For

$$|\delta|, \ |\kappa(t)| \ll \omega,$$

with detuning $\delta \equiv \Delta - m\omega$, the state $|\psi'\rangle$ changes much slower than the rotation with frequency ω. Then we can perform a rotating-wave approximation

$$H'(t) \approx \frac{1}{T}\int_0^T dt\, H'(t) \quad (\text{time-independent}),$$

so that

$$\begin{aligned} H_F &\approx \frac{1}{T}\int_0^T dt\, H'(t) \\ &= \delta|1\rangle\langle 1| + (\kappa_m |0\rangle\langle 1| + \underbrace{\kappa_{-m}}_{\kappa_m^*} |1\rangle\langle 0|) \\ &= \vec{h}\cdot\vec{\sigma} + \delta/2 \end{aligned}$$

with vector of Pauli operators and

$$\vec{h} = \begin{pmatrix} \mathrm{Re}\{\kappa_m\} \\ \mathrm{Im}\{\kappa_m\} \\ \delta/2 \end{pmatrix} \equiv |\vec{h}| \begin{pmatrix} \sin(\vartheta)\cos(\varphi) \\ \sin(\vartheta)\sin(\varphi) \\ \cos(\vartheta) \end{pmatrix}.$$

We thus obtain eigenstates of H_F

$$|u^F_\pm\rangle = \begin{cases} \cos(\vartheta/2)|0\rangle + e^{i\varphi}\sin(\vartheta/2)|1\rangle \\ \sin(\vartheta/2)|0\rangle - e^{i\varphi}\cos(\vartheta/2)|1\rangle \end{cases}$$

with eigenvalues

$$\varepsilon_\pm = \delta/2 \pm |\vec{h}| = \delta/2 \pm \sqrt{|\kappa_m|^2 + (\delta/2)^2}.$$

The Floquet modes then read

$$|u_\pm(t)\rangle \approx \mathcal{U}_0(t)\,|u_\pm^F\rangle$$

$$= \begin{cases} \cos(\vartheta/2)\,|0\rangle + e^{i\varphi - im\omega t}\sin(\vartheta/2)\,|1\rangle \\ \sin(\vartheta/2)\,|0\rangle - e^{i\varphi - im\omega t}\cos(\vartheta/2)\,|1\rangle \end{cases}.$$

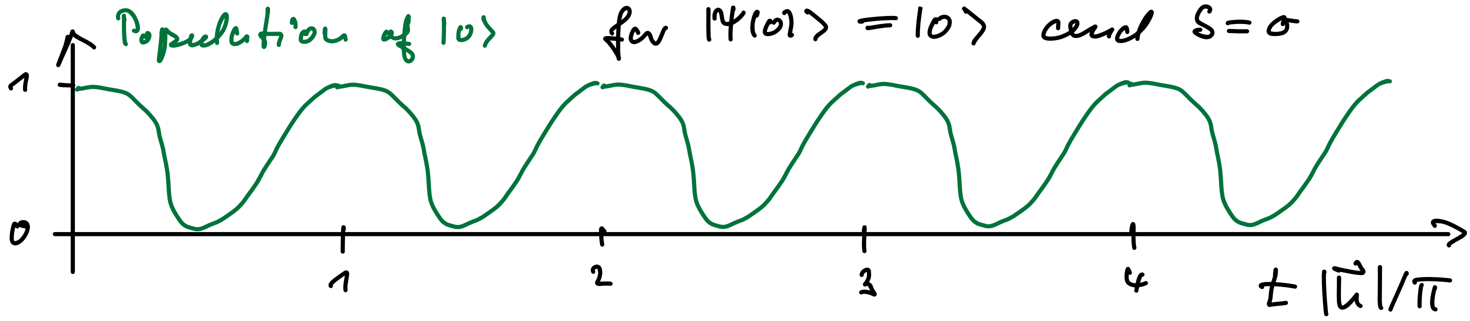
Thus even for $|\Delta| \gg |h(t)|$, the Floquet states are superpositions of both $|0\rangle$ and $|1\rangle$.

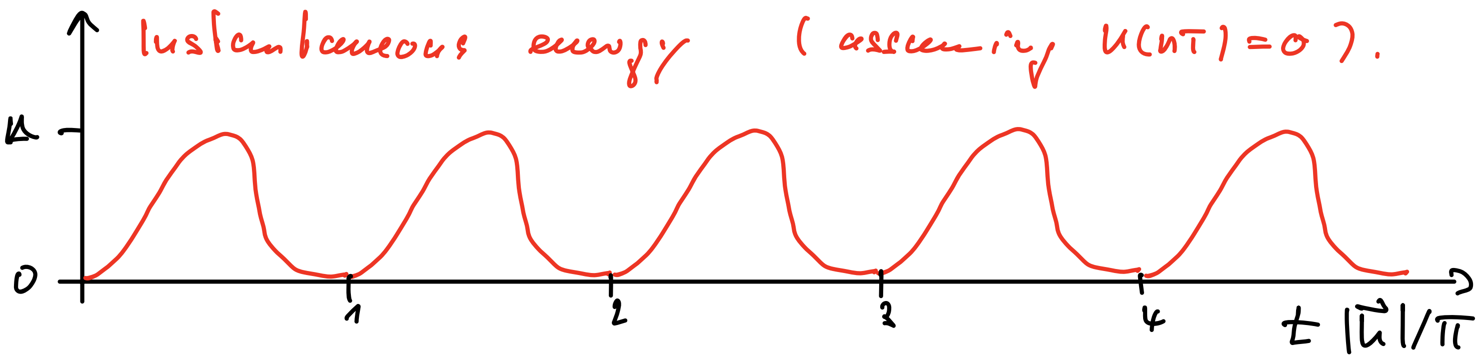
The Floquet Hamiltonian

$$H_F^{(0)} = \mathcal{U}^\dagger(0)\, H_F\, \mathcal{U}(0) = H_F,$$

thus describes large energy changes with time.

Dynamics induced by $H_F^{(0)}$ for $|\psi(0)\rangle = |0\rangle$ and $\delta = 0$





In a (generic) many-body systems resonances will be ubiquotous, so that driving unavoidably induces energy absorbtion (Floquet heating).

Floquet space

Plugging the Floquet states

$$|\psi_i(t)\rangle = |u_{im}(t)\rangle\, e^{-i\varepsilon_{im} t} ,$$

with $|u_{im}(t)\rangle = |u_i(t)\rangle e^{im\omega t}$ and $\varepsilon_{im} = \varepsilon_i + m\omega$ into the Schrödinger equation,

$$i\partial_t |\psi_i\rangle = i\partial_t |u_i\rangle e^{-i\varepsilon_{im} t} + \varepsilon_{im} |u_i\rangle e^{-i\varepsilon_{im} t}$$
$$= H|u_i\rangle e^{-i\varepsilon_{im} t} ,$$

we obtain

$$\underbrace{(H - i\partial_t)}_{Q} |u_{im}\rangle = \varepsilon_{im} |u_{im}\rangle ,$$

which can be viewed as eigenvalue problem of the quasienergy operator Q in the extended space $\mathcal{F}$ of time-periodic states (Floquet space or Sambe space), being the product space $\mathcal{F} = \mathcal{H} \otimes \mathcal{T}$ of our original state space $\mathcal{H}$ and the space of time-periodic functions $\mathcal{T}$ (with period T). We denote elements of this space by double kets,

$|u\rangle\rangle$ for the whole function $|u(t)\rangle$.

The scalar product is given by

$$\langle\langle v|v\rangle\rangle = \frac{1}{T}\int_0^T dt\ \langle v(t)|u(t)\rangle .$$

Remark:

- In this space $|u_{im}\rangle$ and $|u_{im'}\rangle$ are independent solutions for $m' \neq m$, since $\varepsilon_{im} \neq \varepsilon_{im'}$.

Introducing the basis states

$$|\alpha m\rangle\rangle \quad \text{for} \quad |\alpha m\rangle(t) = |\alpha\rangle e^{i\mu\omega t},$$

with Fourier indizes $\mu \in \mathbb{Z}$ and generic basis $\{|\alpha\rangle\}$ of $\mathcal{H}$, we can expand

$$\begin{aligned} |u_{im}(t)\rangle &= |u_i(t)\rangle e^{im\omega t} \\ &= \sum_\mu |u_i^\mu\rangle e^{i\mu\omega t} e^{im\omega t} \\ &= \sum_\alpha \sum_\mu u_i^{\alpha\mu} e^{i(\mu+m)\omega t} \end{aligned}$$

with $|u_i^\mu\rangle = \frac{1}{T}\int_0^T dt\ e^{-i\mu\omega t}|u_i(t)\rangle$ and $u_i^{\alpha\mu} = \langle\alpha|u_i^\mu\rangle$ and compute matrix elements of the quasienergy operator

$$\langle\langle\alpha'\mu'|Q|\alpha\mu\rangle\rangle = \langle\alpha'|H_\mu|\alpha\rangle + \mu\omega\,\delta_{\mu'\mu}\delta_{\alpha'\alpha}$$

(proof: exercise)

$$
\begin{array}{c|cccccc|c}
\mu = & \cdots & 1 & 0 & -1 & -2 & \cdots & \mu' \\
\hline
 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
 & \ddots & H_0 + \hbar\omega & H_1 & H_2 & H_3 & \ddots & 1 \\
Q = & \ddots & H_{-1} & H_0 + 0 & H_1 & H_2 & \ddots & 0 \\
 & \ddots & H_{-2} & H_{-1} & H_0 - \hbar\omega & H_1 & \ddots & -1 \\
 & \ddots & H_{-3} & H_{-2} & H_{-1} & H_0 - 2\hbar\omega & \ddots & -2 \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots
\end{array}
$$

where $H_\mu = \int_0^T dt \, e^{-i\mu\omega t} H(t) = H_{-\mu}^{\dagger}$.

Remarks:

- We can see that Q is hermitian.
- The problem is highly redundant.
- The problem resembles that of a system with time-independent Hamiltonian H_0 coupled to a "photon"-like mode, with "photon" number μ (relative to a large classical background) and energy $\mu\omega$.

Advantages

⊕ Effective time-independent problem

⊕ Hermitian eigenvalue problem

⇒ Use intuition and methods from undriven systems

Price:

⊖ Larger (infinite) state space

Instead of fully diagonalizing Q to obtain the Floquet states, we can also just block diagonalize Q with respect to μ via a unitary rotation $\tilde{\mathcal{U}}$ in Floquet space. Since a block diagonal $Q' = \tilde{\mathcal{U}}^\dagger Q \mathcal{U}$ implies $H'_{\mu\nu} = 0$ for $\mu \neq \nu$, the desired transformation corresponds to the transformation to the rotating frame

$$H_F = \mathcal{U}_F^\dagger H \mathcal{U}_F - i \mathcal{U}_F^\dagger \dot{\mathcal{U}}_F \qquad \text{(time independent)}$$

Namely

$$Q_F = \mathcal{U}_F^\dagger Q \mathcal{U}_F$$

$$= \mathcal{U}_F^\dagger [H - i d_t] \mathcal{U}_F \qquad d_t \mathcal{U}_F = \dot{\mathcal{U}}_F + \mathcal{U}_F d_t$$

$$= \underbrace{\mathcal{U}_F^\dagger H \mathcal{U}_F^\dagger - i \mathcal{U}_F^\dagger \dot{\mathcal{U}}_F}_{H_F} - i d_t ,$$

so that

$$Q_F = \mathcal{U}_F^\dagger Q \mathcal{U}_F = H_F - i d_t .$$

If for the purpose of doing an experiment that effectively realizes the physics of H_F, we actually do not need to compute the Floquet states. The accomplishing the block diagonalization is enough.

In Floquet space, finding H_F (or Q_F) and $\mathcal{U}_F$, is mapped to a well known problem, the block diagonalization of a hermitian operator/matrix. This allows us to employ known methods, with a prominent one being quasi-degenerate perturbation theory. Considering the "photon energy" $\mu\omega$ as unperturbed problem, we find a systematic high-frequency expansion.

$$\mathcal{U}_F(t) = \exp\left(\sum_{\nu=1}^{\infty} G_F^{(\nu)}(t)\right) \simeq \exp\left(\sum_{\nu=1}^{\nu_{max}} G_F^{(\nu)}(t)\right) \equiv \mathcal{U}_F^{HF}(t)$$

$$H_F = \sum_{\nu=1}^{\infty} H_F^{(\nu)} \simeq \sum_{\nu=1}^{\nu_{max}} H_F^{(\nu)} \equiv H_F^{(HF)}$$

with leading terms

$$G_F^{(1)}(t) = \sum_{\mu\neq 0} \frac{e^{i\mu\omega t}}{\mu\omega} H_\mu$$

and

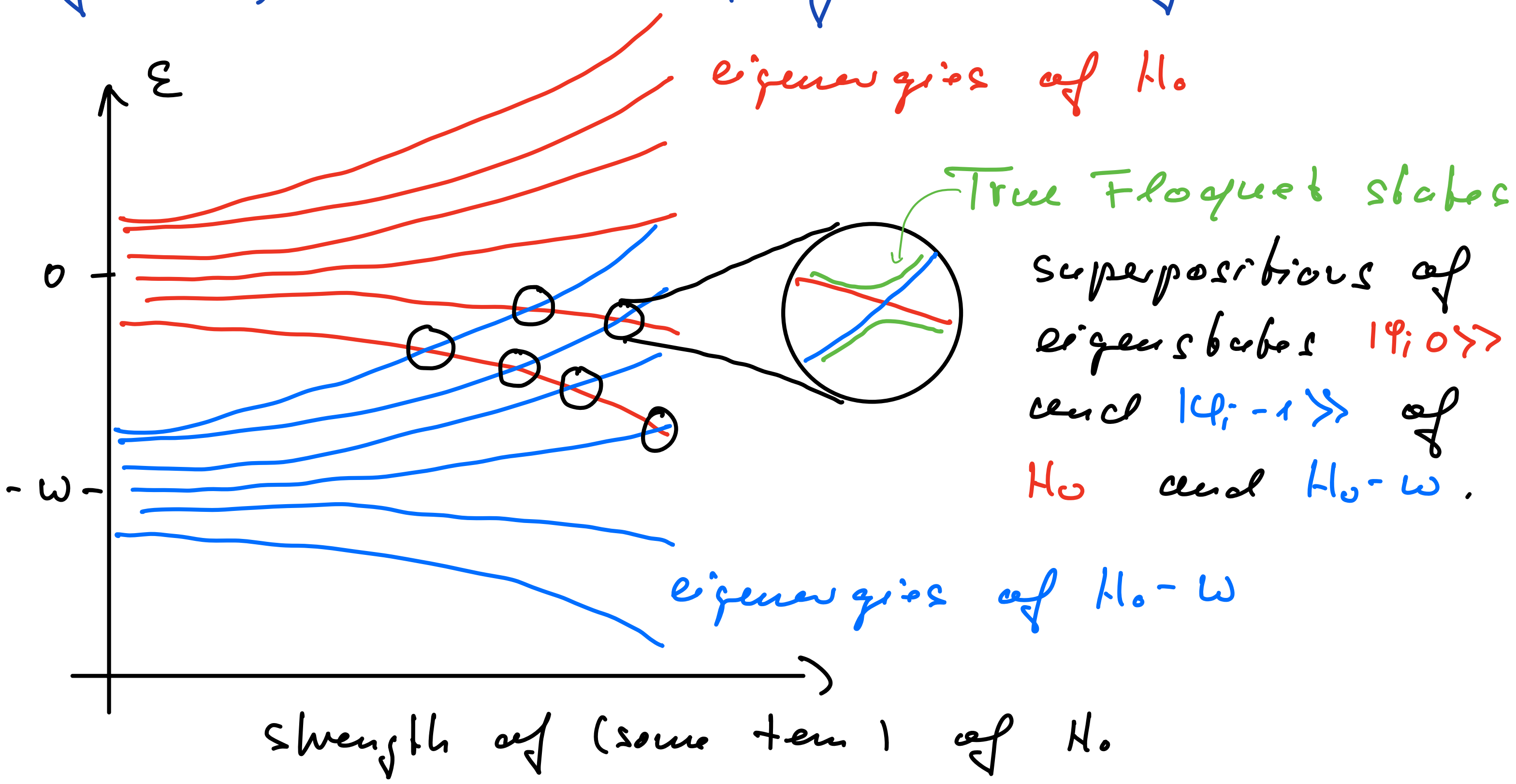
$$H_F^{(1)} = H_0 \quad , \quad H_F^{(2)} = \sum_{\mu \neq F} \frac{H_\mu H_{-\mu}}{\mu \omega} .$$

In first order, this corresponds to a rotating-wave approximation

$$H_F \approx H_0 = \frac{1}{T} \int_0^T dt \; H(t) ,$$

which is valid as long as the dynamics induced by $H(t)$ is slow compared to the driving frequency ω.

In Floquet space this condition appears as the requirement that the spectra of H_0 and $H_0 \pm \omega$ remain well separated and do not overlap (while also H_μ remain small). If they still overlap, this gives rise to resonant coupling of excited states of $H_0 - \mu\omega$ to low lying states of H_0.



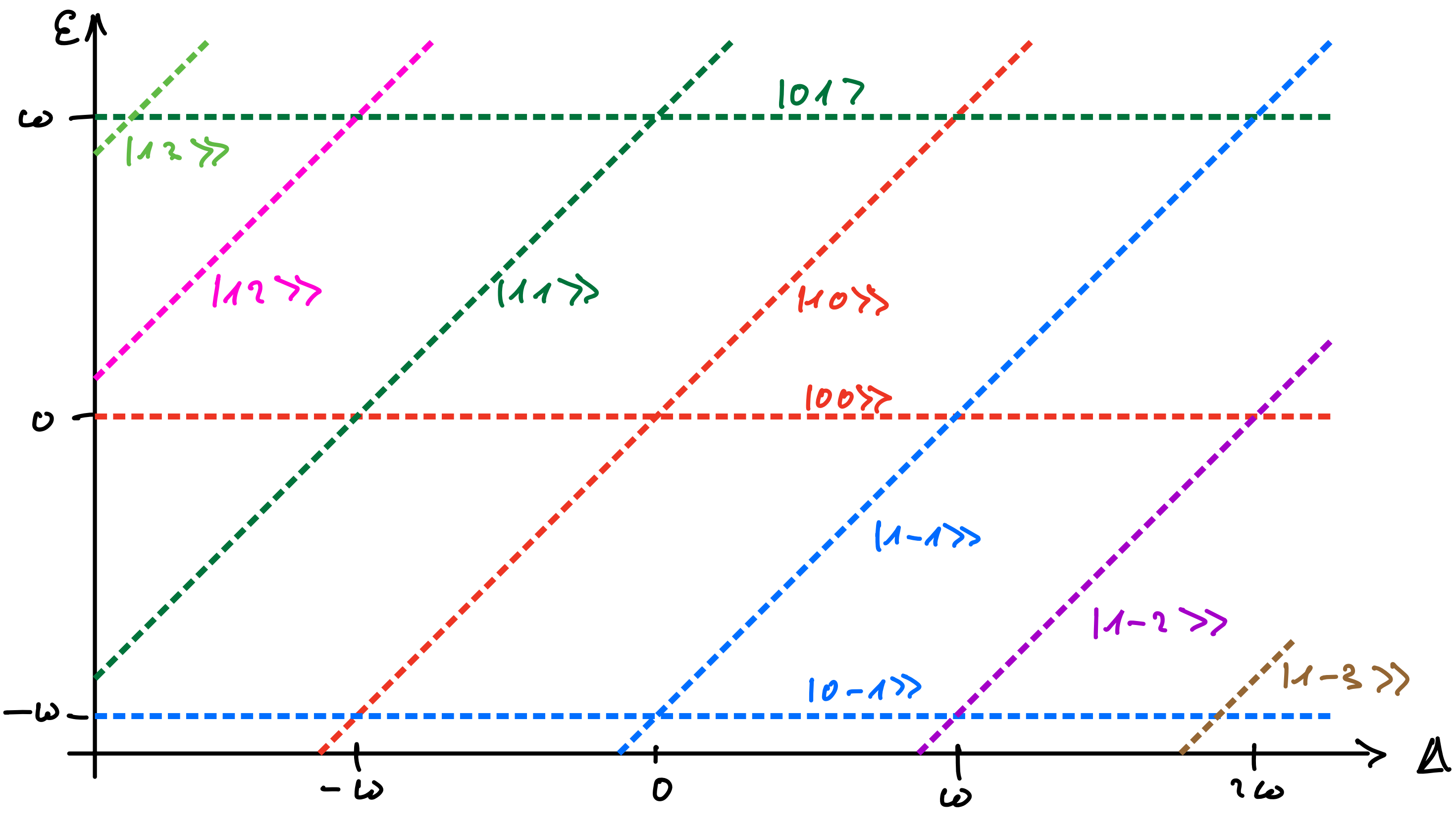
Where eigenstates of H_0 and $H_0 - \omega$ cross, avoided crossings are formed, which reflect their resonant coupling. Such resonant coupling can describe Floquet heating. It is not captured by the high-frequency approximation $H_F \approx H_0$. This physics is captured in a minimal fashion by the simple two-level model introduced above. So let's treat it again in Floquet space.

Example (part 2)

For the example from before, $H(t) = \Delta |1\rangle\langle 1| + \kappa(t)\,(|0\rangle\langle 1| + h.c.)$, we find

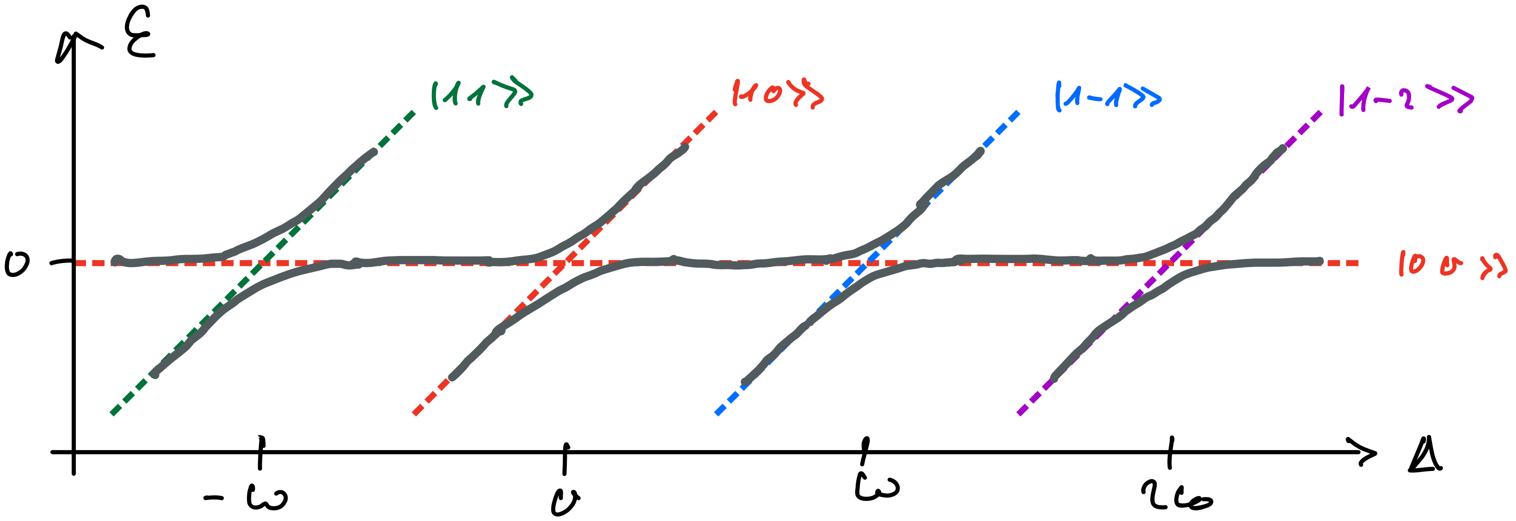
$$H_\mu = \delta_{\mu,0}\,\Delta \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \kappa_\mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Unperturbed quasienergy spectrum for $\kappa_\mu = 0$



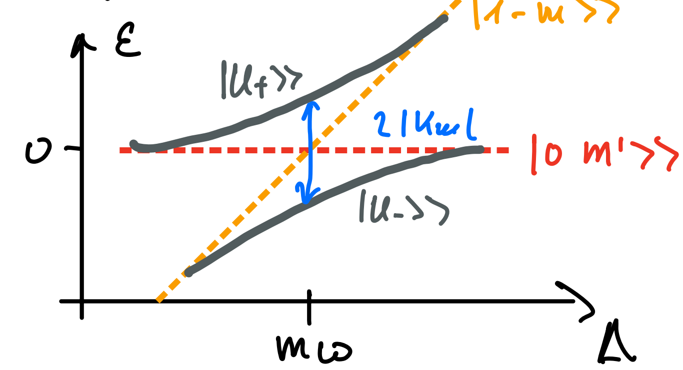
The dashed lines indicate states $|0\,m\rangle\rangle$ (or $|0\rangle e^{im\omega t}$) and $|1\,m\rangle\rangle$ (or $|1\rangle e^{im\omega t}$).

Full quasienergy spectrum for $K_m \neq 0$



We can treat a single resonance near $\varepsilon = m'\omega$ and $\Delta = m\omega$ by neglecting all but the almost degenerate states $|1\,{-m}\rangle\rangle$ and $|0\,0\rangle\rangle$. This corresponds to first order degenerate state perturbation theory and gives, locally

Single resonance:



$$Q_{0m} = \underbrace{(\Delta - m\omega)}_{\equiv \delta} |1\,{-m}\rangle\rangle\langle\langle 1\,{-m}| + K_m |0\,0\rangle\rangle\langle\langle 1\,{-m}| + \underbrace{K_{-m}}_{K_m^*} |1\,{-m}\rangle\rangle\langle\langle 0\,0|$$

$$\equiv \vec{h}\cdot\vec{\sigma} + \frac{\delta}{2} \quad \text{with} \quad \vec{h} = \left(\mathrm{Re}\{K_m\},\ \mathrm{Im}\{K_m\},\ \frac{\delta}{2}\right)^T$$

This is exactly the problem we got from the rotating-wave approximation.

The eigenstates

$$|u_\pm\rangle\rangle = \begin{cases} \cos(\vartheta/2)\, |00\rangle + e^{i\varphi} \sin(\vartheta/2)\, |1{-}m\rangle\rangle \\ \sin(\vartheta/2)\, |00\rangle - e^{i\varphi} \cos(\vartheta/2)\, |1{-}m\rangle\rangle, \end{cases}$$

with $\vec{h} = |\vec{h}| (\sin\vartheta \cos\varphi, \sin\vartheta \sin\varphi, \cos\vartheta)^T$
and corresponding quasienergies

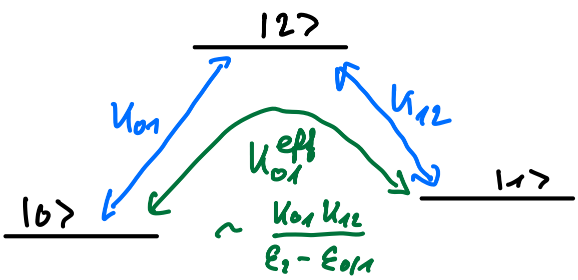
$$\varepsilon_\pm = \frac{\Delta - m\omega}{2} \pm \sqrt{|K_m|^2 + (\Delta - m\omega)^2/4}\,.$$

To obtain the time-dependent solutions, we replace

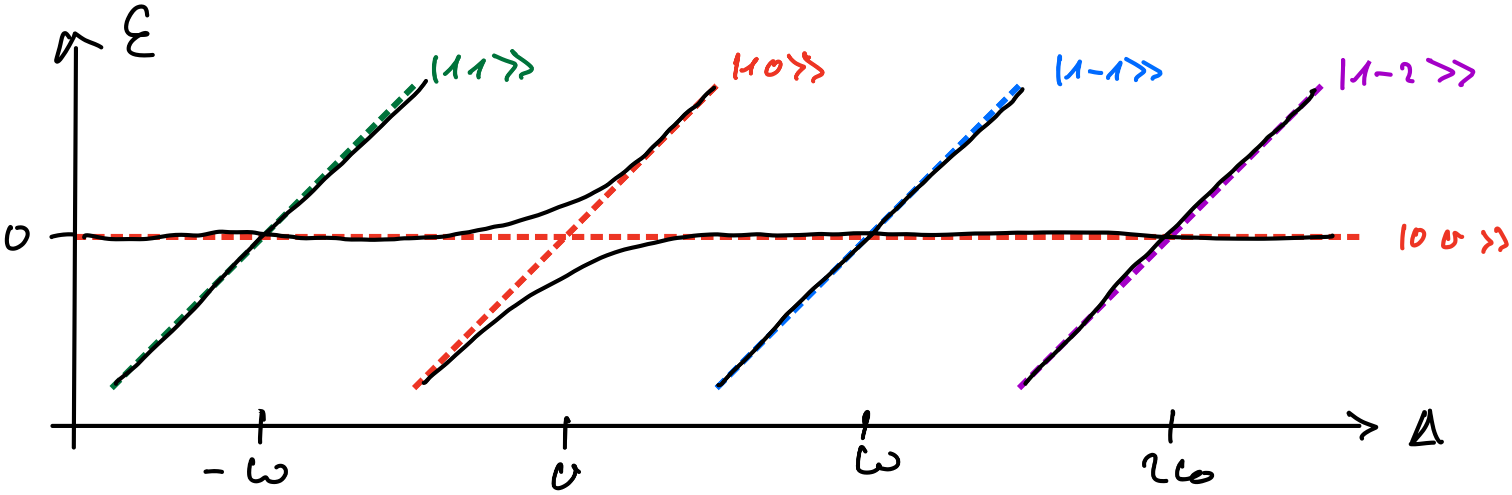
$$|u_\pm\rangle\rangle \longrightarrow |u_\pm(t)\rangle \quad \text{and} \quad |\alpha\, \mu\rangle\rangle \longrightarrow |\alpha\rangle e^{i\mu\omega t}.$$

Remarks:

- The rotating-wave approximation corresponds to first-order degenerate state perturbation theory in Floquet space
- However, in Floquet space, we can also compute higher-order corrections. This allows, for instance, to derive the coupling between states that are not directly coupled by the drive via so-called virtual transitions. Floquet heating is usually associated with such slow high-order processes.



- The zero-"photon" coupling K_0 is part of H_0. Its treatment within degenerate state perturbation theory corresponds to the approximation $H_F \approx H_0$, which is visualized in this picture



c) Master equation in Floquet representation

So far, we essentially followed the procedure for undriven systems and obtained an equation of motion, which, apart from the periodic time dependence of $H(t)$ and $\bar{A}(t)$, looks formally identical to that for time-independent Hamiltonians.

Floquet representation

Let us now write the equation of motion in terms of the Floquet states $|u_i(t)\rangle$ of the unperturbed Hamiltonian $H(t)$,

$$\rho(t) = \sum_{ij} \rho_{ij}(t)\, |u_i(t)\rangle\langle u_j(t)|$$

with $\rho_{ij}(t) = \langle u_i(t)|\,\rho\,|u_j(t)\rangle$. For that purpose, we take matrix elements $\langle u_i|\cdot|u_j\rangle$ of the Redfield equation

$$\dot{\rho}(t) = \frac{1}{i}\left[H(t), \rho(t)\right] - \left[A\bar{A}(t)\rho(t) - \bar{A}(t)\rho(t)A + \text{h.c.}\right],$$

For the left-hand side we obtain

$$\langle u_i|\dot{\rho}|u_j\rangle = \langle u_i|\sum_{kl}\Big(\dot{\rho}_{kl}|u_k\rangle\langle u_l| + \rho_{kl}|\dot{u}_k\rangle\langle u_l| + \rho_{kl}|u_k\rangle\langle \dot{u}_l|\Big)$$

$$\frac{d}{dt}e^{i\varepsilon_k t}|\psi_k\rangle = \left(i\varepsilon_k + \frac{1}{i}H\right)\underbrace{e^{i\varepsilon_k t}|\psi_k\rangle}_{|u_k\rangle} \qquad \langle u_l|\left(-i\varepsilon_l - \frac{1}{i}H\right)$$

$$= \dot{\rho}_{ij} + i\underbrace{(\varepsilon_i - \varepsilon_j)}_{\equiv\,\varepsilon_{ij}}\rho_{ij} + \frac{1}{i}\langle u_i|\left[H, \rho\right]|u_j\rangle$$

Here the last term cancels with the first term on the right-hand side, so that we only need to evaluate the second term

$$-\langle u_i|\left[A\bar{A}\rho-\bar{A}\rho A+\rho\bar{A}^{\dagger}A-A\rho\bar{A}^{\dagger}\right]|u_j\rangle$$

$$=-\sum_{kl}\left[A_{ik}\bar{A}_{kl}\rho_{lj}-\bar{A}_{ik}\rho_{kl}A_{li}+\rho_{ik}\bar{A}^{\dagger}_{kl}A_{li}-A_{ik}\rho_{kl}\bar{A}^{\dagger}_{li}\right],$$

where $A_{ik}=A_{ik}(t)$ and $\bar{A}_{kl}=\bar{A}_{kl}(t)$, with

$$A_{ik}(t)=\langle u_i(t)|A|u_k(t)\rangle\equiv\sum_{\mu}e^{i\mu\omega t}A_{ik}^{(\mu)}$$

$$A^{\dagger}_{ik}(t)\equiv(A^{\dagger})_{ik}(t)=A_{ik}(t)=\sum_{\mu}e^{i\mu\omega t}A_{ik}^{(\mu)}$$

where $\omega=2\pi/T$, and

$$\bar{A}_{kl}(t)=\int_0^\infty d\tau\, C(\tau)\langle u_k|\,\mathcal{U}^{\dagger}(t-\tau,t)\,A\,\underbrace{\mathcal{U}(t-\tau,t)\overbrace{|u_l\rangle}^{|\psi_l(t)\rangle e^{i\varepsilon_l t}}}_{\underbrace{|\psi_l(t-\tau)\rangle}_{e^{-i\varepsilon_l(t-\tau)}|u_l(t-\tau)\rangle}e^{i\varepsilon_l t}=|u_l(t-\tau)\rangle e^{i\varepsilon_l\tau}}$$

$$=\int_0^\infty d\tau\, C(\tau)\, e^{-i\varepsilon_{kl}\tau}A_{kl}(t-\tau)$$

$$\equiv\sum_{\mu}e^{i\mu\omega t}A_{kl}^{(\mu)}\int_0^\infty d\tau\, C(\tau)\, e^{-i(\mu+\varepsilon_{kl})\tau}$$

$$\equiv\sum_{\mu}e^{i\mu\omega t}A_{kl}^{(\mu)}\,G(\mu+\varepsilon_{kl}).$$

Here, we have introduced the half-sided Fourier transform of the bath correlation function

$$G(\varepsilon)=\int_0^\infty d\tau\, e^{-i\varepsilon\tau}\,C(\tau)=\tfrac{1}{2}g(\varepsilon)+iS(\varepsilon)$$

We also find

$$\bar{A}^+_{ij}(t) \equiv \left(\bar{A}^+\right)_{ij}(t)$$

$$= \int_0^\infty d\tau\, C^*(\tau) \langle u_k | \mathcal{U}^\dagger(t-\tau, t)\, A\, \mathcal{U}(t-\tau, t) | u_l \rangle$$

$$= \int_0^\infty d\tau\, C^*(\tau)\, e^{-i\varepsilon_{kl}\tau} A_{kl}(t-\tau)$$

$$\equiv \sum_\mu e^{i\mu\omega t} A^{(\mu)}_{kl} \int_0^\infty d\tau\, C^*(\tau)\, e^{-i(\mu+\varepsilon_{kl})\tau}$$

$$= \sum_\mu e^{i\mu\omega t} A^{(\mu)}_{kl}\; G^*(-\mu - \varepsilon_{kl})$$

Combining everything, the master equation reads

$$\dot{\rho}_{ij} + i\varepsilon_{ij}\rho_{ij} = - \sum_{kl}\left[A_{ik}\bar{A}_{kl}\rho_{lj} - \bar{A}_{ik}\rho_{kl}A_{lj} + c.c.\right].$$

$$= - \sum_{kl}\left[R_{ik,lk}\,\rho_{lj} - R_{lj,ki}\,\rho_{kl} + c.c.\right]$$

with $R_{ij,lk} = R_{ij,lk}(t) = R_{ij,lk}(t+T)$, where

$$R_{ij,lk}(t) = - A_{ij}(t)\,\bar{A}_{kl}(t)$$

$$= - \sum_{\mu\mu'} e^{i(\mu'+\mu)\omega t} A^{(\mu')}_{ij} A^{(\mu)}_{kl}\, G(\mu + \varepsilon_{kl})$$

$$= - \sum_m e^{im\omega t} \underbrace{\sum_\mu A^{(m-\mu)}_{ij} A^{(\mu)}_{kl}\, G(\mu+\varepsilon_{kl})}_{\equiv R^{(m)}_{ij,lk}}.$$

D) Long-time (quasi-) steady state

Time-periodic quasi steady state

As the derived master equation is a homogeneous linear differential equation with time-periodic coefficients $R_{ij,mn}(t)$, it possesses Floquet solutions of the form

$$\varrho_{ij}^{(n)}(t) = e^{\lambda_n t}\, r_{ij}^{(n)}(t)$$

with time-periodic matrices $r_{ij}^{(n)}(t+T) = r_{ij}^{(n)}(t)$, in terms of which we can express the solution

$$\varrho_{ij}(t) = \sum_n a_n\, \varrho_{ij}^{(n)}(t)$$

with time-independent coefficients a_n.

Generally, (physical) solutions should obey

$$\mathrm{Re}\{\lambda_n\} \leq 0 ,$$

so that $0 \leq \varrho_{ii} \leq 1$ and $\varrho_{ij} \leq \sqrt{\varrho_{ii}\varrho_{jj}}$, as a consequence of the positivity and normalization of $\varrho_{ij}(t)$. Let us assume, we have a unique steady state $\varrho_{ij}^{(0)}(t)$ with

$$\lambda_0 = 0 \quad \text{and} \quad \sum_i r_{ii}^{(0)}(t) = 1 ,$$

where $\mathrm{Im}(\lambda_0) = 0$ ensures $\sum_i \rho_{ii}(t) = e^{i\lambda_0 t} \sum_i v_{ii}^{(0)} = 1 \; \forall t$ (we can restrict $\mathrm{Im}(\lambda_n) \in [-\omega/2, \omega/2]$, as any periodic part $e^{\lambda v t}$ can be included in $v_{ij}^{(n)}(t)$).

In turn,

$$\mathrm{Re}\{\lambda_n\} < 0 \quad \text{and} \quad \sum_i v_{ii}^{(n)} = 0 \qquad \text{for } n \neq 0,$$

so that $\sum_i \rho_{ii} = 0$ for all times and

$$\rho_{ij}(t) \simeq \rho_{ij}^{(0)}(t) = v_{ij}^{(0)}(t) = \rho_{ij}(t+T)$$

for late times

$$t \gg \frac{1}{\min_{n \neq 0}(\lambda_n)} .$$

Thus, we find a time-periodic quasi-steady state

$$\rho_0(t) \simeq \sum_{ij} |u_i(t)\rangle \rho_{ij}^{(0)} \langle u_j(t)| = \rho_0(t+T) .$$

Average-rate approximation

Generally, we do not assume that $\gamma_{SE} \ll \varepsilon_{ij}$, since the quasienergy level splitting can become very small at avoided crossings. But we will assume that the rate γ_{SE} for the environment-induced change of ρ is small compared to the driving frequency ω,

$$\Gamma_{SE} \ll \omega .$$

Fourier expanding the quasi-steady state

$$\varrho_{ij}^{(0)} = \sum_m \varrho_{ij}^{(0)}(m)\, e^{im\omega t} ,$$

and plugging this solution into our master equation, we find

$$i(\underbrace{m\omega + \varepsilon_{ij}}_{\mathcal{O}(\omega) \text{ for } m \neq 0})\, \varrho_{ij}^{(0)}(m) = -\underbrace{\sum_{m'}\sum_{k\ell}\left[R_{ik,\ell k}^{(m-m')}\varrho_{\ell j}^{(0)}(m') - R_{\ell j,ki}^{(m-m')}\varrho_{k\ell}^{(0)}(m') + \text{c.c.}\right]}_{\mathcal{O}(\Gamma_{SE}\, \varrho^{(0)})}$$

We find, therefore,

$$\varrho_{ij}^{(0)}(m) = \mathcal{O}\left(\Gamma_{SE}/\omega\right) \ll 1 \quad \text{for } m \neq 0 .$$

Neglecting the $m \neq 0$ contributions, we have a time-independent steady-state matrix

$$\varrho_{ij}^{(0)} \simeq \varrho_{ij}^{(0)}(0) ,$$

which solves

$$i\varepsilon_{ij}\, \varrho_{ij}^{(0)} = -\sum_{k\ell}\left[R_{ik,\ell k}^{(0)}\varrho_{\ell j}^{(0)} - R_{\ell j,ki}^{(0)}\varrho_{k\ell}^{(0)} + \text{c.c.}\right] ,$$

where the right-hand side is obtained for $m = 0$, neglecting contributions with $m' \neq 0$.

This equation can be viewed as a rotating-wave approximation

$$R_{ij,\ell k}(t) \simeq R^{(0)}_{ij,\ell k} = \frac{1}{T}\int_0^T dt' \, R_{ij,\ell k}(t') .$$

It is different from the rotating-wave approximation, which corresponds to the secular approximation for $\varepsilon_{ij} \gg \Gamma_{SE}$ (see next section), we, thus, call it average-rate approximation (it is also known as modified rotating-wave approximation).

From now on, we are only concerned with the steady state. Therefore, we drop the superscript "(0)" and simply write $\varrho^{(0)}_{ij} = \varrho_{ij}$ and

$$i\varepsilon_{ij}\, \varrho_{ij} = - \sum_{k\ell} \left[R^{(0)}_{ik,\ell k}\, \varrho_{\ell j} - R^{(0)}_{\ell j,ki}\, \varrho_{k\ell} + c.c. \right].$$

E) Secular-Näherung (Floquet-Pauli eq.)

In case all quasienergy level splittings are large compared to the rate induced by the system-bath coupling

$$\varepsilon_{ij} \gg \Gamma_{SE} \qquad \forall\, i,j \;,$$

we can see that the off-diagonal elements of the density matrix in Floquet representation are small,

$$\rho_{ij} = O\left(\frac{\Gamma_{SE}}{|\varepsilon_{ij}|}\right) \ll 1 \qquad \forall\; i \neq j \;,$$

and can approximately be neglected,

$$\rho(t) \simeq \sum_i \underbrace{\rho_{ii}}_{\equiv p_i} \underbrace{|u_i(t)\rangle\langle u_i(t)|}_{= |\psi_i(t)\rangle\langle\psi_i(t)|}$$

where the diagonal elements ρ_{ii} correspond to the probability p_i for the system to be in state $|u_i(t)\rangle$. They obey the stationary Floquet Pauli equation

$$0 = -\sum_k \left(R_{ki}\, p_i - R_{ik}\, p_k \right) \quad \Big| = \dot{p}_i$$

where the rate for a transition from i to k is given by

$$R_{ki} \equiv R^{(0)}_{ik,ik} \,.$$

This is known as *secular approximation*. It can, alternatively, also be derived from the dynamical equation $\dot{\varrho}_{ij} + i\varepsilon_{ij} = \ldots$, by applying a rotating-wave approximation. Then the left-hand side is given by $\dot{P}_i$ and the off-diagonal elements decouple from the diagonal ones and decay.

The rates read

$$R_{ki} = \frac{1}{T}\int_0^T dt\, A_{ik}(t)\bar{A}_{ki}(t)$$

$$= \sum_\mu \underbrace{A_{ik}^{(-\mu)}}_{A_{ki}^{(\mu)*}} A_{ki}^{(\mu)}\, G(\mu+\varepsilon_{ki}) \qquad G = \tfrac{1}{2}g + iS \approx \tfrac{1}{2}g$$

$$\approx \sum_\mu \underbrace{|A_{ki}^{(\mu)}|^2 \tfrac{1}{2} g(\mu+\varepsilon_{ki})}_{\equiv R_{ki}^{(\mu)}},$$

where in the last step, we have neglected the imaginary part of $G(\varepsilon)$, which results from the Lamb-shift Hamiltonian.

The real part corresponds to the full Fourier transform of the bath correlation function,

$$g(\varepsilon) = \frac{2}{2}\left(G(\varepsilon) + G^*(\varepsilon)\right)$$

$$= \left(\int_0^\infty d\tau\, e^{i\varepsilon\tau} c(\tau) + \underbrace{\int_0^\infty d\tau}_{\int_\infty^0 d\tau'}\, e^{-i\varepsilon\tau} \underbrace{c^*(\tau)}_{c(-\tau)}\right)$$

$$= \int_{-\infty}^\infty d\tau\, e^{i\varepsilon\tau} c(\tau). \qquad \text{f. } \tau' = -\tau$$

Expressing $C(\tau)$ in terms of the eigenstates $|\nu\rangle$ of H_E with energies E_ν,

$$
\begin{aligned}
C(\tau) &= \langle \tilde{B}(t)\tilde{B}(0)\rangle_0 \\
&= \mathrm{tr}_E\{\rho_E e^{iH_E t} B e^{-iH_E t} B\} \\
&= \sum_{\nu\nu'} \frac{1}{Z} e^{-\beta E_\nu} e^{iE_\nu t} \langle \nu|B|\nu'\rangle e^{-iE_{\nu'} t} \langle \nu'|B|\nu\rangle \\
&= \sum_{\nu\nu'} \underbrace{\frac{1}{Z} e^{-\beta E_\nu}}_{P_\nu} \underbrace{e^{i(E_\nu - E_{\nu'})t}}_{e^{iE_{\nu\nu'}t}} \underbrace{|\langle \nu|B|\nu'\rangle|^2}_{|B_{\nu\nu'}|^2} ,
\end{aligned}
$$

where $E_{\nu\nu'} = E_\nu - E_{\nu'}$ is the energy change in the in the environment associated with the transition $\nu' \to \nu$, we find

$$
\begin{aligned}
g(\varepsilon) &= \int_{-\infty}^{\infty} d\tau \, e^{-i\varepsilon\tau} C(\tau) \\
&= \sum_{\nu\nu'} P_\nu |B_{\nu\nu'}|^2 \underbrace{\int_{-\infty}^{\infty} d\tau \, e^{-i(\varepsilon - E_{\nu\nu'})t}}_{2\pi\,\delta(\varepsilon - E_{\nu\nu'})} \\
&= 2\pi \sum_{\nu\nu'} P_\nu |B_{\nu\nu'}|^2 \delta(\varepsilon - E_{\nu\nu'})
\end{aligned}
$$

as well as

$$
\begin{aligned}
g(-\varepsilon) &= 2\pi \sum_{\nu\nu'} \underbrace{P_\nu}_{e^{-\beta E_{\nu\nu'}} P_{\nu'}} \underbrace{|B_{\nu\nu'}|^2}_{|B_{\nu'\nu}|^2} \underbrace{\delta(-\varepsilon - E_{\nu\nu'})}_{\delta(\varepsilon + E_{\nu\nu'}) = \delta(\varepsilon - E_{\nu'\nu})} \\
&= 2\pi \sum_{\nu\nu'} \underbrace{e^{-\beta E_{\nu'\nu}}}_{e^{\beta\varepsilon}} P_\nu |B_{\nu\nu'}|^2 \delta(\varepsilon - E_{\nu\nu'}) \\
&= e^{\beta\varepsilon} g(\varepsilon) ,
\end{aligned}
$$

so that the rates read

$$R_{ki} = \sum_{\mu} \pi \, |A^{(\mu)}_{ki}|^2 \sum_{\nu\nu'} p_{\nu} \, |B_{\nu\nu'}|^2 \, \delta(\mu\omega + \varepsilon_{ki} - E_{\nu\nu'})$$

$$\equiv \sum_{\mu} R^{(\mu)}_{ki} \, .$$

This expression has the form of the Floquet-variant of Fermi's golden rule. Each term $R^{(\mu)}_{ki}$ describes joint processes of system and environment

$$\begin{aligned} &\text{system} && |u_k(t)\rangle \to |u_i(t)\rangle \\ &\text{environment} && |\nu'\rangle \to |\nu\rangle \, , \end{aligned}$$

with energy change $E_{\nu\nu'} = -\varepsilon_{ki} - \mu\omega$.

Remarks:

- For the individual processes we have

$$R^{(\mu)}_{ki} = \frac{1}{2} \underset{|A^{(-\mu)}_{ik}|^2}{|A^{(\mu)}_{ki}|^2} \, \underbrace{g(\mu + \varepsilon_{ki})}_{e^{\beta(\mu+\varepsilon_{ki})} \, g(-\mu \underbrace{-\varepsilon_{ki}}_{+\varepsilon_{ik}})} = e^{\beta(\mu+\varepsilon_{ki})} R^{(-\mu)}_{ik}$$

- But generally for the full rates R_{ki},

$$\frac{R_{ki}}{R_{ik}} \neq e^{\beta \varepsilon_{ki}} \Rightarrow p_i \neq \frac{1}{Z} e^{-\beta \varepsilon_i} ,$$

as it was the case for the undriven case.

- One might however achieve

$$\frac{R_{ki}}{R_{ik}} \approx e^{\beta \varepsilon_{ki}} \Rightarrow p_i \propto e^{-\beta \varepsilon_i}$$

by suppressing processes with $\mu \neq 0$ by choosing an environment with spectral width $\Delta_E < \omega$.