Experiments on cold atoms and optical cavities

1 – Design of an optical cavity for atom-light interaction

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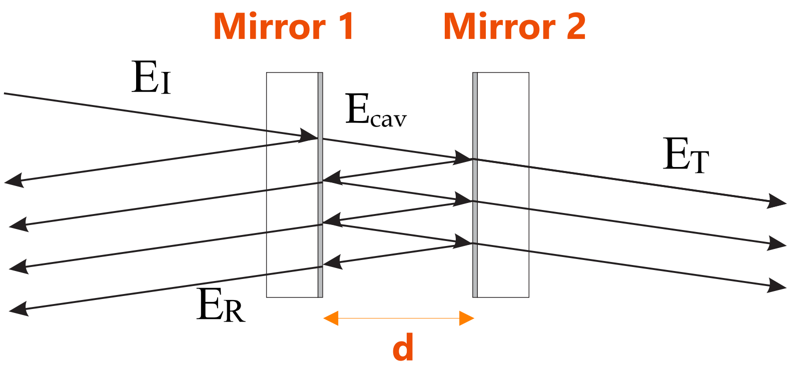
Longitudinal modes of optical cavities

Fabry-Perot resonator for plane waves:

$$\mathcal{E}_I = \mathcal{E}_0 \mathrm{e}^{i\omega t - ikz}$$



r_j, t_j: reflection and transmission coefficients of mirror j

$$|r_j|^2 + |t_j|^2 = 1$$ **[NO LOSSES]**

After treating the two mirrors as beam splitters, we find

$$\mathcal{E}_T = \frac{t_1 t_2 \,\mathrm{e}^{-ikd}}{1 - r_1 r_2 \,\mathrm{e}^{-2ikd}} \mathcal{E}_I \qquad \mathcal{E}_R = \left[\frac{t_1^2 r_2 \,\mathrm{e}^{-2ikd}}{1 - r_1 r_2 \,\mathrm{e}^{-2ikd}} - r_1 \right] \mathcal{E}_I$$

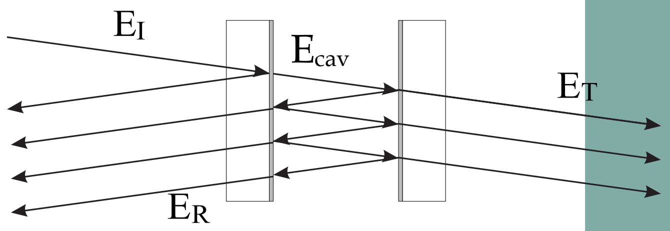
And we get the relative intensity reflection and transmission as

$$T = |\mathcal{E}_T|^2 \qquad R = |\mathcal{E}_R|^2$$

Longitudinal modes of optical cavities

For identical mirrors, no losses,

$$\mathcal{T}_{cav} = \frac{T^2}{(1-R)^2 + 4R\sin^2\left(\frac{\pi\nu}{\Delta_{\mathcal{FSR}}}\right)} = \frac{1}{1+\left(\frac{2\mathcal{F}}{\pi}\right)^2 \sin^2\left(\frac{\pi\nu}{\Delta_{\mathcal{FSR}}}\right)}$$



$$R = |r_j|^2 \qquad T = |t_j|^2$$

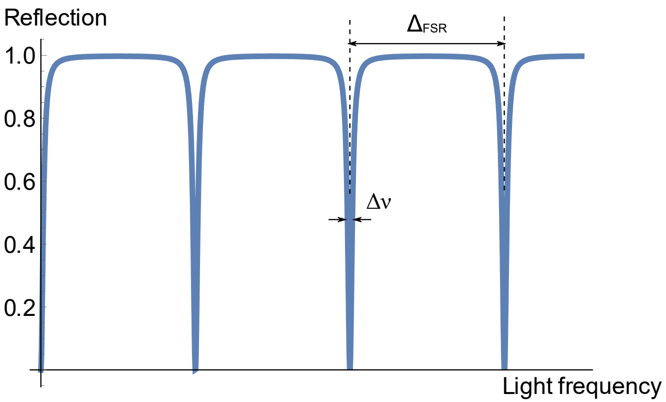
Free-spectral range $\Delta_{\mathcal{FSR}} = \frac{c}{2d}$

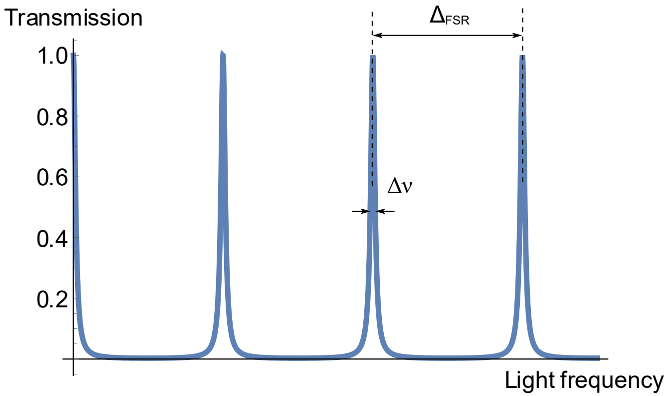
Resonances $\nu_j = j\Delta_{\mathcal{FSR}}$

Finesse $\mathcal{F} = \frac{\pi\sqrt{R}}{1-R}$

Resonance width

$$\Delta\nu = \frac{\Delta_{\mathcal{FSR}}}{\mathcal{F}}$$

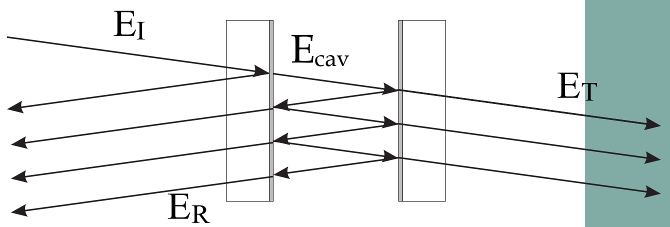




Longitudinal modes of optical cavities

For identical mirrors, no losses,

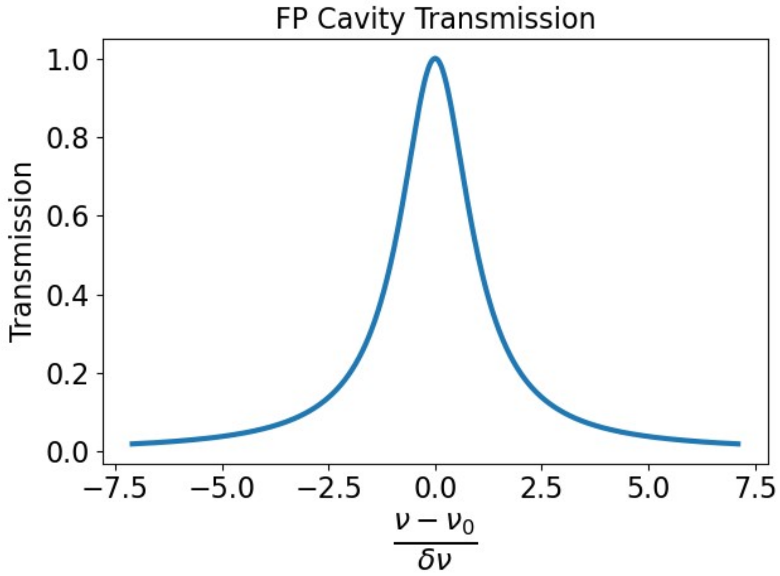
$$\mathcal{T}_{cav} = \frac{T^2}{(1-R)^2 + 4R\sin^2\left(\frac{\pi\nu}{\Delta_{\mathcal{FSR}}}\right)} = \frac{1}{1 + \left(\frac{2\mathcal{F}}{\pi}\right)^2 \sin^2\left(\frac{\pi\nu}{\Delta_{\mathcal{FSR}}}\right)}$$



With $\mathcal{F} \gg 1,$ close to resonance j we can write

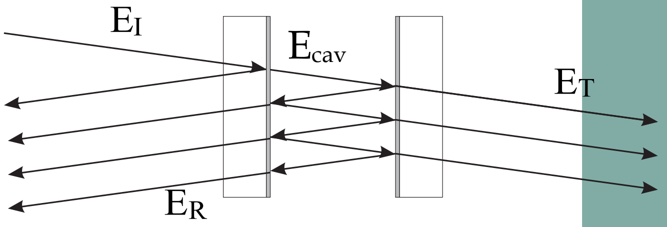
$$\mathcal{T}_{cav} \simeq \frac{1}{1 + \left(\frac{2\mathcal{F}}{\pi}\right)^2 \left[\frac{\pi(\nu-\nu_j)}{\Delta_{\mathcal{FSR}}}\right]^2}$$

Lorentzian profile



Longitudinal modes of optical cavities

Electric field inside cavity:



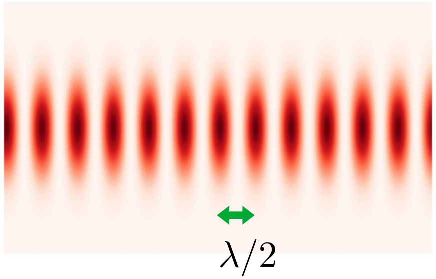
$$\mathcal{E}_{cav} = \mathcal{E}_{cav}^{(+)} + \mathcal{E}_{cav}^{(-)} = \frac{\mathcal{E}_T}{t_2}\mathrm{e}^{ikz} + r_2\mathrm{e}^{-ikz}\frac{\mathcal{E}_T}{t_2}$$

$$|\mathcal{E}_{cav}|^2 = \left|\frac{\mathrm{e}^{ikz} + +r_2\mathrm{e}^{-ikz}}{t_2}\right|^2 \mathcal{T}\,|\mathcal{E}_I|^2$$

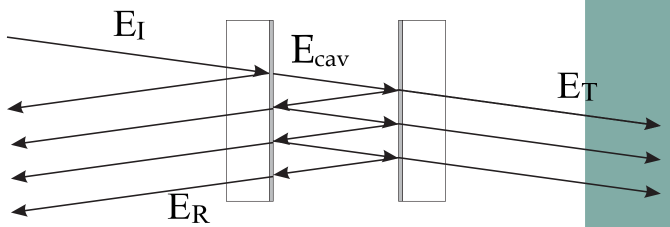
For identical mirrors,

$$\frac{I_{cav}}{I_I} = \frac{|\mathcal{E}_{cav}|^2}{|\mathcal{E}_{in}|^2} = \frac{1 + R + 2\sqrt{R}\cos(2kz)}{1-R}\,\frac{1}{1 + \left(\frac{2\mathcal{F}}{\pi}\right)^2 \sin\left(\frac{\pi\nu}{\Delta_{\mathcal{FSR}}}\right)}$$

Standing wave formed by the counterpropagating beams inside the cavity



Longitudinal modes of optical cavities



Averaging the intensity inside the cavity,

$$\frac{I_{cav}}{I_I} = \frac{1+R}{1-R}\frac{1}{1+\left(\frac{2\mathcal{F}}{\pi}\right)^2\sin\left(\frac{\pi\nu}{\Delta_{\mathcal{FSR}}}\right)}$$

$$\frac{I_{cav}}{I_I} = \frac{\sqrt{1+\left(\frac{2\mathcal{F}}{\pi}\right)^2}}{1+\left(\frac{2\mathcal{F}}{\pi}\right)^2\sin\left(\frac{\pi\nu}{\Delta_{\mathcal{FSR}}}\right)}$$

For high finesse, in resonance,

$$\frac{I_{cav}}{I_I} \simeq \frac{2\mathcal{F}}{\pi}$$

Finesse of cavity:

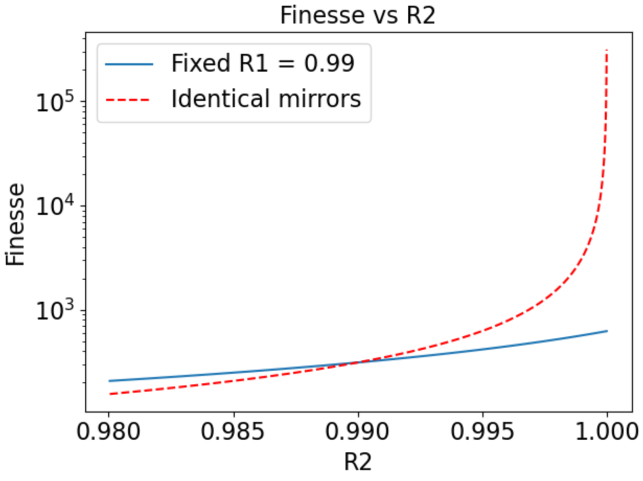
- Frequency sensitivity
- Electric field amplification

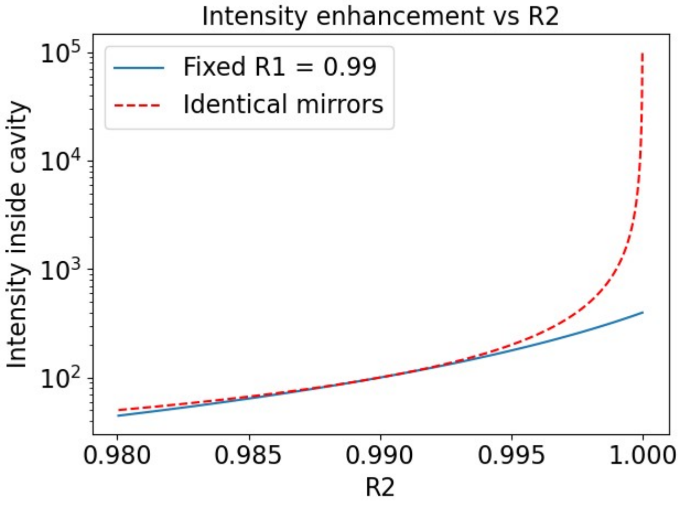
Longitudinal modes of optical cavities

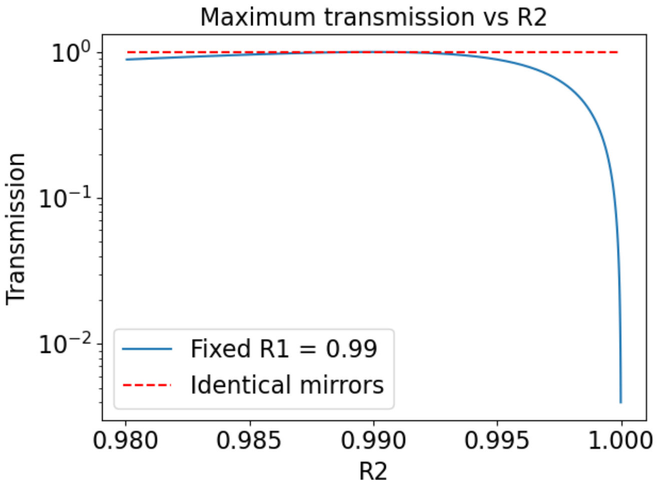
Nonidentical mirrors limit finesse, intensity enhancement, transmission:

$$\mathcal{T}_{cav} = \frac{T_1\,T_2}{(1-\sqrt{R_1 R_2})^2}\,\frac{1}{1+\left(\frac{2\mathcal{F}_{12}}{\pi}\right)^2 \sin^2\left(\frac{\pi\nu}{\Delta_{\mathcal{FSR}}}\right)}$$

$$\mathcal{F}_{12} = \frac{\pi\,(R_1 R_2)^{1/4}}{1-\sqrt{R_1 R_2}}$$







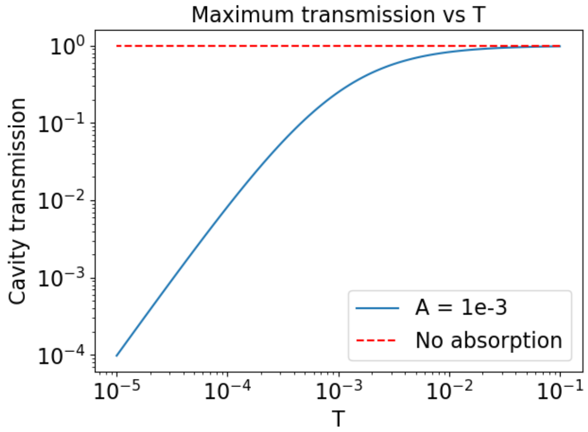
Longitudinal modes of optical cavities

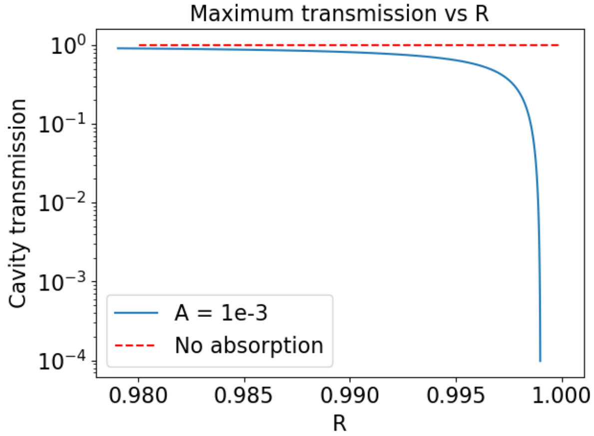
In presence of **losses** (supposing identical mirrors): $R + T + A = 1$

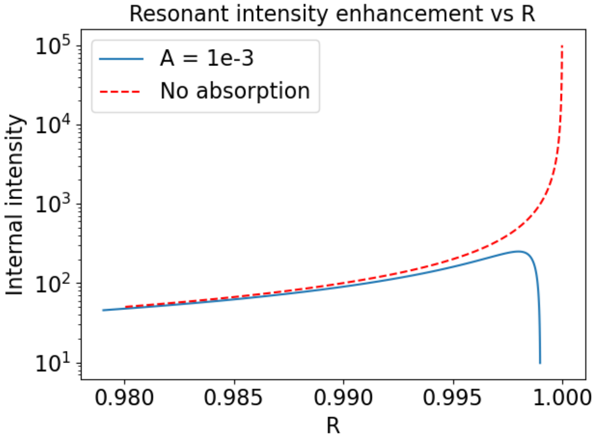
$$\mathcal{T}_{cav} = \frac{T^2}{(1-R)^2 + 4R\sin^2\left(\frac{\pi\nu}{\Delta_{\mathcal{FSR}}}\right)} = \frac{(1-A-R)^2}{(1-R)^2}\frac{1}{1+\left(\frac{2\mathcal{F}}{\pi}\right)^2\sin^2\left(\frac{\pi\nu}{\Delta_{\mathcal{FSR}}}\right)}$$

Maximum mirror reflection: $R_{max} = 1 - A$

Maximum finesse: $\mathcal{F}_{max} = \pi\frac{\sqrt{R_{max}}}{1-R_{max}} = \pi\frac{\sqrt{1-A}}{A}$



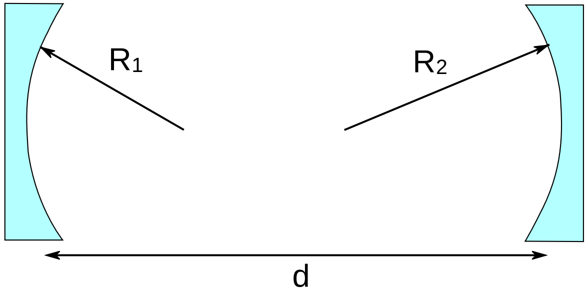


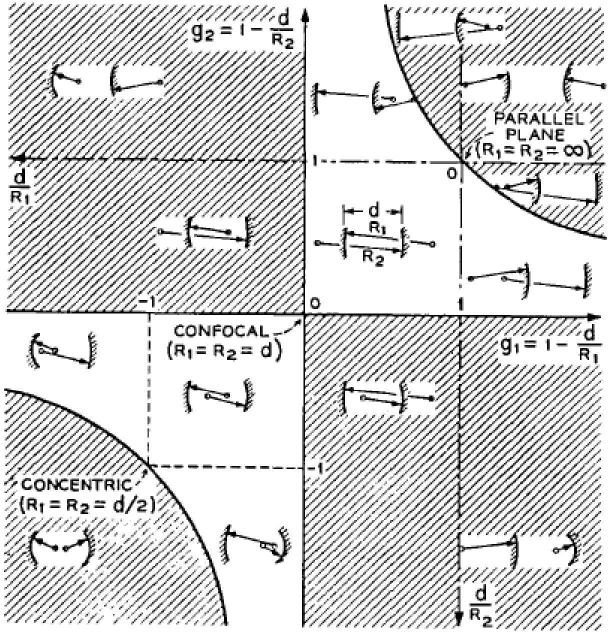


Transverse spatial modes of optical cavities

For transversely finite beams: **diffraction**

Need to consider stability of beam after round trip: curved mirrors





Kogelnik, Herwig, and Tingye Li. "Laser beams and resonators." Applied optics 5.10 (1966): 1550-1567.

Stable region of parameters: admits solution of paraxial equation of light propagation

Transverse spatial modes of optical cavities

Paraxial equation of light propagation

Wave equation: $\frac{1}{c^2}\frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2} - \vec{\nabla}^2 \vec{\mathcal{E}} = 0$, with field $\mathcal{E} = \psi(x,y,z)\,\mathrm{e}^{i\omega t - ikz}$

Slowly varying envelope approximation → paraxial equation: $2ik\frac{\partial \psi}{\partial z} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$

Basis of solutions: **Gaussian beams**

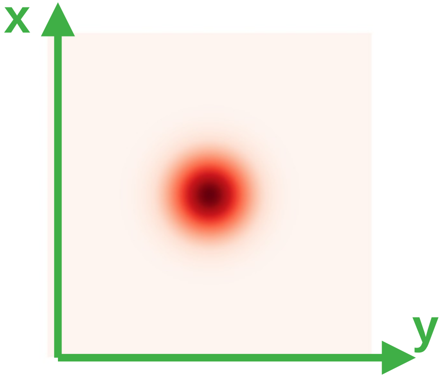
$$\mathcal{E}(x,y,z) = \mathcal{E}_0\, g(x,y)\,\frac{w_0}{w(z)}\,\mathrm{e}^{\frac{-(x^2+y^2)}{w(z)^2}\;-i\left(k\frac{x^2+y^2}{2R(z)} - \phi_G(z)\right)}\,\mathrm{e}^{i\omega t - ikz}$$

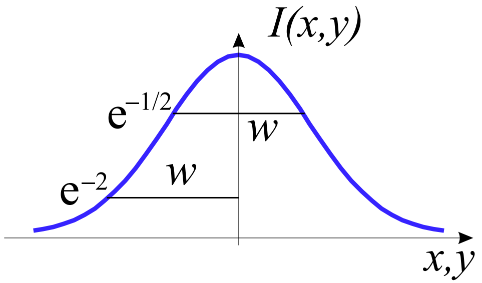
Transverse spatial modes of optical cavities

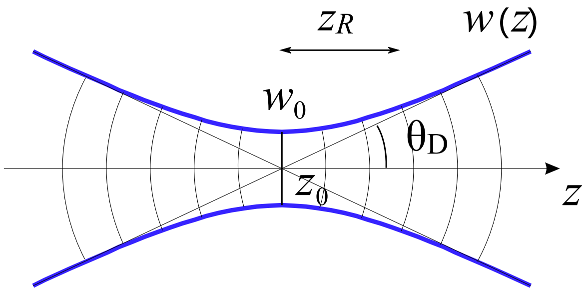
Fundamental solution: **Gaussian beam**

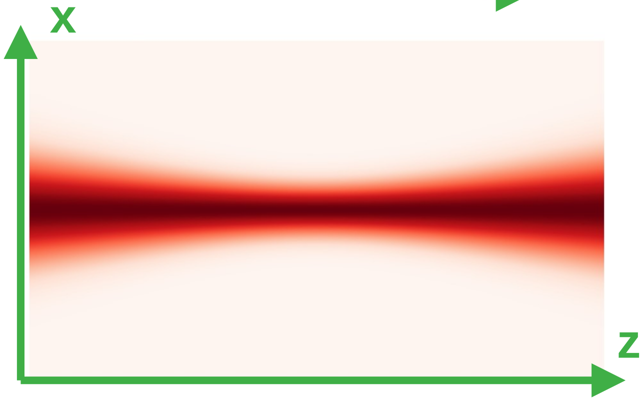
$$\mathcal{E}_{00}(x,y,z) = \mathcal{E}_0\,\frac{w_0}{w(z)}\,\mathrm{e}^{\frac{-(x^2+y^2)}{w(z)^2}\;-i\left(k\frac{x^2+y^2}{2R(z)}-\phi_G(z)\right)}\,\mathrm{e}^{i\omega t-ikz}$$

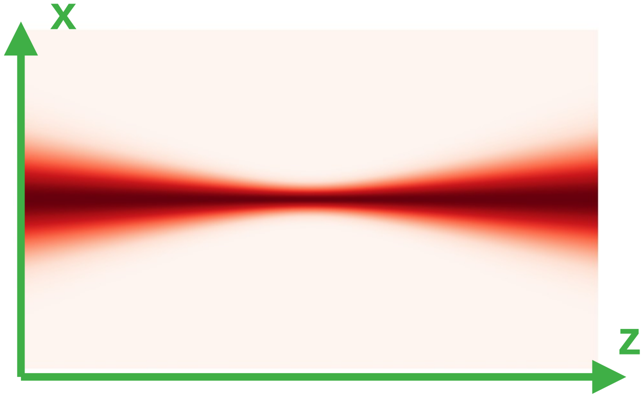
In free space:

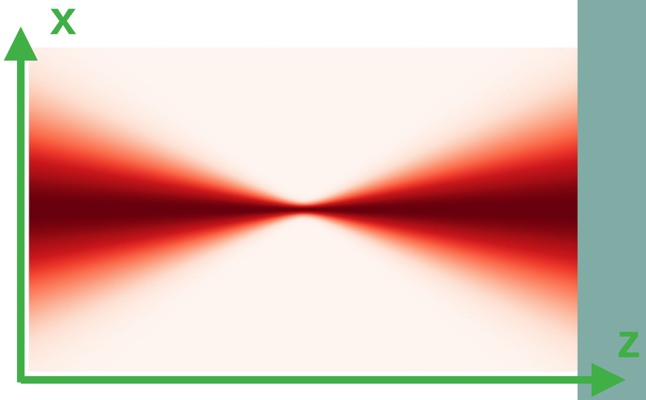






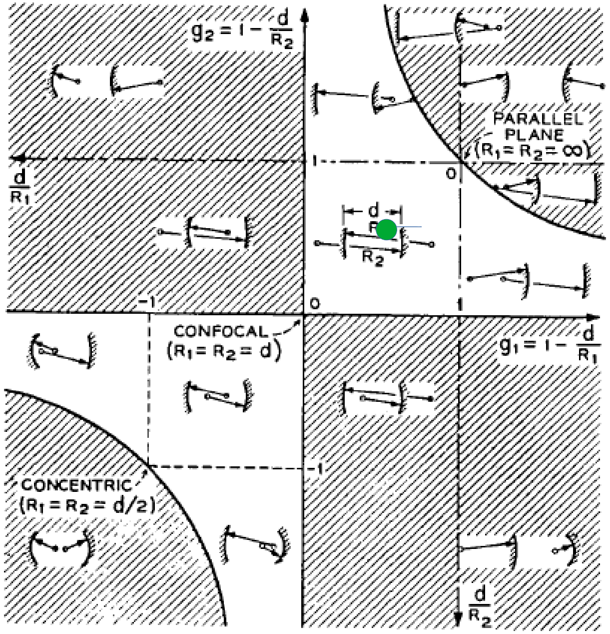






Transverse spatial modes of optical cavities

In resonators: well-defined spatial mode, obtained by matrix formalism



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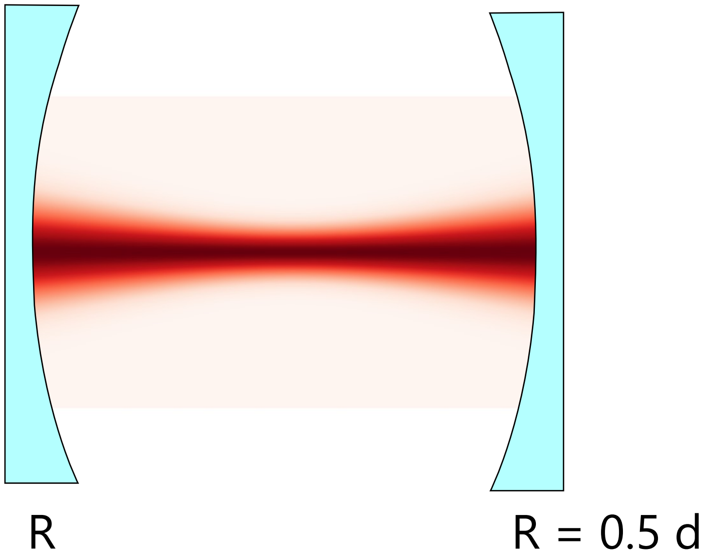

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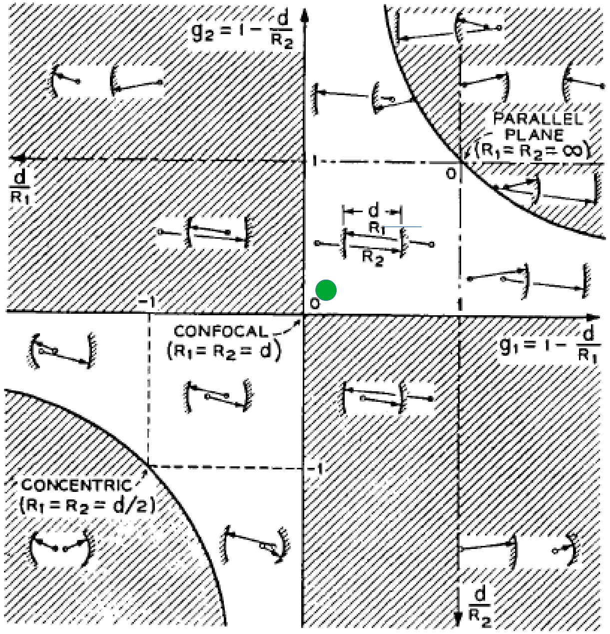

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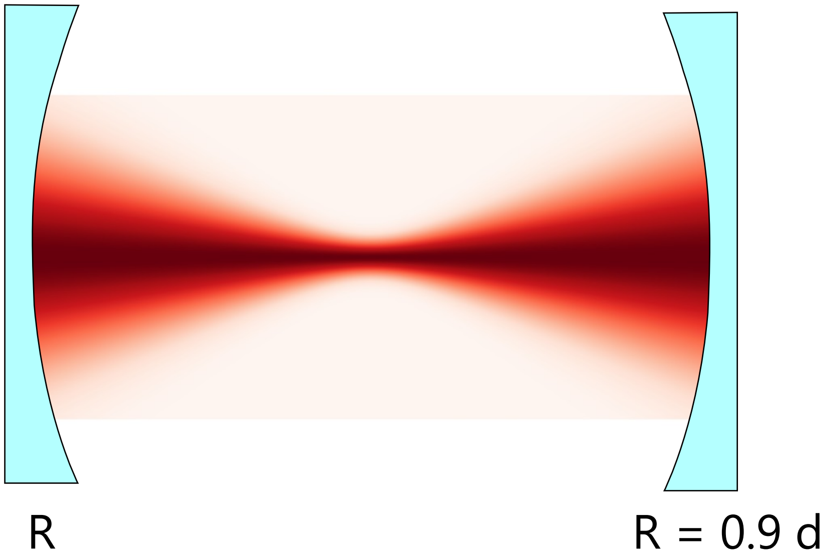

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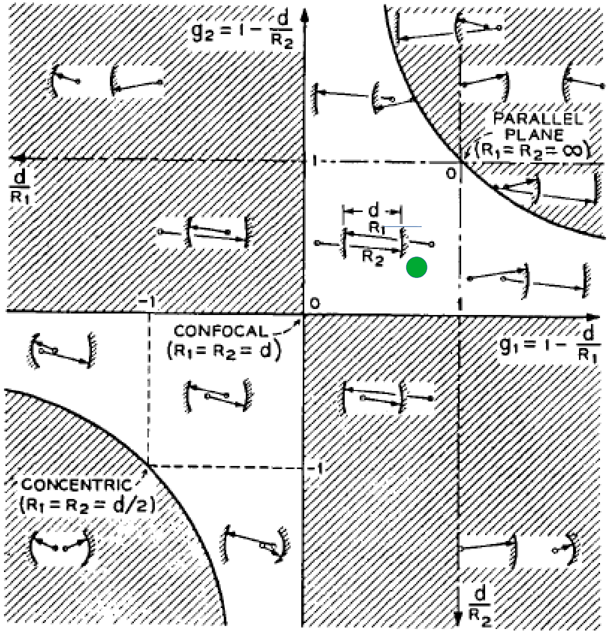

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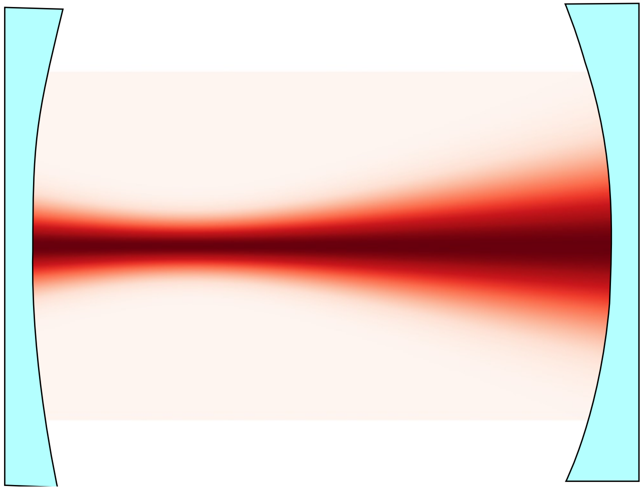
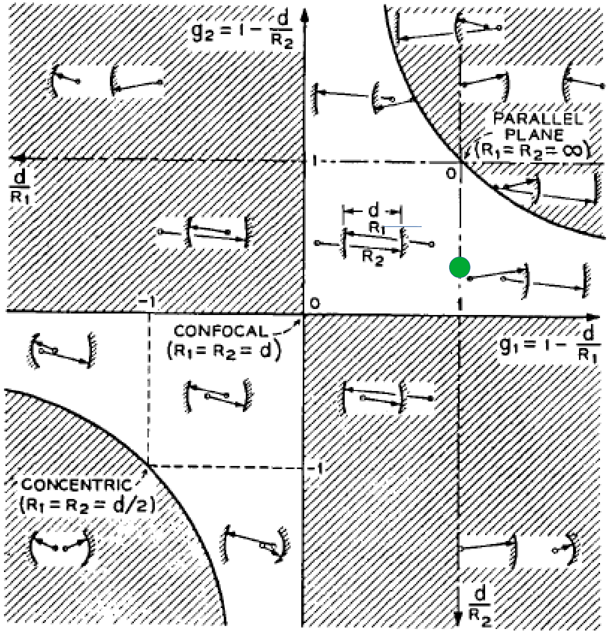


Transverse spatial modes of optical cavities

In resonators: well-defined spatial mode, obtained by matrix formalism



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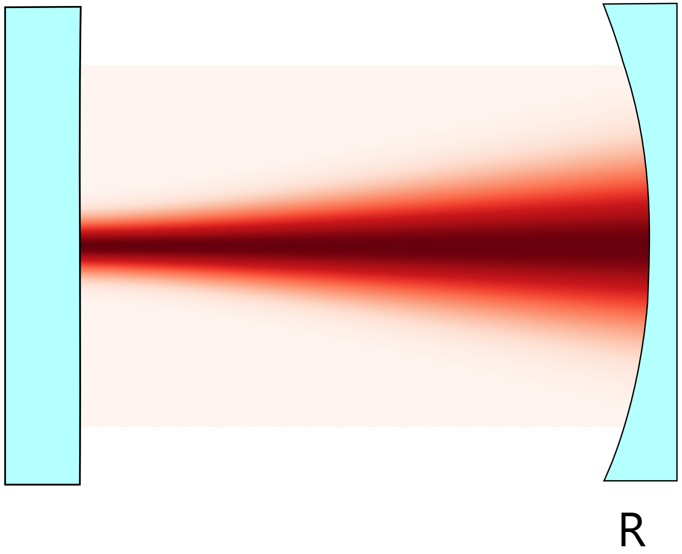
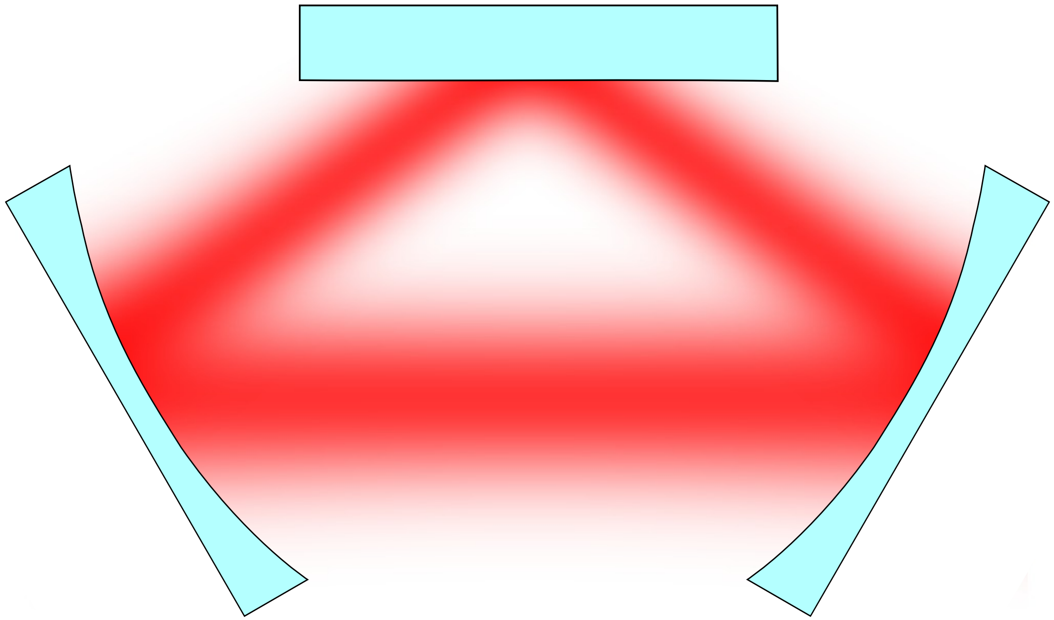


Transverse spatial modes of optical cavities

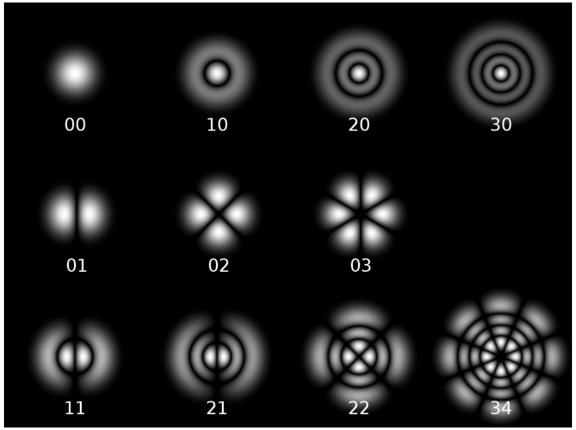
In resonators: well-defined spatial mode, obtained by matrix formalism

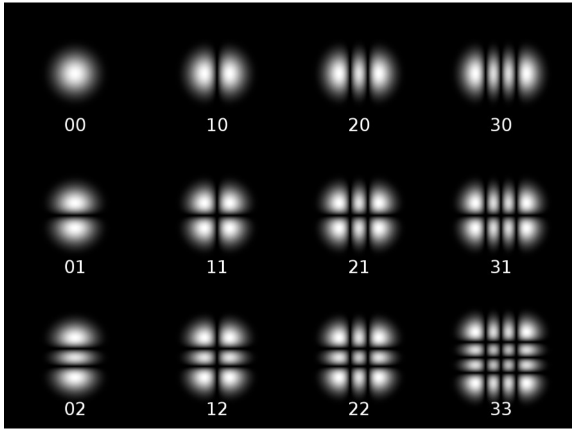


Transverse spatial modes of optical cavities

Basis of solutions: Gaussian modes

$$\mathcal{E}(x,y,z) = \mathcal{E}_0 \boxed{g(x,y)} \frac{w_0}{w(z)}\, \mathrm{e}^{\frac{-(x^2+y^2)}{w(z)^2} - i\left(k\frac{x^2+y^2}{2R(z)} \boxed{-\phi_G(z)}\right)} \mathrm{e}^{i\omega t - ikz}$$





Laguerre-Gauss basis (cylindrical symmetry) OR Hermite-Gauss basis (cartesian symmetry)

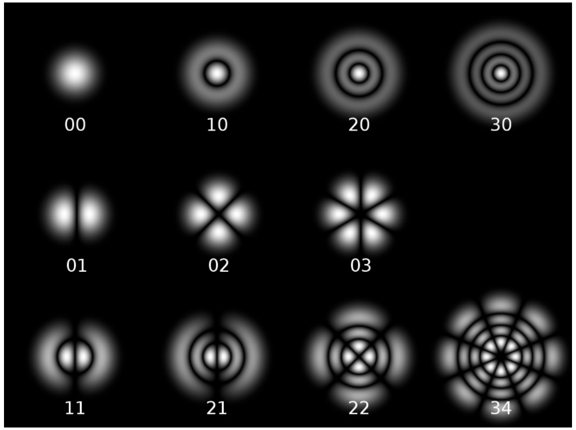
… or other

Transverse spatial modes of optical cavities

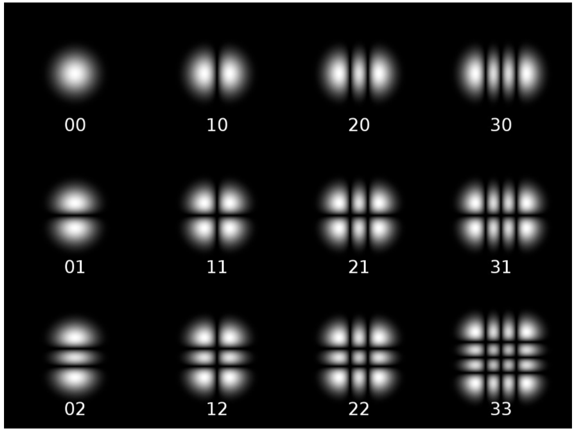
Basis of solutions: Gaussian modes

Gouy phase

$$\mathcal{E}(x,y,z) = \mathcal{E}_0 \boxed{g(x,y)} \frac{w_0}{w(z)} \mathrm{e}^{\frac{-(x^2+y^2)}{w(z)^2} - i\left(k\frac{x^2+y^2}{2R(z)} \boxed{-\phi_G(z)}\right)} \mathrm{e}^{i\omega t - ikz}$$



$$\phi^{LG}_{G,(l,p)}(z) = (|l| + 2p + 1)\arctan\left(\frac{z-z_0}{z_R}\right)$$



$$\phi^{HG}_{G,(m,n)}(z) = (m + n + 1)\arctan\left(\frac{z-z_0}{z_R}\right)$$

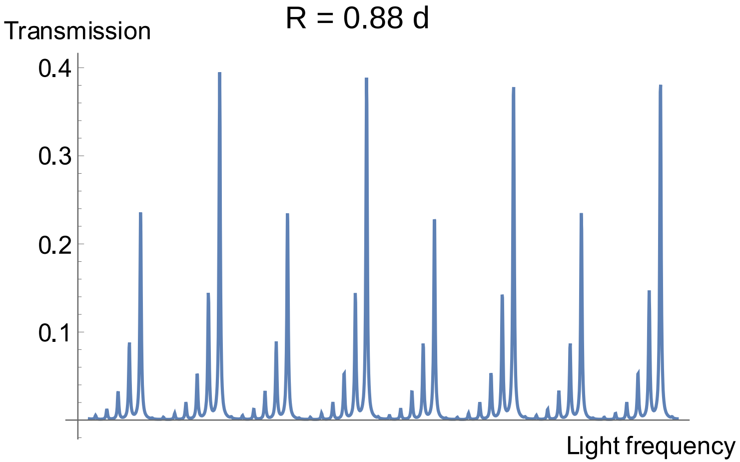
→ Different transverse resonances for the spatial modes:

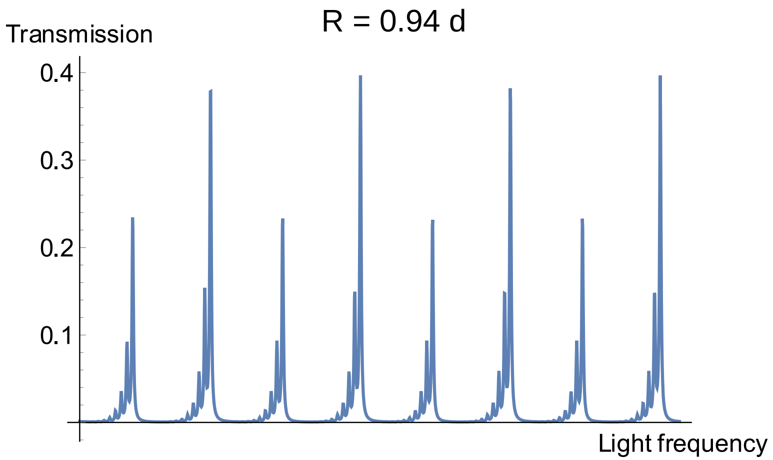
$N = m + n$ or $N = |l| + 2p$

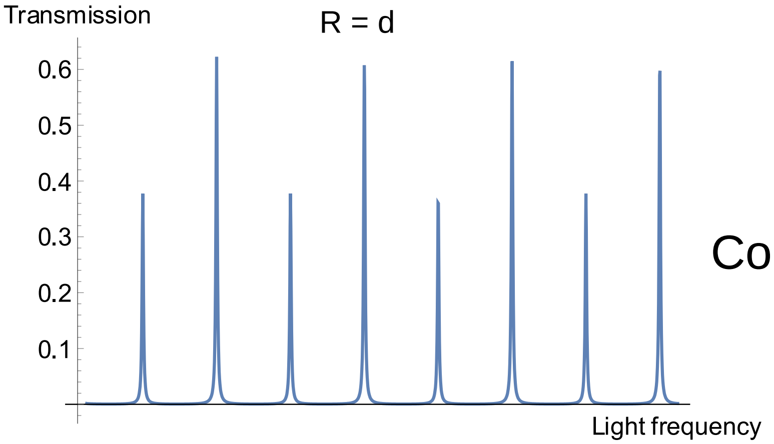
$$\nu_j = \Delta_{\mathcal{FSR}}\left[j + \frac{1+N}{\pi}\arccos\left(1 - \frac{d}{R}\right)\right]$$

for confocal cavity

Transverse spatial modes of optical cavities



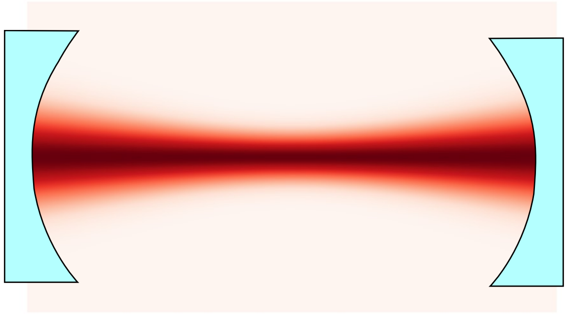




Confocal cavity

Quantization of cavity mode

Cavity modes are a **natural basis** for the quantization of light inside cavity



- j[th] longitudinal mode $\nu_j = j\Delta_{\mathcal{FSR}}$
- Transverse mode (m,n)

$\hat{a}_{j,m,n}$, $\hat{a}^\dagger_{j,m,n}$: annihilation and creation operators associated to mode (j,m,n)

Hamiltonian $\hat{H}_{\text{cav}} = \sum_{j,m,n} \hbar\omega_j \, \hat{a}_{j,m,n} \, \hat{a}^\dagger_{j,m,n}$

$$\left[\hat{a}_{j,m,n} \, , \, \hat{a}^\dagger_{j',m',n'}\right] = \delta_{j,j'}\delta_{m,m'}\delta_{n,n'}$$

Electric field operator at center of cavity: $\hat{\mathrm{E}}(\mathbf{r}_C) = i \sum_{j,m,n} \mathcal{E}^{(1)}_{j,m,n} \left[\hat{a}_{j,m,n} \, \mathrm{e}^{ik_j z} - \hat{a}^\dagger_{j,m,n} \, \mathrm{e}^{-ik_j z}\right]$

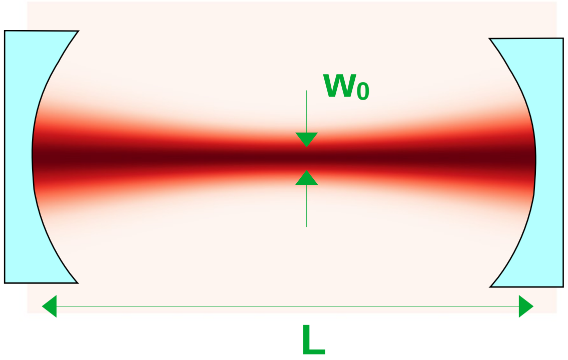
(only one polarization mode)

$\mathcal{E}^{(1)}_{j,m,n} = \sqrt{\dfrac{\hbar\omega_j}{2\epsilon_0 V_{\text{eff}}}}$: single photon electric field

Quantization of cavity mode

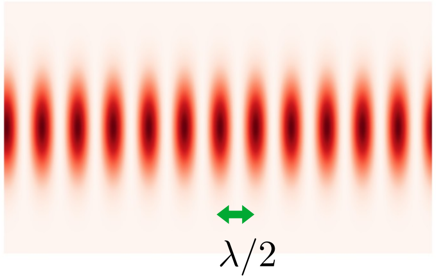
Effective volume of the cavity mode: Purcell definition

$$V_{\mathrm{eff}} = \frac{\int_V \epsilon(\mathbf{r})\,|\mathcal{E}(\mathbf{r})|^2\,\mathrm{d}\mathbf{r}}{\epsilon(\mathbf{r}_C)\,|\mathcal{E}(\mathbf{r}_C)|^2}$$



For the fundamental Gaussian mode, $V_{\mathrm{eff}} = \frac{\pi}{2} w_0^2 L$

BUT: Standing wave formed by incoming and reflected beam:



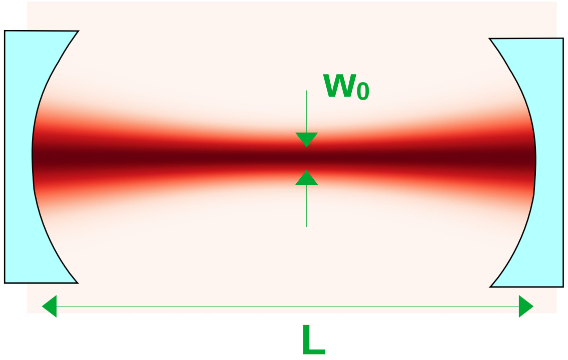
$$\hat{\mathrm{E}}(\mathbf{r}) = i \sum_{j,m,n} \mathcal{E}^{(1)}_{j,m,n}(\mathbf{r}) \left[\hat{a}_{j,m,n}\,\mathrm{e}^{ik_j z} - \hat{a}^{\dagger}_{j,m,n}\,\mathrm{e}^{-ik_j z} \right]$$

$$\mathcal{E}^{(1)}_{j,m,n} = \sqrt{\frac{\hbar\omega_j}{2\epsilon_0 V_{\mathrm{eff}}}}\,\cos^2(k_0 z) \qquad V_{\mathrm{eff}} = \frac{\pi}{4} w_0^2 L$$

Quantization of cavity mode

Effective volume of the cavity mode: Purcell definition

$$V_{\text{eff}} = \frac{\int_V \epsilon(\mathbf{r}) \left|\mathcal{E}(\mathbf{r})\right|^2 \mathrm{d}\mathbf{r}}{\epsilon(\mathbf{r}_C) \left|\mathcal{E}(\mathbf{r}_C)\right|^2}$$



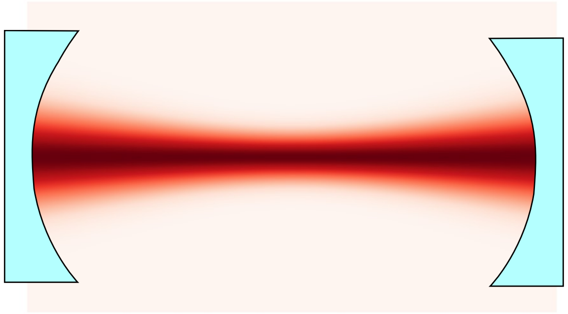
For the fundamental Gaussian mode, $V_{\text{eff}} = \frac{\pi}{2} w_0^2 L$

BUT 2: For lossy cavities: generalized mode volume.

- Lossy channels: Quasi-normal modes
- nontrivial normalization
- Generalized mode volume

P. T. Kristensen, S. Hughes, ACS Photonics 2014, 1, 1, 2–10

Coupling light to the cavity



Impinging light near-resonant with a cavity mode $\hat{a}$: new effective Hamiltonian term **(non-Hermitian!!)**

$$\hat{H}_{\rm in} = i\hbar\,\eta\,\mathrm{e}^{i\omega t}\hat{a}^\dagger - i\hbar\eta^*\mathrm{e}^{-i\omega t}\hat{a}$$

η : Pump rate of the electric field
ω : Frequency of incoming field

The cavity is an open system – Lindbladian:
Jump operator $\hat{a}$, field decay rate κ

$$\hat{H} = \hat{H}_{\rm cav} + \hat{H}_{\rm in}$$

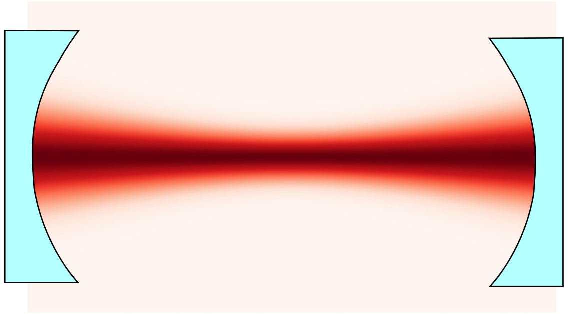
Total Hamiltonian: $\hat{H} = \hat{H}_{\rm cav} + \hat{H}_{\rm in}$

Rotating frame of incoming light: $\mathrm{e}^{-i\omega t}\hat{a} \rightarrow \hat{a}$, $\mathrm{e}^{i\omega t}\hat{a}^\dagger \rightarrow \hat{a}^\dagger$

In the rotating frame, keeping only quasi-resonant mode, $\hat{H} = -\hbar\Delta_C\hat{a}^\dagger\hat{a} + i\hbar\left(\eta\hat{a}^\dagger - \eta^*\hat{a}\right)$

$\Delta_C = \omega - \omega_C$: Cavity detuning

Coupling light to the cavity



Evolution equation for an operator:

$$\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} = \frac{i}{\hbar}\left[\hat{H},\hat{A}\right] + \kappa\left(2\hat{a}^\dagger\hat{A}\hat{a} - \hat{a}^\dagger\hat{a}\hat{A} - \hat{A}\hat{a}^\dagger\hat{a}\right)$$

Evolution equation for the density matrix:

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\frac{i}{\hbar}\left[\hat{H},\hat{A}\right] + \kappa\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}\right)$$

Mean-field approximation: evolution equation for $\alpha = \langle\hat{a}\rangle$ and for the average photon number $\langle\hat{a}^\dagger\hat{a}\rangle$:

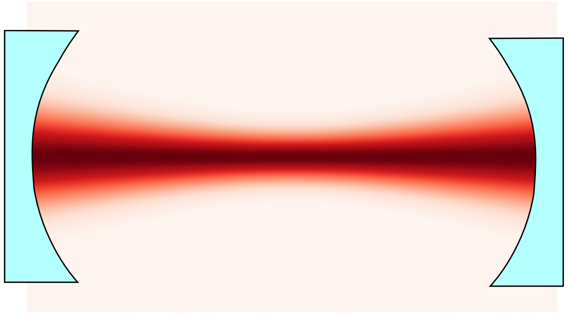
$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = (i\Delta_C - \kappa)\,\alpha + \eta \qquad\qquad \frac{\mathrm{d}\left\langle\hat{a}^\dagger\hat{a}\right\rangle}{\mathrm{d}t} = \eta\left(\langle\hat{a}\rangle + \left\langle\hat{a}^\dagger\right\rangle\right) - 2\kappa\left\langle\hat{a}^\dagger\hat{a}\right\rangle$$

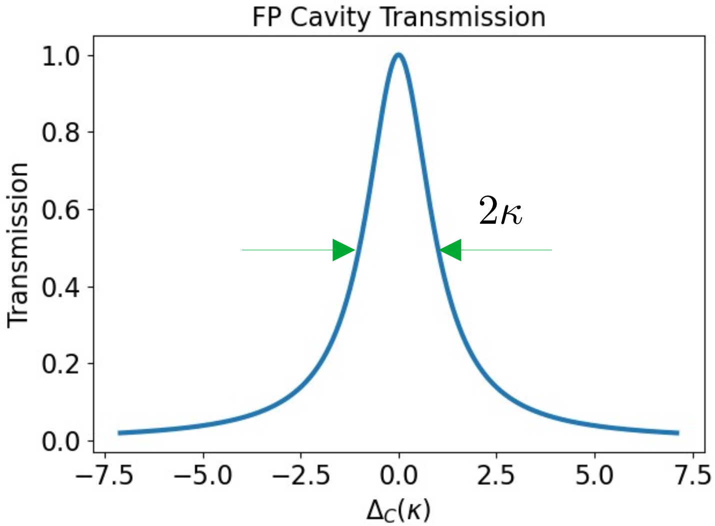
$$\alpha(t) = [\alpha(0) - \alpha(\infty)]\,\mathrm{e}^{(i\Delta_C - \kappa)t} + \alpha(\infty) \quad , \quad \alpha(\infty) = \frac{\eta}{\kappa + i\Delta_C}$$

Coupling light to the cavity

In the mean field approximation,

$$\left\langle \hat{a}\hat{a}^\dagger \right\rangle(\infty) = |\alpha(\infty)|^2 = \frac{|\eta|^2}{\kappa^2 + \Delta_C^2}$$





$$2\kappa = \Delta\omega = 2\pi\Delta\nu = \frac{\pi c}{L\mathcal{F}}$$

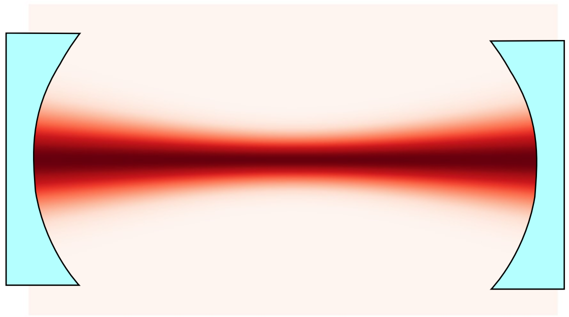
Valid in the limit of high reflectivity and small losses

Cavity linewidth ~ Lorentzian

Coupling light to the cavity

In resonance,

$$\left\langle \hat{a}\hat{a}^{\dagger} \right\rangle (\infty) = |\alpha(\infty)|^2 = \frac{|\eta|^2}{\kappa^2}$$



In the case of negligible losses and for identical good mirrors, there is perfect impedance match with incoming light (no light reflection):

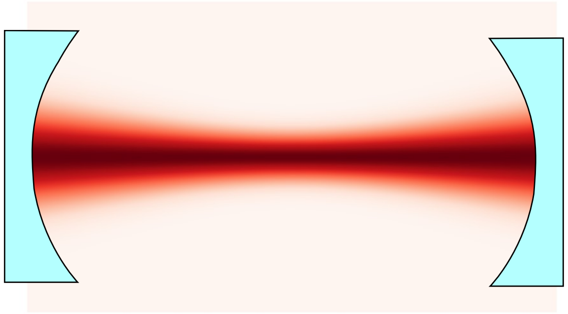
$$P_{\mathrm{inc}} = P_{\mathrm{transm}} = \left| \left(\frac{\mathrm{d}E}{\mathrm{d}t} \right)_{\mathrm{losses}} \right| = 2\kappa E$$

$$P_{\mathrm{inc}} = 2\kappa\hbar\omega_C \left\langle \hat{a}\hat{a}^{\dagger}(\infty) \right\rangle$$

$$P_{\mathrm{inc}} = 2\frac{|\eta|^2 \hbar\omega_C}{\kappa}$$

$\frac{|\eta|^2}{\kappa^2}$: Rate of photons coupled to cavity

Classical limit of light inside cavity



Let's consider now a time-dependent amplitude of the incoming field:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = [i\Delta_C - \kappa]\,\alpha + \eta(t)$$

$$\alpha(t) = \mathrm{e}^{i(\Delta_C-\kappa)t}\alpha(0) + \int_0^t \mathrm{e}^{i(\Delta_C-\kappa)(t-\tau)}\eta(\tau)\,\mathrm{d}\tau$$

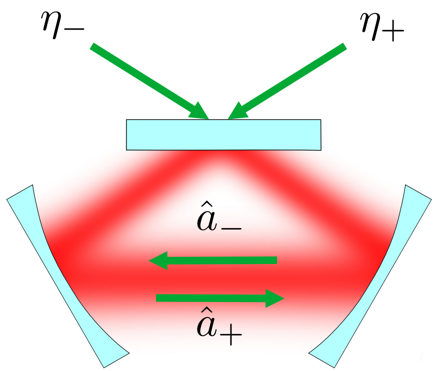
$G(t) = \mathrm{e}^{i(\Delta_C-\kappa)t}$: Green's function that represents a **low-pass filter** (and dephasing) of the incoming amplitude function

Spectral filtering ↔ Temporal filtering

Particularities of ring cavities

Cavities composed of > 2 mirrors present some particularities with respect to linear cavities:

- Two resonant modes of **opposite propagation**
- Break of degenerescency between s and p polarization



r_S, r_P : Reflection coefficients of mirrors with different amplitude and phase

$\rightarrow$ Modes of different finesse and resonant frequency

$\hat{a}_\pm$ are propagating modes, not standing waves $\rightarrow$ uniform coupling to atoms

An atom in free space

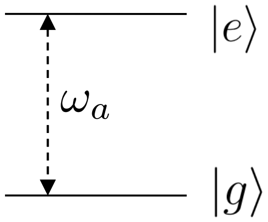
Two-level atom coupled to the vacuum modes of light (Free Space):

Atomic dipolar transition of frequency ω_a + light vacuum

$$\hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g|$$

$$\hat{\sigma}_- = |g\rangle\langle e|$$

$$\hat{\sigma}_+ = (\hat{\sigma}_-)^\dagger$$



Hamiltonian: $$\hat{H} = \hbar\omega_a \frac{\hat{\sigma}_z}{2} + \sum_{\mathbf{k},\epsilon} \hbar\omega_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k},\epsilon}\hat{a}_{\mathbf{k},\epsilon} - i\hbar \sum_{\mathbf{k},\epsilon} (\hat{\sigma}_- + \hat{\sigma}_+)\left(g_{\mathbf{k},\epsilon}\hat{a}_{\mathbf{k},\epsilon} - g^*_{\mathbf{k},\epsilon}\hat{a}^\dagger_{\mathbf{k},\epsilon}\right)$$

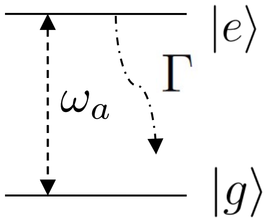
$$g_{\mathbf{k},\epsilon} = \sqrt{\frac{\omega_{\mathbf{k}}}{2\hbar\epsilon_0 V}}\mathbf{d}\cdot\boldsymbol{\epsilon} \qquad \mathbf{d} = \langle g|\hat{\mathbf{d}}|e\rangle$$

The evolution of the atom and light operators can be calculated, and within the Markov approximation (i.e. tracing over the degrees of freedom of the atom) one finds an effective irreversible atomic decay, with decay rate given by

$$\Gamma = 2\pi \sum_{\mathbf{k},\epsilon} |g_{\mathbf{k},\epsilon}|^2\,\delta(\omega_{\mathbf{k}} - \omega_a)$$ Fermi Golden Rule

An atom in free space

The sum can be replaced by an integral and worked out as



$$\Gamma = 2\pi \sum_{\mathbf{k},\epsilon} |g_{\mathbf{k},\epsilon}|^2 \, \delta(\omega_{\mathbf{k}} - \omega_a) \longrightarrow 2\pi \sum_{\epsilon} \frac{V}{(2\pi)^3} \int \mathrm{d}^3\mathbf{k} \, |g_{\mathbf{k},\epsilon}|^2 \, \delta(\omega_{\mathbf{k}} - \omega_a)$$

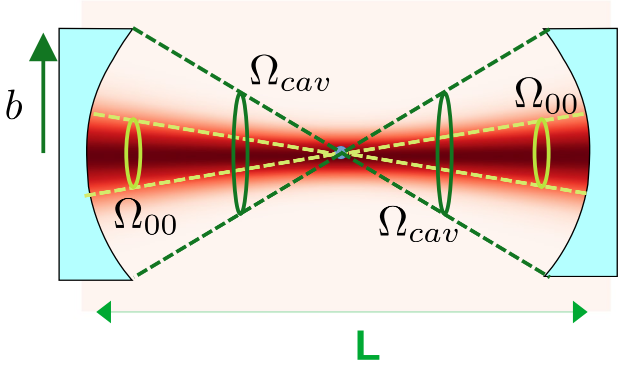
$$\Gamma = \frac{V}{(2\pi)^2} \int_0^\infty \mathrm{d}k \, k^2 \int_0^\pi \mathrm{d}\theta \, \sin\theta \int_0^{2\pi} \mathrm{d}\varphi \, \frac{\omega_{\mathbf{k}}}{2\hbar\epsilon_0 V} \left(\sum_{\epsilon} |\mathbf{d} \cdot \boldsymbol{\epsilon}|^2 \right) \frac{\delta(k - k_a)}{c}$$

$$\Gamma = \frac{d^2}{8\pi^2 \hbar\epsilon_0} \frac{\omega_a^3}{c^3} \int_0^\pi \mathrm{d}\theta \, \sin\theta \int_0^{2\pi} \mathrm{d}\varphi \, \sin^2\theta$$

$$\Gamma = \frac{d^2 \omega_a^3}{3\pi\epsilon_0 \hbar c^3}$$

Spontaneous emission rate for an excited state of a **free-space atom**, coupled to a lower one by electric dipolar coupling

An atom inside a cavity



The presence of the cavity **changes the vacuum** around the atom

Density of modes increased or decreased around the solid angle Ω_{cav} determined by the cavity mirrors

$$\Omega_{cav} = 2\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\arcsin(2b/L)} \mathrm{d}\theta \sin\theta = 4\pi \left[1 - \sqrt{1 - \frac{4b^2}{L^2}}\right]$$

$$\Omega_{cav} \simeq \frac{8\pi b^2}{L^2}$$

- Reduction of spontaneous emission for an atom not in resonance with the cavity (**Purcell effect**):

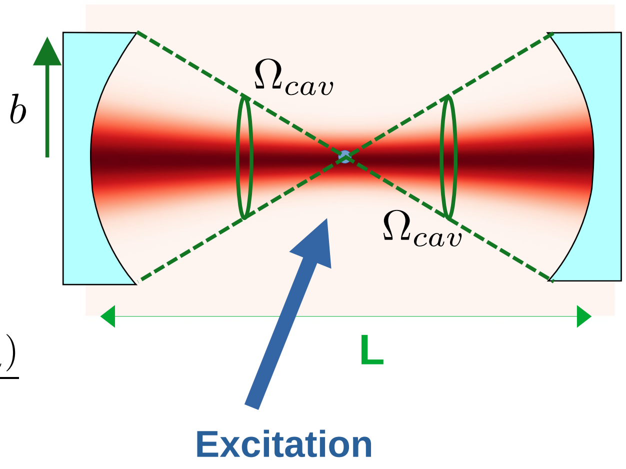
$$\Gamma_{free \notin cav} = 2\pi \sum_{\epsilon} \frac{V}{(2\pi)^3} \int_{\mathcal{S}-\Omega_{cav}} \mathrm{d}^3\mathbf{k} \, |g_{\mathbf{k},\epsilon}|^2 \, \delta(\omega_{\mathbf{k}} - \omega_a)$$

An atom inside a cavity

$$\Gamma_{free \notin cav} = \Gamma - 2\pi \sum_{\epsilon} \frac{V}{(2\pi)^3} \int_{\Omega_{cav}} \mathrm{d}^3\mathbf{k} \, |g_{\mathbf{k},\epsilon}|^2 \, \delta(\omega_{\mathbf{k}} - \omega_a)$$

For small b/L, $\sum_{\epsilon} |\mathbf{d} \cdot \boldsymbol{\epsilon}|^2 \simeq \sin^2(\theta_{in})$ is nearly constant, and then

$$\Gamma_{free \notin cav} = \Gamma - \frac{V}{(2\pi)^2} \int_0^{\infty} \mathrm{d}k \, k^2 \iint_{\Omega_{cav}} \mathrm{d}\varphi \, \mathrm{d}\theta \, \sin\theta \, \frac{\omega_{\mathbf{k}}}{2\hbar\epsilon_0 V} \, \sin^2\theta_{in} \, \frac{\delta(k - k_a)}{c}$$



$$\Gamma_{free \notin cav} = \Gamma \left[1 - \frac{3}{8\pi} \Omega_{cav} \sin^2\theta_{in} \right]$$

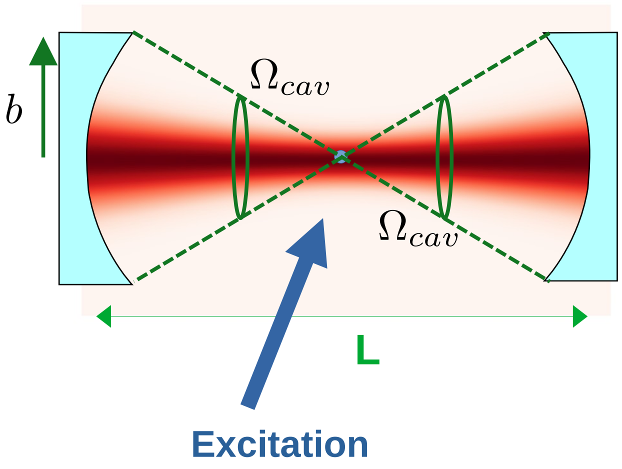
Geometrical reduction of the spontaneous emission rate of an atom: Purcell effect.

An atom inside a cavity

- Increase of spontaneous emission in the cavity mode in resonance:

If confocal cavity:

$$\Gamma_{cav} = 2\pi \sum_{\epsilon} \frac{V}{(2\pi)^3} \int_{\Omega_{cav}} \mathrm{d}^3\mathbf{k} \, |g_{\mathbf{k},\epsilon}|^2_{cav} \, \delta(\omega_{\mathbf{k}} - \omega_a)$$



The coupling to the cavity is modulated with respect to the free space one:

$$\frac{|g_{\mathbf{k},\epsilon}|^2_{cav}}{|g_{\mathbf{k},\epsilon}|^2} = \frac{|\mathcal{E}_{cav}|^2}{|\mathcal{E}_I|^2} = \frac{\sqrt{1+\left(\frac{2\mathcal{F}}{\pi}\right)^2}}{1+\left(\frac{2\mathcal{F}}{\pi}\right)^2 \sin\left(\frac{\pi\nu}{\Delta_{\mathcal{FSR}}}\right)}$$

$$\Gamma_{cav} = \frac{3}{8\pi}\Omega_{cav} \sin^2\theta_{in}\Gamma \frac{\sqrt{1+\left(\frac{2\mathcal{F}}{\pi}\right)^2}}{1+\left(\frac{2\mathcal{F}}{\pi}\right)^2 \sin\left(\frac{\pi\nu}{\Delta_{\mathcal{FSR}}}\right)}$$

In resonance and for high finesse,

$$\Gamma_{cav} = \frac{3\mathcal{F}}{4\pi^2}\Omega_{cav} \sin^2\theta_{in}\,\Gamma$$

An atom inside a cavity

Out of resonance:

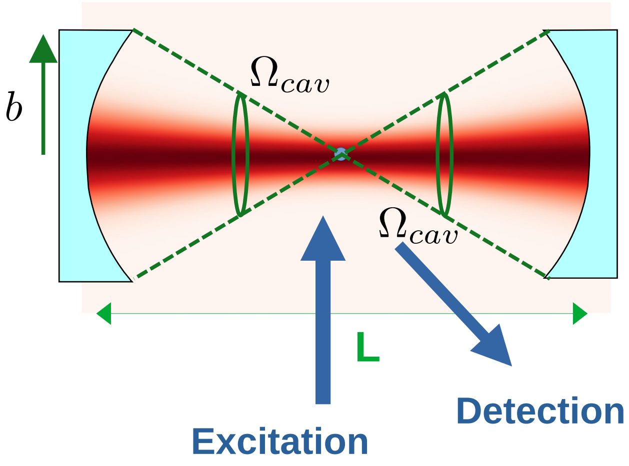
$$\Gamma_{\overline{res}} = \Gamma_{free \notin cav} = \Gamma\left[1 - \frac{3}{8\pi}\Omega_{cav}\sin^2\theta_{in}\right]$$

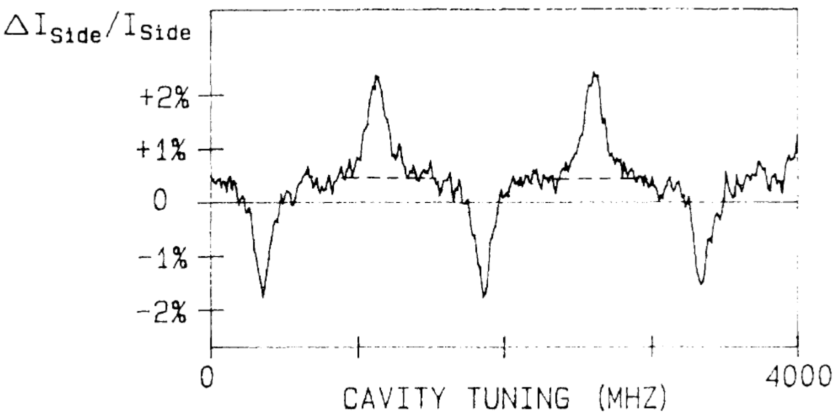
In resonance, confocal:

$$\Gamma_{res} = \Gamma + \Gamma_{cav} = \Gamma\left[1 + \frac{3\mathcal{F}}{4\pi^2}\Omega_{cav}\sin^2\theta_{in}\right]$$

A change in Γ also changes the population of atomic excited state, and the rate of emitted photons:

$$\rho_{ee} \simeq \frac{\left|\mathbf{d}\cdot\mathcal{E}_{exc}\right|^2}{\Gamma^2}$$





Heinzen, D. J., et al. PRL 58, 1320 (1987).

An atom inside a cavity

Out of resonance:

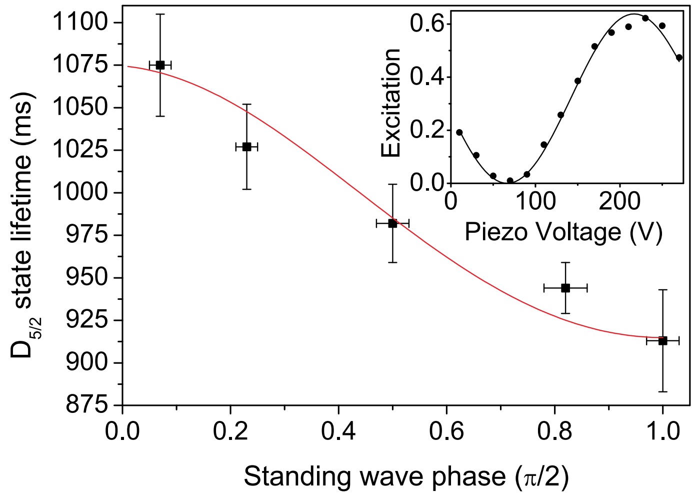
$$\Gamma_{\overline{res}} = \Gamma_{free \notin cav} = \Gamma \left[1 - \frac{3}{8\pi} \Omega_{cav} \sin^2 \theta_{in} \right]$$

In resonance, confocal:

$$\Gamma_{res} = \Gamma + \Gamma_{cav} = \Gamma \left[1 + \frac{3\mathcal{F}}{4\pi^2} \Omega_{cav} \sin^2 \theta_{in} \right]$$

The lifetime of the excited state can also be probed directly:

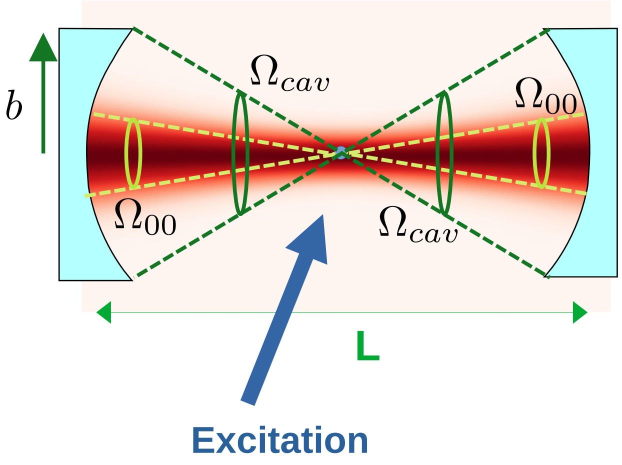
$$\tau_{sp} = \frac{1}{\Gamma_{sp}}$$



Kreuter, A., et al. "Spontaneous emission lifetime of a single trapped Ca+ ion in a high finesse cavity." PRL 92, 203002 (2004).

An atom inside a cavity

- Increase of spontaneous emission in the cavity mode in resonance:



If not confocal, supposing only resonant with fundamental mode: One must define the spherical angle Ω_{00}

→ transverse Fourier transform of beam:

$$(x, y) \longrightarrow (k_x, k_y) \simeq k(\theta_x, \theta_y)$$

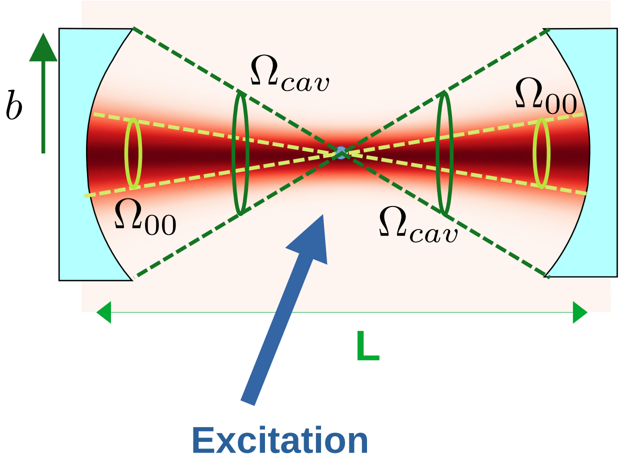
$$\mathcal{E}_{00}(x, y, z) = \mathcal{E}_0 \, \frac{w_0}{w(z)} \, \mathrm{e}^{\frac{-(x^2+y^2)}{w(z)^2}} \, \mathrm{e}^{-i\left(k\frac{x^2+y^2}{2R(z)} - \phi_G(z)\right)} \, \mathrm{e}^{-ikz}$$

$$\mathcal{E}_{00}(k_x, k_y, z) = \int_{\mathbb{R}^2} \mathrm{d}x \, \mathrm{d}y \, \mathrm{e}^{ik_x x + ik_y y} \, \mathcal{E}_{00}(x, y, z) = \mathcal{E}_0 \, \pi w_0^2 \, \mathrm{e}^{\frac{-w_0^2(k_x^2+k_y^2)}{4}} \, \mathrm{e}^{i\frac{\left(k_x^2+k_y^2\right)z}{2k}} \, \mathrm{e}^{-ikz}$$

$$\Omega_{00} = 2 \, \frac{1}{k^2} \int_{\mathbb{R}^2} \mathrm{d}k_x \, \mathrm{d}k_y \, \frac{\left|\mathcal{E}_{00}(k_x, k_y, z)\right|^2}{\left|\mathcal{E}_{00}(0, 0, z)\right|^2} = \frac{4\pi}{k^2 w_0^2}$$

An atom inside a cavity

- Increase of spontaneous emission in the cavity mode in resonance:



Inserting it at Γ_{cav} :

$$\Gamma_{00} = \frac{3}{8\pi}\Omega_{00}\,\sin^2\theta_{in}\,\Gamma\,\frac{\sqrt{1+\left(\frac{2\mathcal{F}}{\pi}\right)^2}}{1+\left(\frac{2\mathcal{F}}{\pi}\right)^2\sin\left(\frac{\pi\nu}{\Delta_{\mathcal{FSR}}}\right)}$$

$$\Gamma_{cav} = \frac{3}{2k^2w_0^2}\,\sin^2\theta_{in}\,\Gamma\,\frac{\sqrt{1+\left(\frac{2\mathcal{F}}{\pi}\right)^2}}{1+\left(\frac{2\mathcal{F}}{\pi}\right)^2\sin\left(\frac{\pi\nu}{\Delta_{\mathcal{FSR}}}\right)}$$

In resonance and for high finesse,

$$\Gamma_{cav} = \frac{3\mathcal{F}}{\pi k^2 w_0^2}\sin^2\theta_{in}\,\Gamma$$

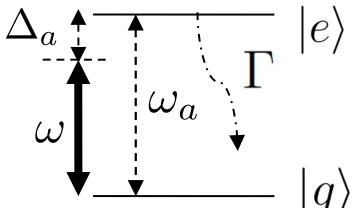
For a dipole perpendicular to the cavity axis,

$$\Upsilon = \frac{\Gamma_{cav}}{\Gamma} = \frac{3\mathcal{F}}{\pi k^2 w_0^2}$$

Purcell factor or **Cooperativity parameter**

Coherent coupling between atom and cavity

For highly cooperative systems, coupling between atom and cavity is **coherent**



- Rabi frequency of coupling between atom and vacuum mode of cavity: $\omega_a \simeq \omega_C \simeq \omega$

$$g = \sqrt{\frac{\omega_C}{2\hbar\epsilon_0 V_{eff}}}\,d = \sqrt{\frac{2\omega_C}{\pi\hbar\epsilon_0 w_0^2 L}}\,d$$

$$g^2 = \frac{2\omega_C}{\pi\hbar\epsilon_0 w_0^2 L} d^2 = 4\Upsilon\kappa\Gamma \quad \longrightarrow \quad \Upsilon = \frac{1}{4}\frac{g^2}{\kappa\Gamma}$$

The cooperativity parameter measures the strength of the coupling with respect to the dissipative mechanisms

Strong coupling regime: $\Upsilon \gg 1$

Memo:

$$V_{\text{eff}} = \frac{\pi}{4} w_0^2 L$$

$$\Gamma = \frac{d^2\omega_a^3}{3\pi\epsilon_0\hbar c^3}$$

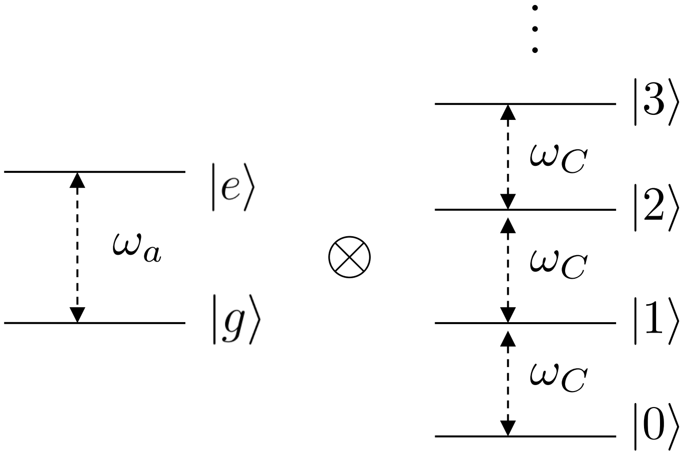
$$\kappa = \frac{\pi c}{2L\mathcal{F}}$$

$$\Upsilon = \frac{3\mathcal{F}}{\pi k^2 w_0^2}$$

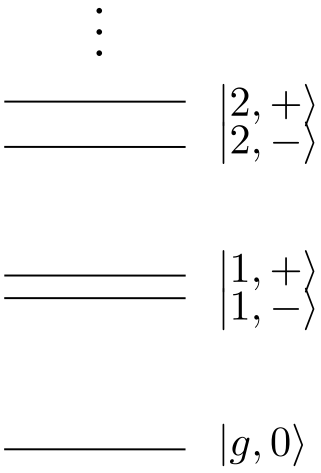
Jaynes-Cummings Hamiltonian

Coupling of 1 atom and cavity mode (in rotating-wave approximation) : Jaynes-Cummings Hamiltonian

$$\hat{H}_{JC} = \hbar\omega_C\, \hat{a}^\dagger \hat{a} + \hbar\omega_a \frac{\hat{\sigma}_z}{2} + \hbar g \left(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_-\right)$$



Eigenstates: coupled atom x light states within the manifolds of constant excitation $\{|e, n-1\rangle\,, |g, n\rangle\}$



$$E_\pm(n) = \hbar\omega_C\left(n - \frac{1}{2}\right) \pm \frac{\hbar}{2}\sqrt{4ng^2 + \Delta_{aC}^2} = \hbar\omega_n^\pm$$

$$|n,-\rangle = \sin(\theta_n)\,|g,n\rangle - \cos(\theta_n)\,|e,n-1\rangle$$

$$|n,+\rangle = \cos(\theta_n)\,|g,n\rangle + \sin(\theta_n)\,|e,n-1\rangle$$

$$\Delta_{aC} = \Delta_C - \Delta_a \qquad \theta_n = \tan^{-1}\left(\frac{g\sqrt{n}}{\Delta_{aC}}\right)$$

For $\Delta_{aC} = 0\,, \quad \omega_n^+ - \omega_n^- = 2\sqrt{n}g$

Jaynes-Cummings Hamiltonian

Vacuum Rabi oscillations:

If $\Delta_{aC} = 0\,,\ |\psi(t=0)\rangle = |e,0\rangle$,

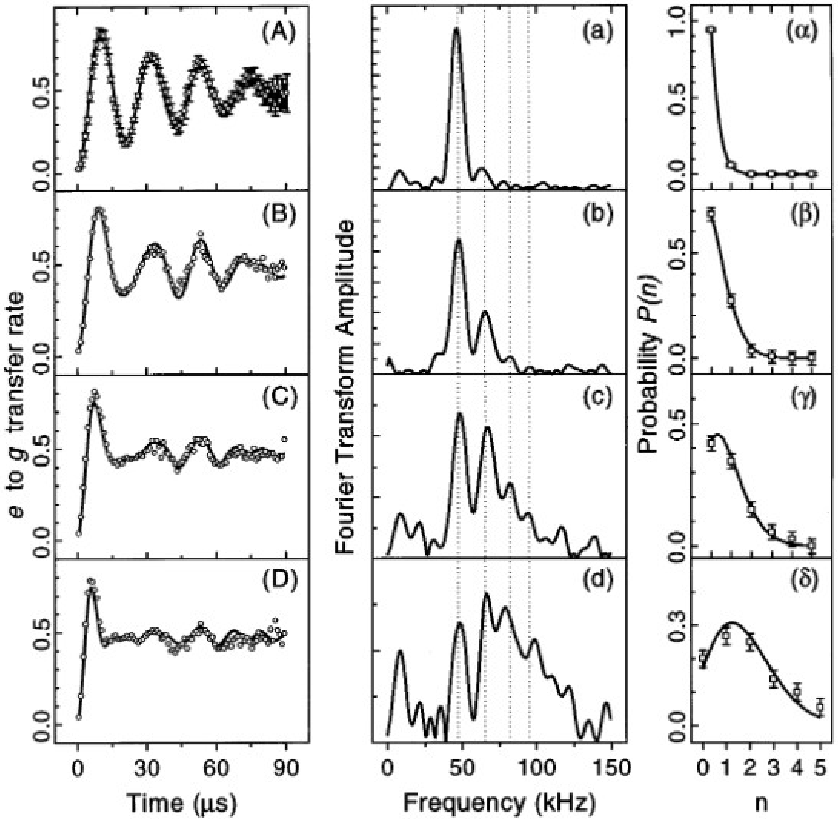
$$|\psi(t)\rangle = \cos(gt)\,|e,0\rangle - i\sin(gt)\,|g,1\rangle$$

Generalized Rabi oscillations:

For a distribution of number of photons within the cavity,

$$|\psi(t=0)\rangle = \sum_n c_n\,|e,n\rangle$$

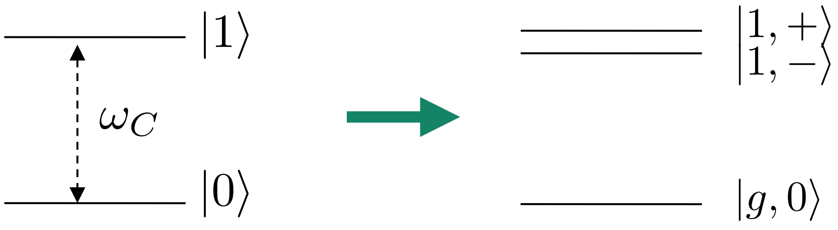
$$|\psi(t)\rangle = \sum_n \left[\cos(\sqrt{n}gt)\,|e,n\rangle - i\sin(\sqrt{n}gt)\,|g,n+1\rangle\right]$$

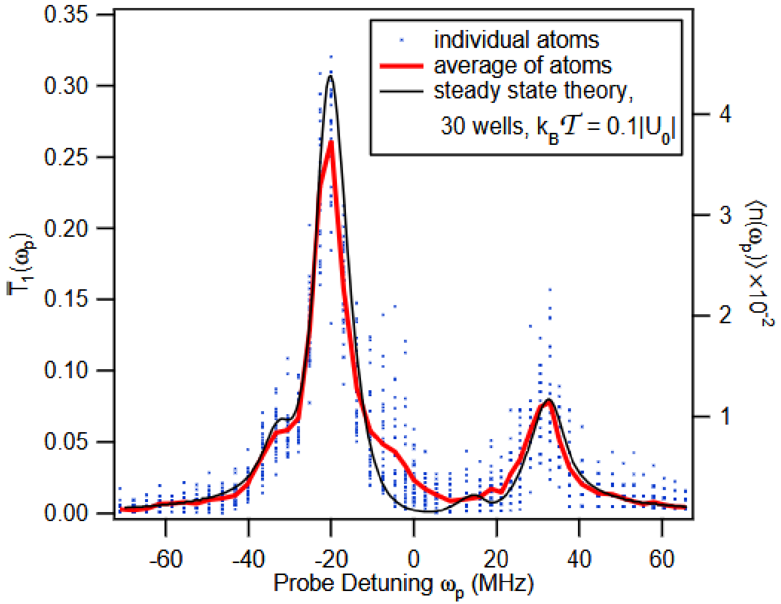


Brune, Michel, et al. "Quantum Rabi oscillation: a direct test of field quantization in a cavity." PRL 76, 1800 (1996).

Vacuum Rabi splitting

Vacuum Rabi splitting of cavity resonance

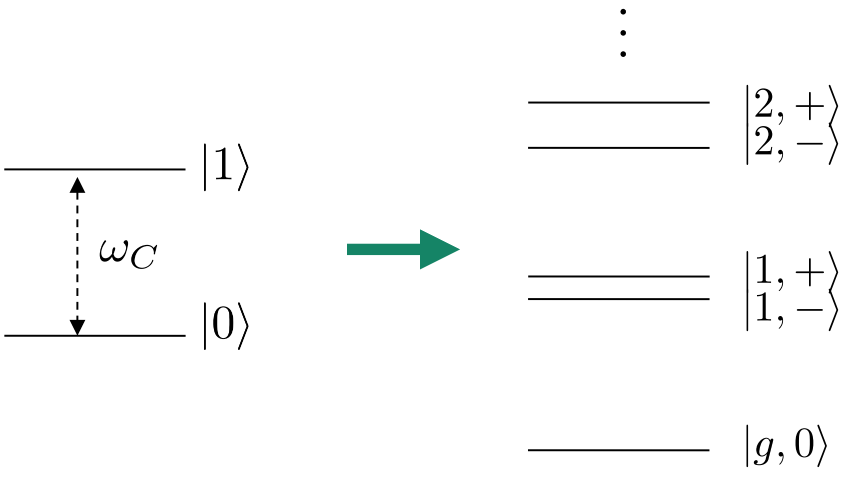


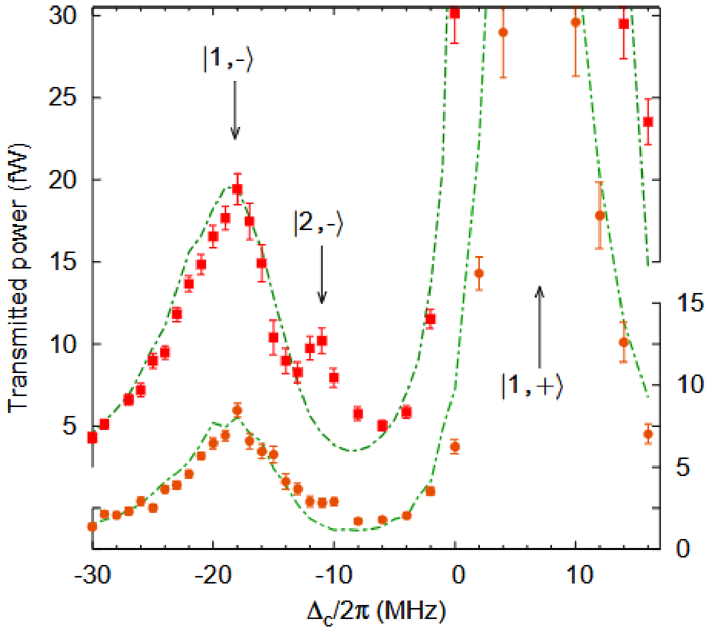


Boca, A., et al. "Observation of the vacuum Rabi spectrum for one trapped atom." PRL 93, 233603 (2004).

Nonlinear Rabi splitting

Vacuum Rabi splitting of cavity resonance





Schuster, Ingrid, et al. "Nonlinear spectroscopy of photons bound to one atom." Nature Physics 4, 382-385 (2008).