Exact equation of motion

The starting point of our theory is the von Neumann equation

$$\dot{\varrho}_{tot} = \frac{1}{i}[H_{tot}, \varrho_{tot}]$$

This corresponds to the time evolution

$$\varrho_{tot}(t) = U_{tot}(t)\,\varrho_{tot}(0)\,U_{tot}^{+}(t) ,$$

with $U_{tot}(t) = \mathcal{T} \exp\left(-i \int_0^t dt' H_{tot}(t')\right)$.

As we will treat the system-bath coupling H_{SB} perturbatively, it is convenient to transform to the interaction picture:

$$\varrho_{tot}(t) \longrightarrow \tilde{\varrho}_{tot}(t) = U_0^{+}(t)\,\varrho_{tot}(t)\,U_0(t)$$

which undoes part of the evolution of $U_{tot}(t)$, so that in the new (rotating) reference frame, with unperturbed Hamiltonian and evolution operator

(not the Fourier transform!)

$$H_0(t) = H(t) + H_E$$
$$U_0(t) = \mathcal{T} \exp\left(-i \int_0^t dt' H_0(t')\right) = U(t)\,U_E(t) .$$

To preserve expectation values $\langle X \rangle = tr\{\varrho_{tot} X\} \doteq tr\{\tilde{\varrho}_{tot} \tilde{X}\}$, observables $X = X^{+}$ have to transform like

$$X \to \tilde{X} = U_0^+ X U_0 = \tilde{X}^+$$

$$XY \to \tilde{X}\tilde{Y} = U_0^+ X Y U_0 = U_0^+ X \underbrace{U_0 U_0^+}_{1} Y U_0 = \tilde{X}\tilde{Y}$$

The time evolution is then governed by the (transformed) coupling operator (the perturbation) alone

$$\dot{\tilde{S}} = \dot{U}_0^+ S U_0 + U_0^+ \dot{S} U_0 + U_0^+ S \dot{U}_0$$

$\dot{S} = \frac{1}{i}[H_{tot}, S]$; $\dot{U}_0^+ = (\frac{1}{i} H_0 U_0)^+ = -\frac{1}{i} U_0^+ H_0 = -\frac{1}{i} H_0 U_0^+$; $\dot{U}_0 = \frac{1}{i} H_0 U_0 = \frac{1}{i} U_0 H_0$

$$= \frac{1}{i}\left[(\tilde{H}_{tot} - \tilde{H}_0), \tilde{S}\right] = \frac{1}{i}\left[\tilde{H}_{SE}, \tilde{S}\right],$$

with

$$\tilde{H}_{SE} = U_0^+ H_{SE} U_0 = \underbrace{U^+(t) A(t) U(t)}_{\tilde{A}(t)} \underbrace{U_E^+ B U_E}_{\tilde{B}(t)},$$

since H and H_E act in different state spaces and therefore commute, $[H, H_E] = 0$.

By formally integrating this equation in time from 0 to t, one obtains

$$\tilde{S}_{tot}(t) - \tilde{S}_{tot}(0) = \frac{1}{i}\int_0^t dt' \left[\tilde{H}_{SB}(t'), \tilde{S}_{tot}(t')\right],$$

which we plug into the original equation to obtain the integro-differential equation

$$\dot{\tilde{S}}_{tot}(t) = \frac{1}{i}\left[\tilde{H}_{SB}(t), \tilde{S}_{tot}(0)\right]$$

$$+ \frac{1}{i} \int_0^t dt' \left[\hat{H}_{SB}(t), \left[H_{SB}(t'), \tilde{\rho}_{tot}(t') \right] \right] .$$

To get an equation of motion for the reduced density operator of the system, $\rho = tr_E \{ \rho_{tot} \}$, we take the trace over the environment

$$\dot{\tilde{\rho}}(t) = tr_E \{ \dot{\tilde{\rho}}_{tot} \}$$

$$= \frac{1}{i} tr_E \left[\tilde{H}_{SB}(t), \tilde{\rho}_{tot}(0) \right]$$

$$- \int_0^t dt' \, tr_E \left[\hat{H}_{SB}(t), \left[H_{SB}(t'), \tilde{\rho}_{tot}(t') \right] \right] .$$

For simplicity, we assume that

$$tr_E \left[\hat{H}_{SB}(0), \tilde{\rho}_{tot}(0) \right] = 0 ,$$

which is the case as long as

$$\langle \hat{B}(t) \rangle_0 = tr_E \{ \rho_E(0) \hat{B}(t) \} = 0 .$$

For the typical situation $\rho_E(0) \propto \exp(-\beta H_E)$ $H_E = \sum_n \omega_n b_n^\dagger b_n$ and $B = \sum_n c_n (b_n^\dagger + b_n)$, with bosonic annihilation operators b_n, this is fulfilled. In other cases, we can always achieve this by considering this term as part of the system Hamiltonian, by redefining $H_{SB}' = H_{SB} - \gamma A \langle \hat{B}(t) \rangle_0$ and $H' = H + \gamma \langle B(t) \rangle_0 A$.

Note, however, that $\langle \tilde{B}(t) \rangle_0$ can be time-dependent. However, when B commutes with H_E and, thus, with $U_E(t)$, we have $\tilde{B}(t) = B$, which is the case, e.g., for $B = \sum_n c_n \ddot{b}_n b_n$.

With that we arrive at the equation of motion

$$\dot{\tilde{\varrho}}(t) = -\int_0^t dt' \, \mathrm{tr}_E \left[\tilde{H}_{SB}(t), \left[\tilde{H}_{SB}(t'), \tilde{\varrho}_{tot}(t') \right] \right],$$

which is still exact, but not useful, as it contains $\tilde{\varrho}_{tot}$ on the right-hand side.

We will now apply several approximations that rely on weak system-bath coupling H_{SB} to arrive at a closed master equation for the system.

Born approximation

The first approximation is the so-called Born approximation

$$\tilde{\varrho}_{tot}(t) = \tilde{\varrho}(t) \otimes \underbrace{\varrho_E(0)}_{\equiv \varrho_E}.$$

The physical meaning of this approximation is that we neglect more than a single interaction process between system and environment during the time scale τ_E on which the environment relaxes.

With this, and using $\tilde{H}_{SB} = \tilde{A}(t)\,\tilde{B}(t)$, we arrive at

$$\dot{\tilde{\varrho}}(t) = -\int_0^t dt' \Big[\tilde{A}(t)\tilde{A}(t')\tilde{\varrho}(t') \langle \tilde{B}(t)\tilde{B}(t') \rangle_0 - \tilde{A}(t)\,\tilde{\varrho}(t')\,\tilde{A}(t') \langle \tilde{B}(t')\,\tilde{B}(t) \rangle_0 - \tilde{A}(t')\tilde{\varrho}(t')\,\tilde{A}(t) \langle \tilde{B}(t)\,\tilde{B}(t') \rangle_0 + \tilde{\varrho}(t')\tilde{A}(t')\,\tilde{A}(t) \langle \tilde{B}(t')\,\tilde{B}(t) \rangle_0 \Big].$$

Here, the bath correlation function

$$\langle \tilde{B}(t)\tilde{B}(t') \rangle_0 \equiv \mathrm{tr}_E \{ \varrho_E \tilde{B}(t)\tilde{B}(t') \} \equiv C(t-t')$$

depends only on the time difference, as H_E is time independent. Moreover, with $\tilde{B}^\dagger(t) = \hat{B}(t)$

$$C^*(t-t') = \mathrm{tr}_E \{ [\varrho_E \tilde{B}(t)\tilde{B}(t')]^\dagger \}$$

$$= \mathrm{tr}_E \{ \tilde{B}(t')\tilde{B}(t)\varrho_E \} = C(t'-t).$$

cyclic permutations leave trace unchanged

Substituting $t' \to \tau = t - t'$, i.e. $\int_0^t dt' = -\int_t^0 d\tau = \int_0^t d\tau$ the equation becomes

$$\dot{\tilde{\varrho}}(t) = -\int_0^t d\tau \Big\{ C(\tau) \big[\tilde{A}(t)\tilde{A}(t-\tau)\varrho(t-\tau) - \tilde{A}(t-\tau)\varrho(t-\tau)\tilde{A}(t) \big] + \text{h.c.} \Big\}.$$

Markov approximation

The bath correlation function decays on the characteristic relaxation time τ_E of the environment

$$C(\tau) \simeq 0 \quad \text{for} \quad \tau \gg \tau_E .$$

With that, the change of $\tilde{\xi}$ happens with a rate of order

$$\gamma_{SE} = \tau_E \langle A^2 \rangle \langle B^2 \rangle \equiv \tau_{SE}^{-1}$$

As long as the change of $\tilde{\xi}$ is slow compared to the bath relaxation,

$$\gamma_{SE} \tau_E = \tau_E^2 \langle A^2 \rangle \langle B^2 \rangle \ll 1 ,$$

we can replace $\tilde{\xi}(t-\tau) \simeq \tilde{\xi}(t)$ to obtain the time-local master equation.

$$\dot{\tilde{\xi}}(t) = -\int_0^t d\tau \Big\{ C(\tau) \big[\tilde{A}(t) \tilde{A}(t-\tau) \tilde{\xi}(t) - \tilde{A}(t-\tau) \tilde{\xi}(t) \tilde{A}(t) \big] + \text{h.c.} \Big\} ,$$

which is the (time-dependent) Redfield equation.

Remarks:

- We we can generally not approximate

$\tilde{A}(t-\tau)$ by $\tilde{A}(t)$, as $\tilde{A}(t)$ varies on time scales governed by the system Hamiltonian H, which can be faster than τ_{SE}.

- The same condition, $\Gamma_{SE}\tau_E \ll 1$, also ensures the validity of the Born approximation, as processes with two interaction processes happen on the scale $\Gamma_{SE}^{(2)} = \tau_E^3 \langle A^2\rangle^2 \langle B^2\rangle^2 \ll \Gamma_{SE}$ for $\Gamma_{SE}\tau_E \ll 1$ (see, e.g., detailed discussion in Hone et al. 2009).

Redfield equation for $t \gg \tau_E$

It is left to transform back to the Schrödinger picture: With $\rho(t) = U(t)\tilde{\rho}(t)U^\dagger(t)$,

$$\dot{\rho} = \frac{d}{dt} U(t)\tilde{\rho}(t)U^\dagger(t)$$

$$= \underbrace{\dot{U}}_{HU/i}\tilde{\rho}U^\dagger + U\tilde{\rho}\underbrace{\dot{U}^\dagger}_{-U^\dagger H/i} + U\dot{\tilde{\rho}}U^\dagger$$

$$= \frac{1}{i}[H(t), \rho(t)] - \int_0^t d\tau \Big\{ U(t-\tau)\, C(\tau) \Big[\underbrace{U(t)U^\dagger(t)}_{\uparrow} A(t) \underbrace{U(t)U^\dagger(t-\tau)}_{U^\dagger(t-\tau,t)} A(t-\tau) \underbrace{U(t-\tau)U^\dagger(t)}_{\equiv U(t-\tau,t)} \rho(t) \underbrace{U(t)U^\dagger(t)}_{\uparrow}$$

$$\underbrace{U(t)U^\dagger(t-\tau)}_{U^\dagger(t-\tau,t)} A(t-\tau) \underbrace{U(t-\tau)U^\dagger(t)}_{U(t-\tau,t)} \rho(t) \underbrace{U(t)U^\dagger(t)}_{\uparrow} A(t) \underbrace{U(t)U^\dagger(t)}_{\uparrow} \Big] + \text{h.c.} \Big\}$$

gives us the Born-Markov or time-dependent Redfield master equation in the Schrödinger picture

$$\dot{S} = \frac{1}{i}[H(t), S(t)] - \int_0^t d\tau \left\{ C(\tau)\left[A(t)\tilde{A}(t-\tau,t)S(t) - \tilde{A}(t-\tau,t)S(t)A(t)\right] + \text{h.c.}\right\},$$

where

$$\tilde{A}(t_2,t_1) = \underbrace{U^\dagger(t_2,t_1)}_{U(t_1,t_2)} A(t_2) U(t_2,t_1).$$

We can write this equation in compact form

$$\dot{S}(t) = \frac{1}{i}[H(t), S(t)] - \left[A(t)\bar{A}(t)S(t) - \bar{A}(t)S(t)A(t) + \text{h.c.}\right],$$

with the convoluted coupling operator

$$\bar{A}(t) = \int_0^t d\tau\, C(\tau)\, \tilde{A}(t-\tau,t).$$

After a short initial evolution, we can extend the integration to infinity

$$\bar{A}(t) = \int_0^\infty d\tau\, C(\tau)\, \tilde{A}(t-\tau,t) \quad \text{for} \quad t \gg \tau_E.$$

From now on, we will use this expression.

Remarks:

- The derivation of the Redfield equation was done analogously to the case of time-independent Hamiltonians H.
- For time-independent systems, with constant H and A, $\tilde{A}(t-\tau, t) = \tilde{A}(-\tau)$ becomes independent of t, since then $U(t-\tau, t) = U(-\tau)$. With that also

$$\bar{A} = \int_0^\infty d\tau\, C(\tau)\, \tilde{A}(-\tau) \quad \text{for} \quad t \gg \tau_E .$$

becomes time independent, so that we arrive at a (the) time-independent Redfield master equation.

- For time-periodic systems, $H(t+T) = H(t)$ and $A(t+T) = A(t)$, $\tilde{A}(t-\tau, t)$ becomes periodic with respect to t,

$$\tilde{A}(t+T-\tau, t+T) = \tilde{A}(t-\tau, t),$$

since $U(t_1+T, t_2+T) = U(t_1, t_2)$ as a consequence of $H(t+T) = H(t)$, so that

$$\bar{A}(t+T) = \bar{A}(t) \quad \text{for} \quad t \gg \tau_E$$

gives rise to a (the) time-periodic Redfield equation.

Splitting coherent from dissipative dynamics

The second term of the Redfield equation, describing the impact of the system-environment coupling, can be separated into two contributions. From

$$-A\bar{A}\rho + \bar{A}\rho A - \rho\bar{A}^{\dagger}A^{\dagger} + A^{\dagger}\rho\bar{A}^{\dagger}$$

$$= \frac{1}{i}\frac{1}{2}\Big[\big(i\bar{A}^{\dagger}A^{\dagger} - iA\bar{A}\big)\rho - \rho\big(-iA\bar{A} + i\bar{A}^{\dagger}A^{\dagger}\big)\Big]$$
$$+ \bar{A}\rho A - \frac{1}{2}\{A\bar{A}\rho + \rho A\bar{A}\}$$
$$+ A^{\dagger}\rho\bar{A}^{\dagger} - \frac{1}{2}\{\bar{A}^{\dagger}A^{\dagger}\rho + \rho\bar{A}^{\dagger}A^{\dagger}\}$$

where $A^{\dagger} = A$ was not yet employed for making the structure of the equation transparent, we find

$$\dot{\rho} = \frac{1}{i}\big[H(t) + H_{LS}(t), \rho(t)\big] + \mathcal{D}(t)[\rho(t)],$$

with Lamb-shift Hamiltonian $H_{LS}(t)$ and Redfield dissipator $\mathcal{D}(t)[\rho(t)]$ defined as

$$H_{LS}(t) = i\big[\bar{A}^{\dagger}(t)A - A\bar{A}(t)\big]$$

$$\mathcal{D}(t)[\rho(t)] = \Big(\bar{A}(t)\rho(t)A - \frac{1}{2}\{A\bar{A}(t), \rho(t)\}$$
$$+ A\rho\bar{A}^{\dagger} - \frac{1}{2}\{\bar{A}^{\dagger}A, \rho(t)\}$$

where $\{x, y\} = xy + yx$ is the anticommutator.

By introducing $A_1 \equiv A$ and $A_2 \equiv A^\dagger$, we can rewrite the dissipator as

$$\mathcal{D}(t)[\rho(t)] = \sum_{ij} \sigma^x_{ij} \left(A_i \rho A_j - \frac{1}{2} \{A_i A_j, \rho\} \right),$$

with Pauli matrix $\sigma^x_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Remarks:

- The splitting into Hamiltonian and dissipator is not unique. Replacing

$$\begin{cases} A_i \longrightarrow A_i' = A_i - \alpha_i \\ H_{LS} \longrightarrow H_{LS}' = H_{LS} + \sum_{ij} \sigma^x_{ij} \frac{1}{2i} (\alpha_i A_j^\dagger - \alpha_j^* A_i) \end{cases}$$

leaves the master equation invariant (proof: homework).

- At first glance, the Redfield dissipator looks like a Lindblad dissipator, with the Kossakowski matrix given by σ^x_{ij}. However, σ^x_{ij} is not positive semidefinite, but has eigenvalues $+1$ and -1. Thus, we can only bring it into pseudo-Lindblad form*,

$$\mathcal{D}(t)[\rho] = \left[\left(A_+ \rho A_+^\dagger - \frac{1}{2} \{A_+^\dagger A_+, \rho\} \right) - \left(A_- \rho A_-^\dagger - \{A_-^\dagger A_-, \rho\} \right) \right],$$

with the A_- contribution having negative sign, where

$$A_\pm(+1) = \frac{e^{i(\alpha \pm \beta/2)}}{\sqrt{2\cos(\varphi)}} \left[\pm \lambda e^{\pm i\varphi/2} A_1 + \frac{1}{\lambda} e^{\mp i\varphi/2} A_2 \right],$$

with free parameters $\lambda, \alpha, \beta, \varphi \in \mathbb{R}$ [T. Becker, L.-N. Wu, A. Eckardt, PRE 104, 014110 (2021); T. Becker, A. Eckardt, PRE 111, 034132 (2025)].

* This form can be used either

  (i) to find an approximate Lindblad equation by neglecting A_-, which is different from the Lindblad equation obtained from the secular approximation [T. Becker, L.-N. Wu, A. Eckardt, PRE 104, 014110 (2021)],

  or

  (ii) as the starting point for quantum trajectory simulations of the (non-Lindblad) Redfield equation [T. Becker, C. Netzer, A. Eckardt, PRL 131, 160401 (2023)].

  In both cases, it is favorable to use the free parameters to minimize the contribution from the negative A_- term.

C) Isolated quantum Floquet systems

References:

- Formalism of Floquet Theory
  A. Eckardt and E. Anisimovas,
  NJP 17, 093039 (2015)
- Floquet engineering
  A. Eckardt,
  Rev. Mod. Phys. 89, 011004 (2017)

Quantum Floquet systems* are systems that are driven periodically in time, so that their dynamics is governed by a time-periodic Hamiltonian

$$H(t) = H(t+T) = \sum_{\mu=-\infty}^{\infty} H_\mu e^{i\mu\omega t}$$

with driving period T and corresponding (angular) frequency $\omega = 2\pi/T$. The Fourier components of the Hamiltonian are defined as $H_\mu = \frac{1}{T}\int_0^T dt\, e^{-i\mu\omega t} H(t)$.

* The Schrödinger equation is then a differential equation of the Floquet type, i.e. first order, linear, homogeneous, with periodic coefficients, which implies the existence of Floquet states via Floquet's theorem. Hence, the name "Floquet system".

Floquet systems are non-equilibrium systems. However, thanks to their discrete translation symmetry in time, they still retain a lot of structure in their dynamics (quasi-steady states, description via an effective Hamiltonian, etc.). This permits to better understand their behaviour (using the formalism of quantum Floquet theory) and it can be exploited for controlling systems (this is known as Floquet engineering).

Floquet engineering

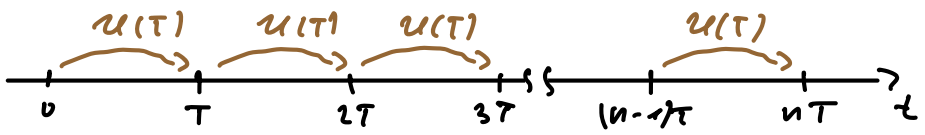
When looking at the dynamics of a Floquet system stroboscopically in steps of the driving period T,

$$|\psi(nT)\rangle = \underbrace{U(nT,0)}_{U(nT)} |\psi(0)\rangle,$$

we can use the discrete time-translation invariance of the evolution operator

$$U(t_2+T, t_1+T) = U(t_2,t_1) \quad \Big| = \mathcal{T} e^{-i\int_{t_1}^{t_2} dt\, H(t)}$$

to write



$$U(nT) = \underbrace{U(nT,(n-1)T)}_{U(T)} \underbrace{U((n-1)T,(n-2)T)}_{U(T)} \cdots \underbrace{U(T,0)}_{U(T)}$$

$$= [U(T)]^n .$$

Employing the fact that every unitary operator U can be written as $\exp(iA)$, with hermitian operator A, we can define

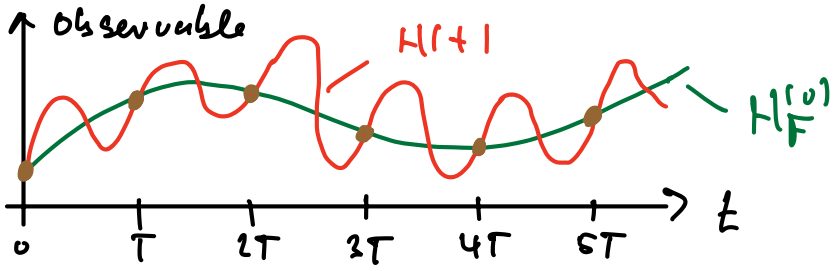
$$U(T) \equiv \exp\left(-iT H_F^{(0)}\right)$$

Label indicating the initial time $t_0 = 0$. Generally: $H_F^{(t_0)}$.

with hermitian time-independent Floquet Hamiltonian $H_F^{(0)}$, so that

$$U(nT) = [U(T)]^n = \exp\left(-inT H_F^{(0)}\right).$$

The evolution looks like that of an undriven system with Hamiltonian $H_F^{(0)}$!



* Not that simple in many-body systems (see below)!

Idea of Floquet engineering*: Tailor drive, so that $H_F^{(0)}$ acquires interesting modified properties (that one hoped to achieve by other means).

Numerically, we can compute $H_F^{(0)}$ by constructing $U(T)$ and taking the logarithm

$$H_F^{(0)} = \frac{i}{T} \ln U(T).$$

But how to get analytical intuition?

Floquet states

The Schrödinger equation,

$$i\,|\dot{\Psi}(t)\rangle = H(t)\,|\Psi(t)\rangle,$$

is then a linear differential equation with time-periodic generator $-iH(t)$. The Floquet theorem, therefore, implies quasi-steady solutions of the form*

$$|\Psi_i(t)\rangle = |u_i(t)\rangle\, e^{-i\varepsilon_i t} \quad \text{with} \quad |u_i(t+T)\rangle = |u_i(t)\rangle,$$

($|u_i(t)\rangle$: micromotion (time-periodic); $e^{-i\varepsilon_i t}$: phase evolution (like in undriven syst.))

which are called Floquet states (hence the name Floquet theory) and characterized by quasienergies ε_i and time-periodic Floquet modes $|u_i(t)\rangle$.

The Floquet states obey Bloch states

$$|\Psi_i(t+T)\rangle = e^{-i\varepsilon_i T}\,|\Psi_i(t)\rangle = U(t+T,t)\,|\Psi(t)\rangle$$

(This relation is equivalent to the previous one and can, therefore, also be used as definition of Floquet states.)

It immediately follows that the Floquet states are the eigenstates of the one-cycle evolution operator $U(t+T,t)$ with t-independent eigenvalues $e^{-i\varepsilon_i T}$.

The fact that $U(t+T,t)$ is unitary implies** additionally that $\exp(-i\varepsilon_i T)$ is a phase,

$$\left| e^{-i\varepsilon_i T} \right| = 1 \quad \Longleftrightarrow \quad \varepsilon_i \in \mathbb{R} ,$$

making the quasienergies ε_i real and that the Floquet states (can be chosen to) form a complete orthonormal basis,

$$\langle \psi_i(t) | \psi_j(t) \rangle = \langle u_i(t) | u_j(t) \rangle = \delta_{ij} ,$$

$$\sum_i |\psi_i(t)\rangle\langle\psi_i(t)| = \sum_n |u_i(t)\rangle\langle u_i(t)| = \mathbb{1} .$$

* A simple proof for the existence of Floquet states is to consider the eigenstates $|\varphi_i^{(t)}\rangle$ of the one-cycle evolution operator $U(t+T,t)$

$$U(t+T,t)\,|\varphi_i^{(t)}\rangle = \lambda_i^{(t)}\,|\varphi_i^{(t)}\rangle$$

and to show that

(i) they (can be defined so that they) solve the Schrödinger equation

$$i\frac{d}{dt}\,|\varphi_i^{(t)}\rangle = H(t)\,|\varphi_i^{(t)}\rangle ,$$

(ii) that the ε_i values are independent of t,

$$\lambda_i^{(t')} = \lambda_i^{(t)} \equiv \lambda_i$$

Then they are Floquet states

$$|\psi_i(t)\rangle = |\varphi_i^{(t)}(t)\rangle \quad \text{with} \quad e^{i\varepsilon_i T} = \lambda_i .$$

The proof is simple. Multiplying the eigenvalue equation $U(T,0)\,|\varphi_i^{(0)}\rangle = \lambda_i^{(0)}|\varphi_i^{(0)}\rangle$ with $U(t,0)$ from the left and introducing $1 = U(0,T)\,U(T,0)$,

$$\underbrace{\underbrace{U(t,0)U(T,0)}_{U(t+T,T)}\underbrace{U(0,t)}_{\text{since } H(t+T)=H(t)}\underbrace{U(t,0)|\varphi_i^{(0)}\rangle}_{\equiv|\varphi_i^{(t)}\rangle}}_{U(t+T,t')} = \underbrace{U(t,0)\,\overbrace{\lambda_i^{(0)}U(t,0)}^{\lambda_i^{(0)}}|\varphi_i^{(0)}\rangle}_{\lambda_i^{(0)}\,|\varphi_i^{(t)}\rangle}$$

we find that $|\varphi_i^{(t)}\rangle = U(t,0)\,|\varphi_i^{(0)}\rangle$ are the sought-after eigenstates that solve the Schrödinger equation and have t-independent eigenvalues $\lambda_i^{(0)} \equiv \lambda_i$.

** To proof that unitary eigenvalue problems,

$$U|\psi_i\rangle = \lambda_i|\psi_i\rangle \quad \text{with} \quad U^\dagger = U ,$$

posses eigenvalues $|\lambda_i| = 1$ as well as orthonormal eigenstates $\langle\psi_i|\psi_j\rangle = \delta_{ij}$ for $i \neq j$, we consider

$$\langle\psi_i|\psi_j\rangle = \langle\psi_i|U^\dagger U|\psi_j\rangle = \lambda_i^* \lambda_j \langle\psi_i|\psi_j\rangle$$

For $i = j$ it follows that $|\lambda_i| = 1$ and for $\lambda_i \neq \lambda_j \Rightarrow \lambda_i^* \lambda_j \neq 1$ one must find $\langle \psi_i | \psi_j \rangle = 0$. Moreover, we can choose $\langle \psi_i | \psi_i \rangle = 1$ and $\langle \psi_i | \psi_j \rangle = 0$ for degenerate eigenvalues $\lambda_i = \lambda_j$.

"Rotating" frame

A conceptually (not yet necessarily practically) easy way of obtaining Floquet states is to perform a rotation to a "rotating" frame,

$$|\psi_F(t)\rangle = U_F^\dagger(t)\,|\psi(t)\rangle,$$

with a time-periodic unitary

$$U_F(t) = U_F(t+T) = U_F^\dagger$$

known as micromotion operator, so that the transformed Hamiltonian*

$$H_F = U_F^\dagger(t)\,H(t)\,U_F(t) - i\hbar\, U_F^\dagger(t)\,\dot{U}_F(t),$$

known as effective Hamiltonian, becomes time-independent

$$\dot{H}_F \stackrel{!}{=} 0.$$

* $i|\dot{\Psi}\rangle = \underbrace{U_F|\dot{\Psi}_F\rangle}_{\equiv -iH_F|\Psi_F\rangle} + \dot{U}_F|\Psi_F\rangle = H\underbrace{|\Psi\rangle}_{U_F|\Psi_F\rangle}$

Multiplying by $iU_F^\dagger$ from the left and noting that $|\Psi_F\rangle$ can be any state, one finds

$$H_F + iU_F^\dagger \dot{U}_F = U_F^\dagger H U_F .$$

Once we have achieved this, we can compute the eigenstates of H_F

$$H_F|u_i^F\rangle = \varepsilon_i |u_i^F\rangle \quad \Big| \quad H_F = \sum_i \varepsilon_i |u_i^F\rangle\langle u_i^F|$$

to obtain the corresponding steady states of the Schrödinger equation,

$$|\Psi_i^F(t)\rangle = |u_i^F\rangle e^{-i\varepsilon_i t}$$

Transforming back to the original frame, they become the Floquet states,

$$|\Psi_i(t)\rangle = U_F(t)|\Psi_i^F(t)\rangle = \underbrace{U_F(t)|u_i^F\rangle}_{|u_F(t)\rangle} e^{-i\varepsilon_i t},$$

where we immediately see that $U_F(t)$ describes the micromotion. The task of computing the Floquet states has been split into two steps: (i) Finding $U_F(t)$ and H_F as well as (ii) diagonalizing H_F.

We can also write down the time-evolution operator in the rotating frame,

$$e^{-i(t_2-t_1)H_F},$$

from which we can obtain the time-evolution operator in the original frame

$$U(t_2,t_1) = U_F(t_2)\, e^{-i(t_2-t_1)H_F}\, U_F^\dagger(t_1).$$

- $U_F(t_2)$: transform to original frame
- $e^{-i(t_2-t_1)H_F}$: evolve within rotating frame
- $U_F^\dagger(t_1)$: transform to rotating frame

In this form micromotion and phase evolution are neatly separated.

Exercise

Express the Floquet Hamiltonian $H_F^{(0)}$ in terms of

(i) $U_F(t)$ and H_F

(ii) $|u_i(t)\rangle$ and ε_i ?

Answer:

$$U(T,0) = \underbrace{U_F^\dagger(T)}_{U_F(0)}\, e^{-iTH_F}\, U_F(0) =$$

$$= \exp\left(-iT\, U_F^\dagger(0) H_F U_F(0)\right)$$

$$\Rightarrow \quad H_F^{(0)} = U_F^\dagger(0) H_F U_F(0)$$

$$= U_F^\dagger(0) \sum_i \varepsilon_i |u_i^F\rangle\langle u_i^F| U_F(0)$$

$$= \sum_i \varepsilon_i |u_i(0)\rangle\langle u_i(0)|$$

Thus the Floquet Hamiltonian is a unitary rotation of the effective Hamiltonian and shares the same spectrum ε_i.

Ambiguity of quasienergies and resonances

The quasienergies are defined modulo $\omega = 2\pi/T$ only; namely, we can alternatively write

$$|\psi_i(t)\rangle = \underbrace{|u_i(t)\rangle e^{im\omega t}}_{\equiv |u_{im}(t)\rangle = |u_{im}(t+T)\rangle} e^{-i\overbrace{(\varepsilon_i + m\omega)}^{\equiv \varepsilon_{im}} t}.$$

This becomes apparent also from the fact that the eigenvalues of $U(T)$ only fix the phase factor

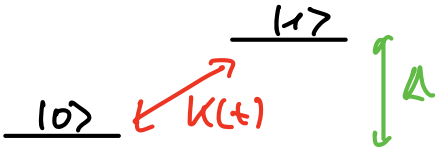
$$e^{-iT\varepsilon_i} = e^{-iT(\varepsilon_i + m\omega)}$$

or, alternatively, from the ambiguity of the operator logarithm

$$H_F^{(0)} = \frac{i}{T} \ln\left(U(T)\right).$$

This ambiguity reflects the possibility of resonant coupling: Consider the unperturbed undriven system H_0 with two eigenstates $|0\rangle$ and $|1\rangle$ with energies E_0 and E_1. Now adding a small time-periodic perturbation, $H(t) = H_0 + V(t)$, with $V(t+T) = V(t)$ that is resonant, $E_1 - E_0 = m\omega$ with integer m, the corresponding unperturbed quasienergies $\varepsilon_0^{[0]} = E_0$ and $\varepsilon_1^{[0]} = E_1 - m\omega$ are degenerate and will be hybridized by $V(t)$, so that the (perturbed) Floquet states will be superpositions of $|0\rangle$ and $|1\rangle$.

Example (part 1)



$$H(t) = \Delta\,|1\rangle\langle 1| + \underbrace{V(t)}_{\sum_\mu V_\mu e^{i\mu\omega t}\;|\;V_{-\mu}=V_\mu^*}\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)$$

To compute the Floquet states, we go to a rotating frame, $|\psi'\rangle = U_0^\dagger|\psi\rangle$, with

$$U_0(t) = \exp\left(-i m\omega t\,|1\rangle\langle 1|\right) = U_0(t+T)$$

so get, with $U_0^\dagger|0\rangle\langle 1|U_0 = e^{-im\omega t}|0\rangle\langle 1|$ and $\dot U_0 = -im\omega\, U_0$,

$$H'(t) = U_0^\dagger(t)\,H(t)\,U_0(t) - i\,U_0^\dagger(t)\,\dot U_0(t)$$
$$= (\Delta - m\omega)|1\rangle\langle 1| + \sum_\mu V_\mu e^{i\mu\omega t}\left(e^{-im\omega t}|0\rangle\langle 1| + \text{h.c.}\right)$$