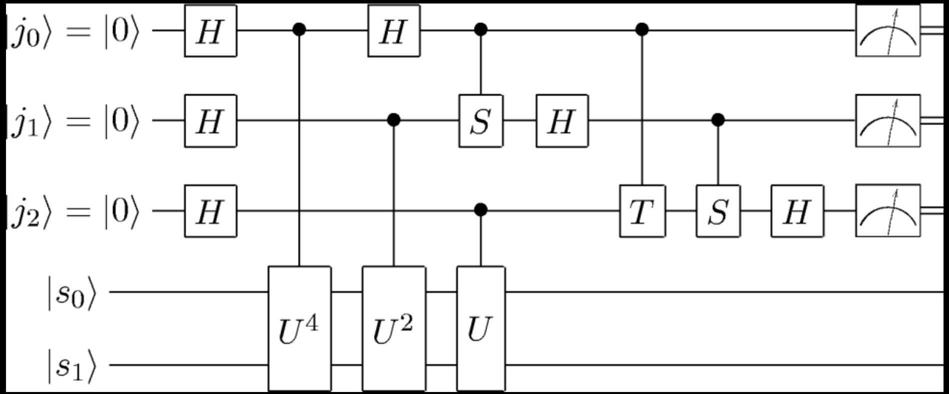
$$\frac{\begin{array}{c}\text{Quantum}\\ \text{Causation and Inference}\end{array}}{\begin{array}{c}\text{Classical}\\ \text{Causation and Inference}\end{array}} \approx \frac{\begin{array}{c}\text{Relativistic}\\ \text{Notions of Space}\\ \text{and Time}\end{array}}{\begin{array}{c}\text{PreRelativistic}\\ \text{Notions of Space}\\ \text{and Time}\end{array}}$$

Quantum Computation





$|\psi\rangle$
$|0\rangle$
H
E
$|\psi\rangle$



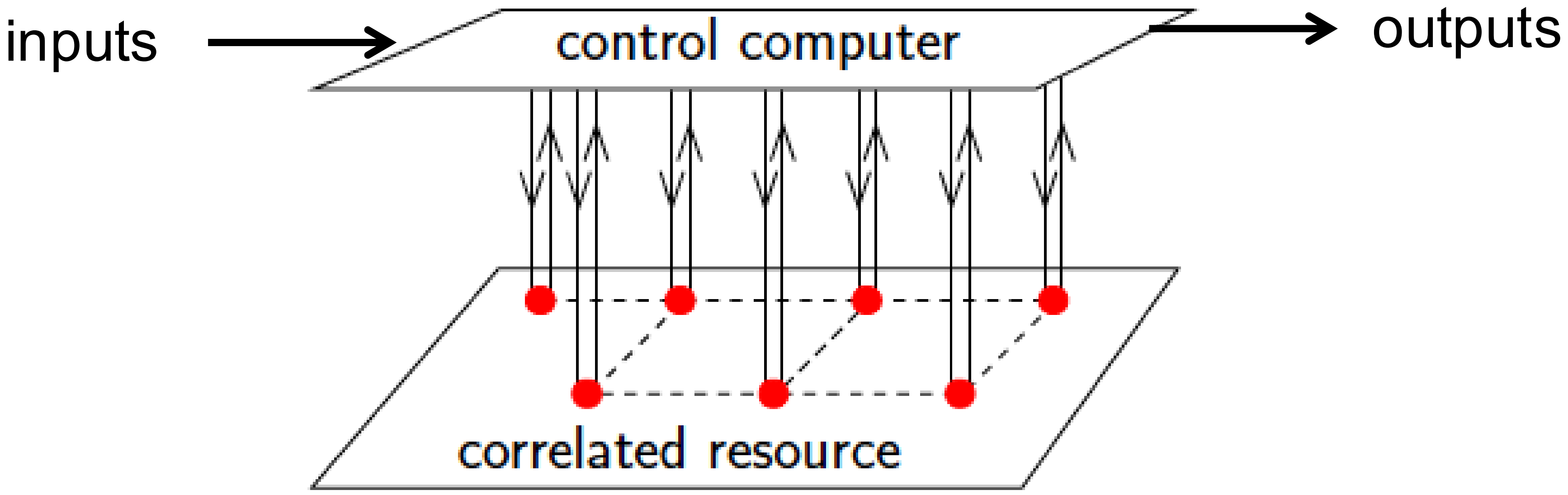
$|j_0\rangle = |0\rangle$
$|j_1\rangle = |0\rangle$
$|j_2\rangle = |0\rangle$
$|s_0\rangle$
$|s_1\rangle$
H
S
T
U^4
U^2
U

Quantum advantage for shallow circuits
Bravyi, Gosset, Koenig, Science **362**, 308 (2018)

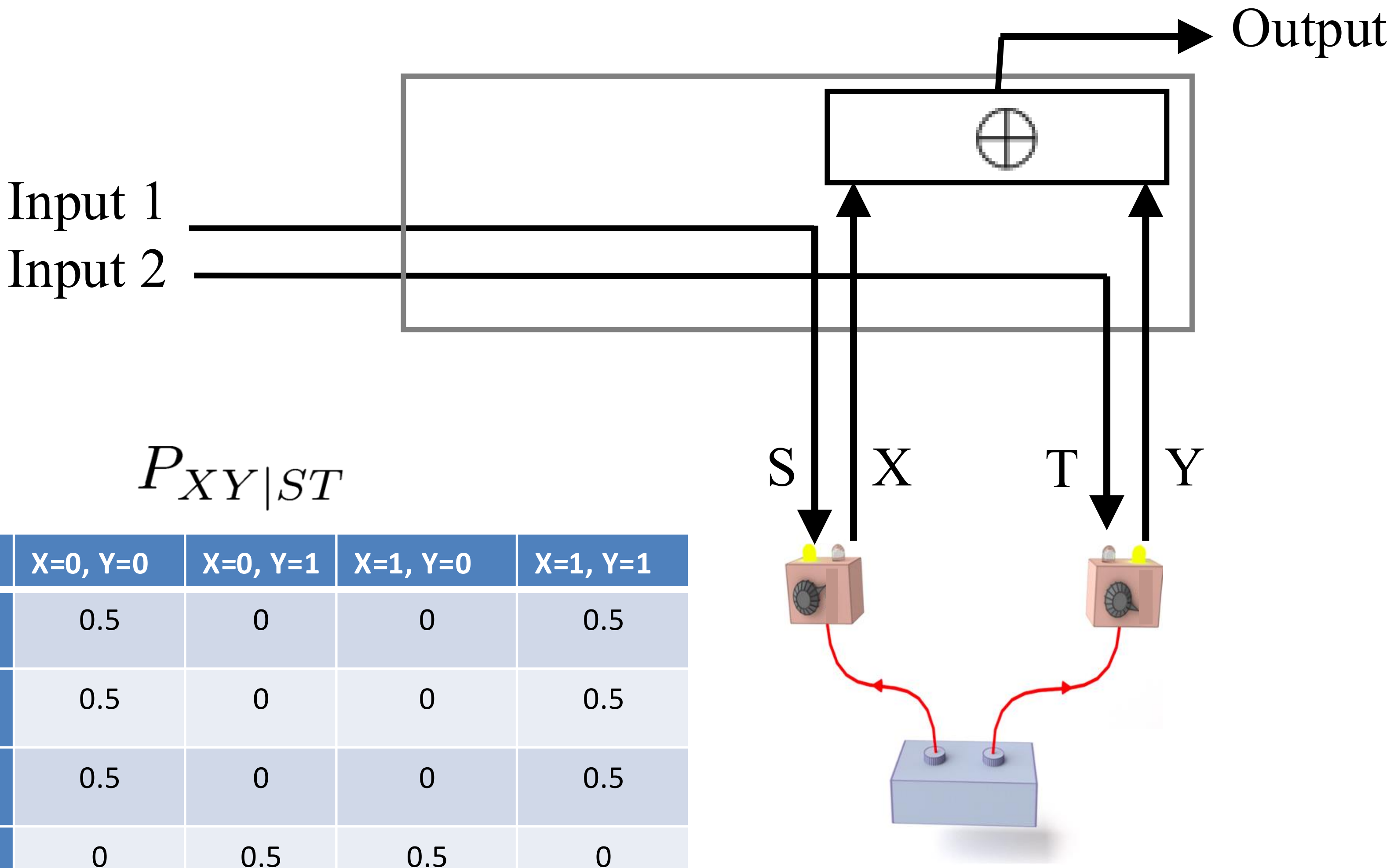
Quantum advantages in measurement-based models of computation
Browne and Anders, Phys. Rev. Lett. 102, 050502 (2009)
Raussendorf, Phys. Rev. A 88, 022322 (2013)

Measurement-based computation

Raussendorf & Briegel PRL 86, 5188 (2001)



Browne and Anders, Phys. Rev. Lett. 102, 050502 (2009)
Raussendorf, Phys. Rev. A 88, 022322 (2013)



$P_{XY|ST}$

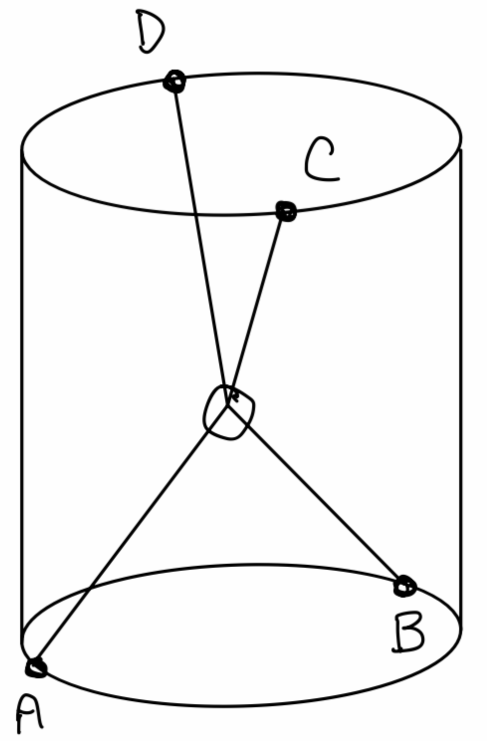
Popescu-Rohrlich (PR) box

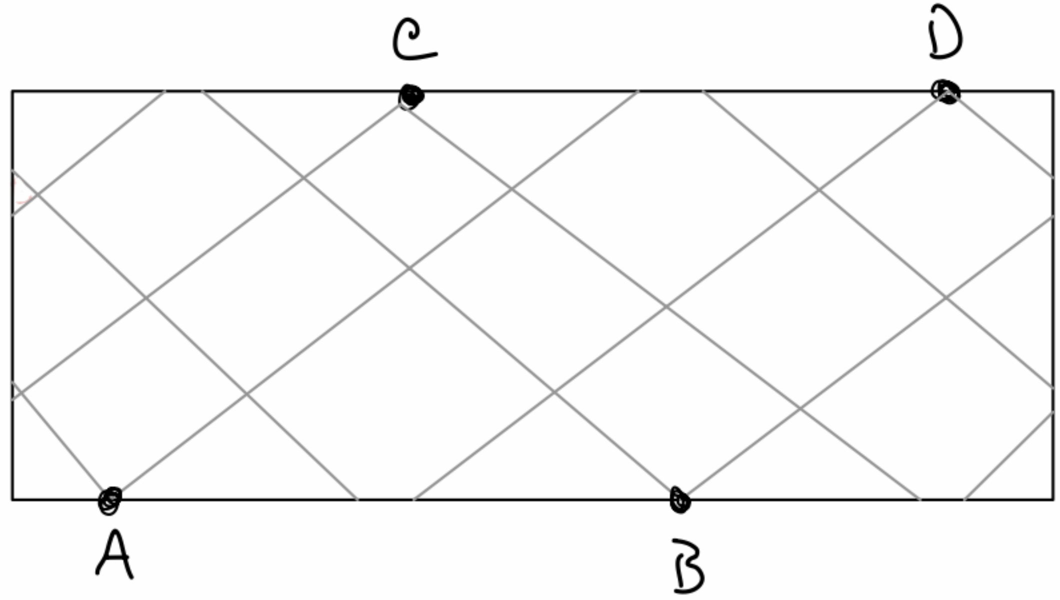
	X=0, Y=0	X=0, Y=1	X=1, Y=0	X=1, Y=1
S=0, T=0	0.5	0	0	0.5
S=0, T=1	0.5	0	0	0.5
S=1, T=0	0.5	0	0	0.5
S=1, T=1	0	0.5	0.5	0

$$\boxed{X \oplus Y = ST}$$

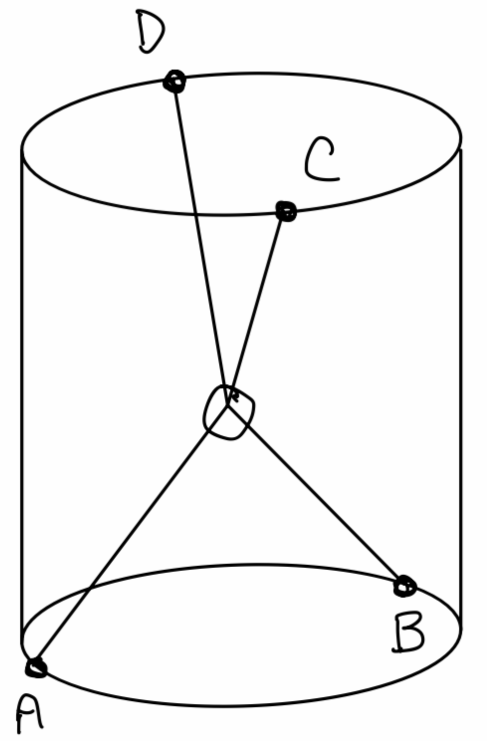
Holography

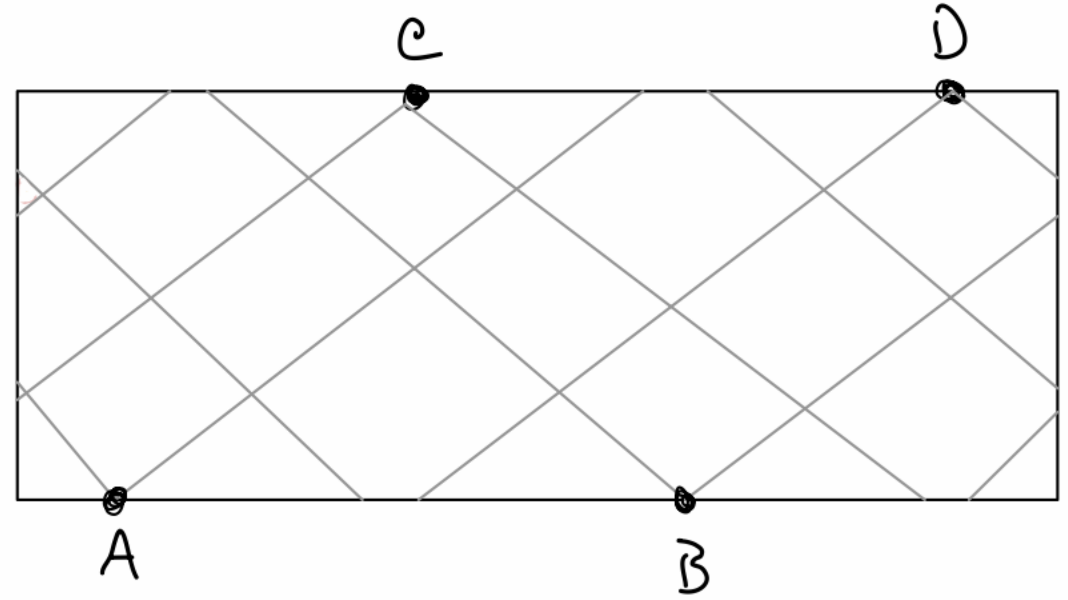
Alex May, Geoff Penington, and Jonathan Sorce.
J. High Energy Physics, 2020(8):1–34, 2020

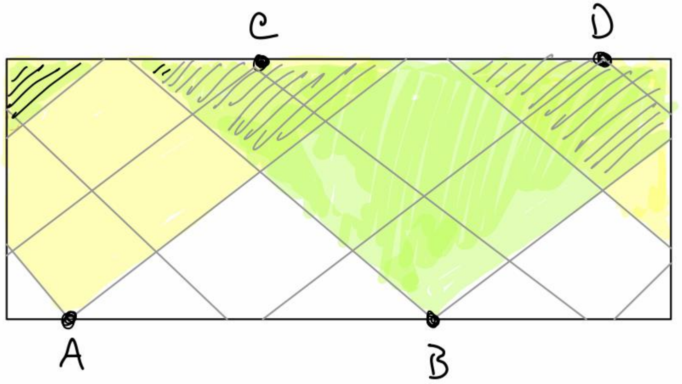


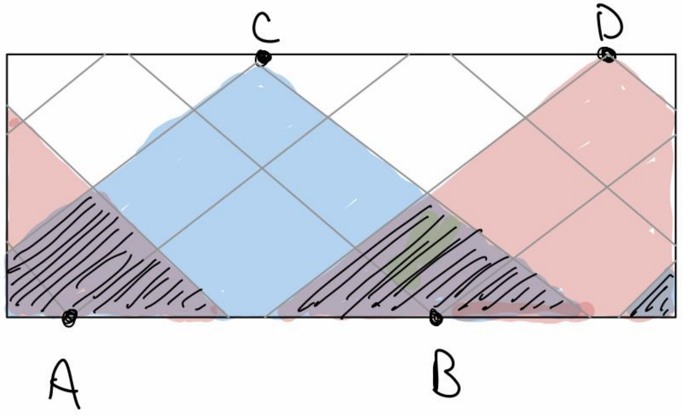


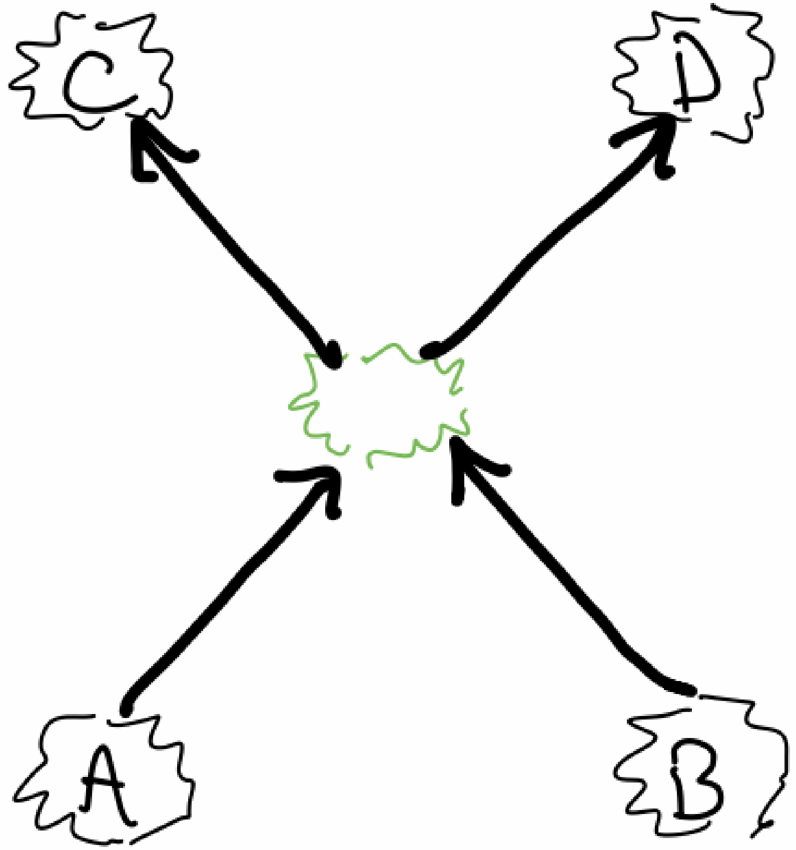
Alex May, Geoff Penington, and Jonathan Sorce.
J. High Energy Physics, 2020(8):1–34, 2020

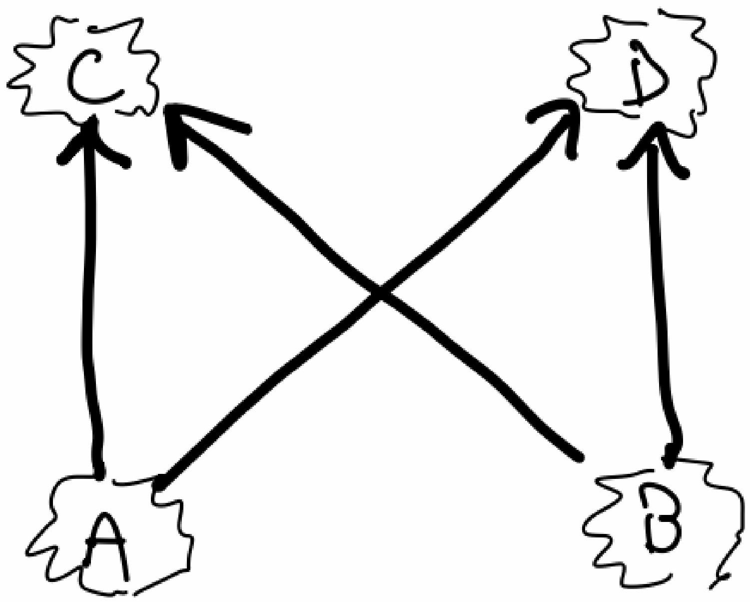


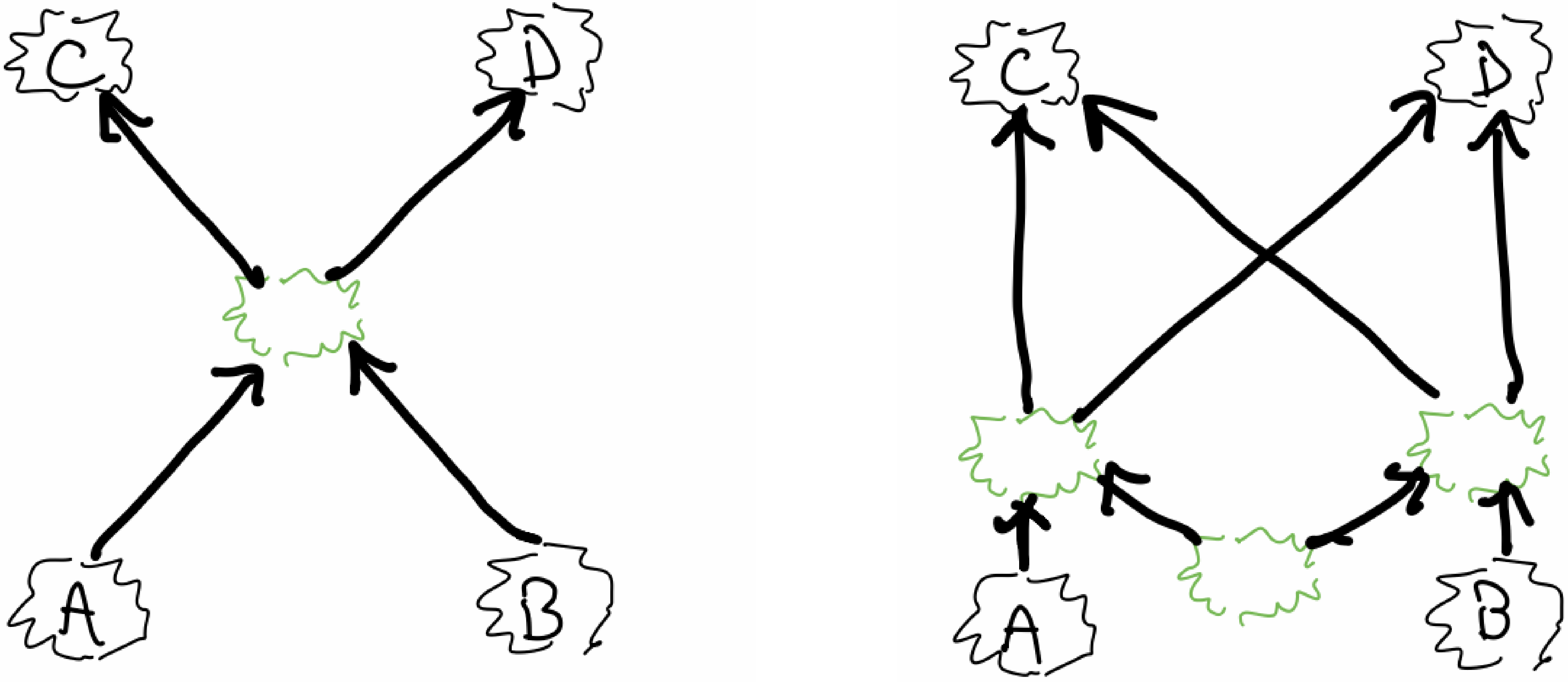







C
D
A
B

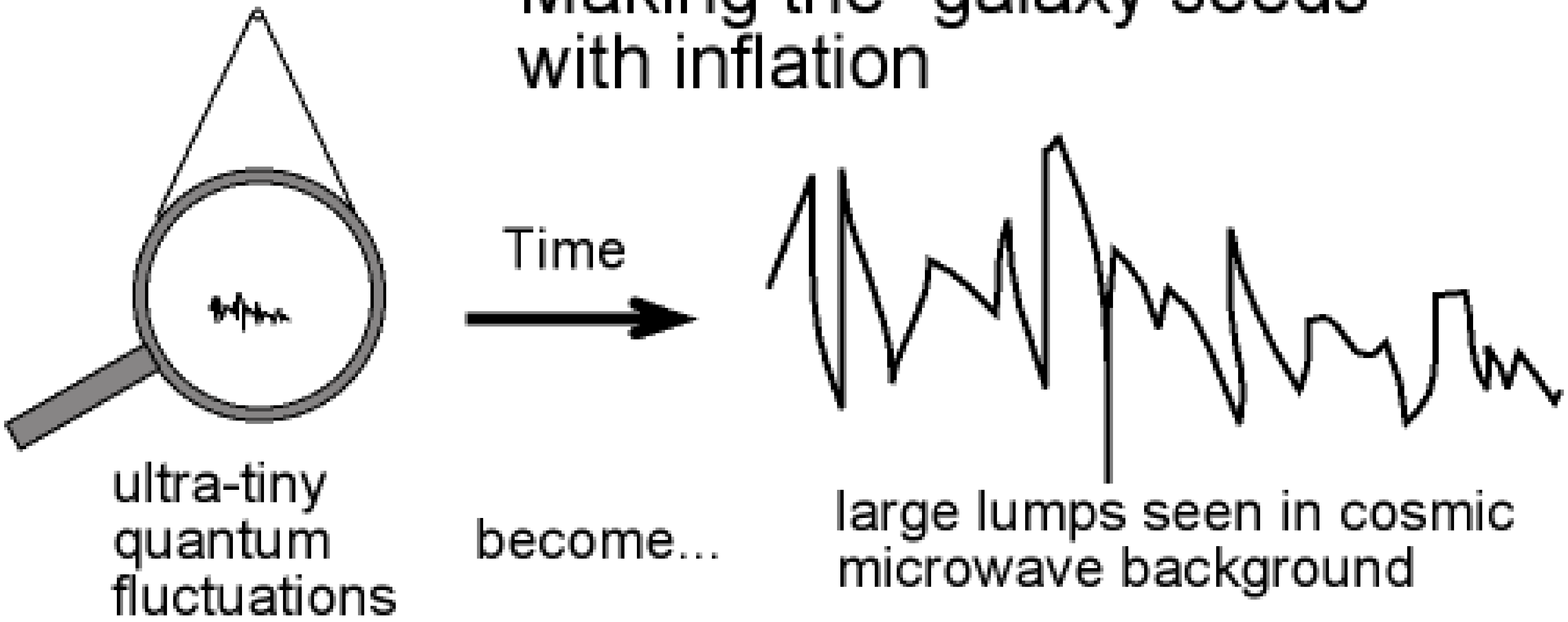

C
D
A
B



C
D
A
B
C
D
A
B

Cosmology

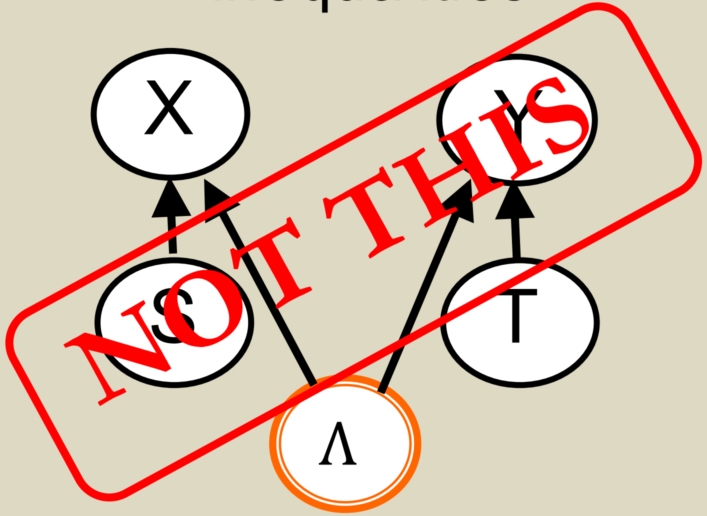
Making the "galaxy seeds" with inflation



Credit: https://people.umass.edu/wqd/strobel/cosmolgy/cosmlgyc.htm

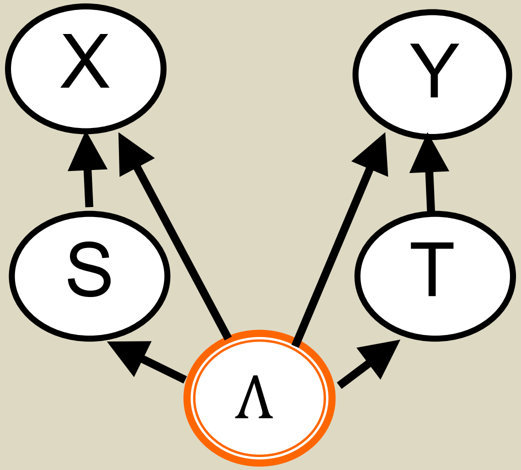
Juan Maldacena, Fortschritte der Physik 64, 10–23 (2016)

Violation of Bell inequalities

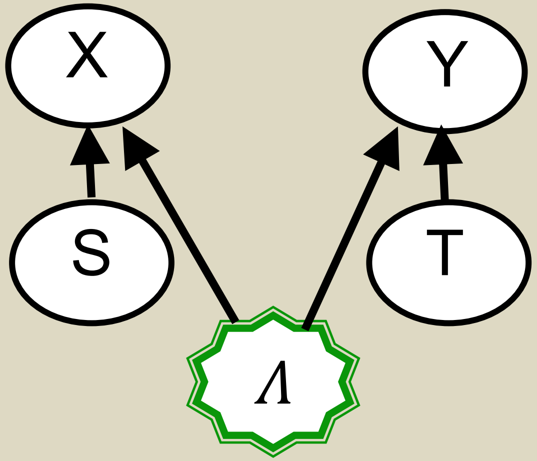


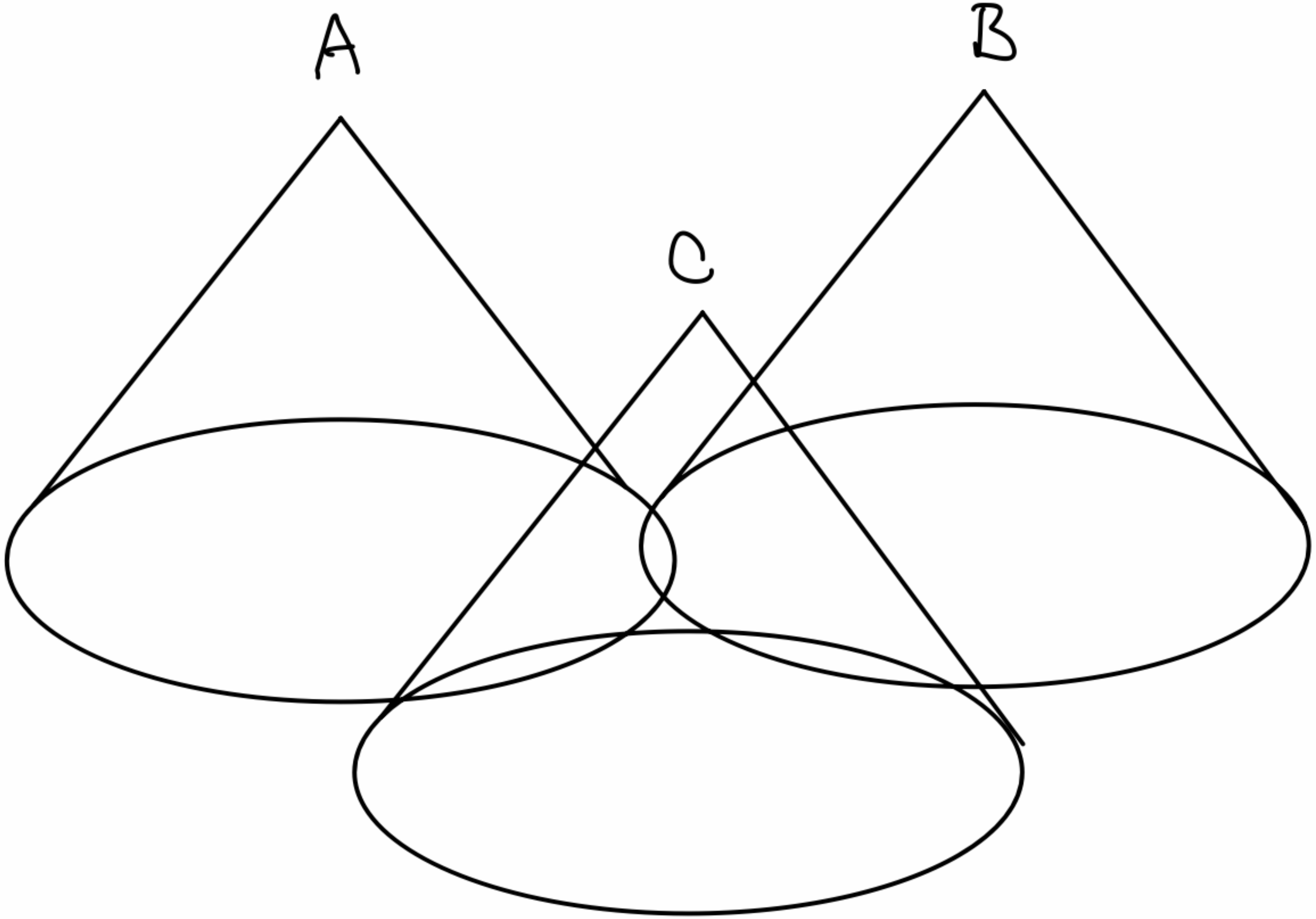
or

Witnessing need for different structure

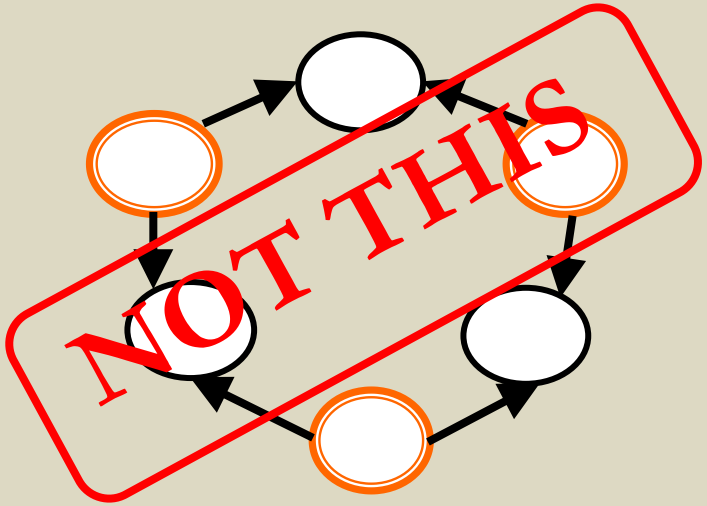


Witnessing quantumness



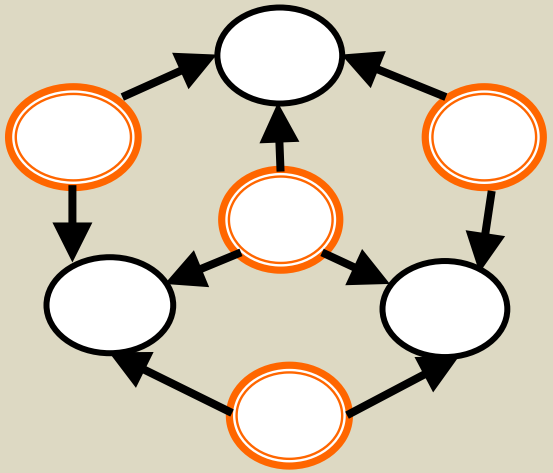

A
B
C

Violation of certain causal compatibility inequalities

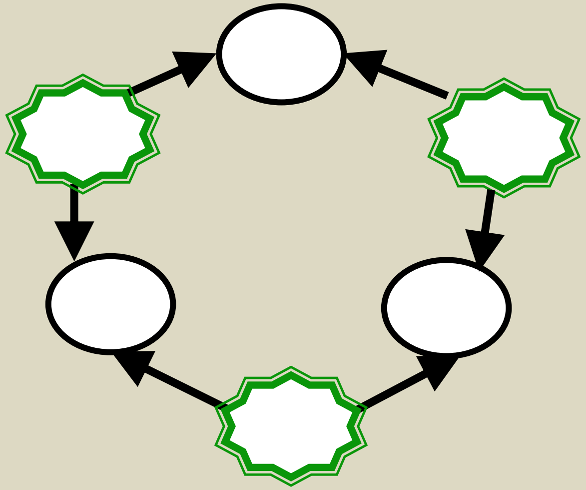


or

Witnessing need for different structure



Witnessing quantumness



Causal Inference: Classical and Quantum

PHYS 777-007
Lecturer: Robert Spekkens
TA: Marina Ansanelli

March 6, 2023

Causarum Investigatio
“Investigate the causes”

15 hours of lectures
Available online at https://pirsa.org/c23016

Thanks!