Experiments on cold atoms and optical cavities
2 – Collective coupling to the cavity

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Tavis-Cummings Hamiltonian

In the regime of small single-atom cooperativity, $\frac{g^2}{\kappa\Gamma} \ll 1,$ we can search for strong **collective** coupling

Tavis-Cummings Hamiltonian with homogeneous coupling between atoms and cavity mode:

$$\hat{H}_{TC} = -\hbar\Delta_C\,\hat{a}^\dagger\hat{a} - \hbar\Delta_a\sum_{j=1}^{N}\frac{\hat{\sigma}_z^{(j)}}{2} + \hbar g\sum_{j=1}^{N}\left(\hat{a}\hat{\sigma}_+^{(j)} + \hat{a}^\dagger\hat{\sigma}_-^{(j)}\right)$$

This Hamiltonian present a symmetry, the Dicke symmetry via the coupling of light to the collective atomic spin:

$$\hat{\mathbf{J}} = \frac{1}{2}\sum_{j=1}^{N}\hat{\boldsymbol{\sigma}}_j = \frac{1}{2}\sum_{j=1}^{N}\hat{\sigma}_x^{(j)}\boldsymbol{\epsilon}_x + \hat{\sigma}_y^{(j)}\boldsymbol{\epsilon}_y + \hat{\sigma}_z^{(j)}\boldsymbol{\epsilon}_z \qquad \hat{J}_\pm = \sum_{j=1}^{N}\hat{\sigma}_\pm^{(j)}$$

$$\hat{H}_{JC} = -\hbar\Delta_C\,\hat{a}^\dagger\hat{a} - \hbar\Delta_a\hat{J}_z + 2\hbar g\left(\hat{a}\hat{J}_+ + \hat{a}^\dagger\hat{J}_-\right)$$

Tavis-Cummings Hamiltonian

Collective atomic spin:

Eigenstates of $\hat{\mathbf{J}}$ and $\hat{J}_z$: $|J, M, \alpha\rangle$ such as

$$\hat{\mathbf{J}}\,|J, M, \alpha\rangle = J(J+1)\,|J, M, \alpha\rangle$$

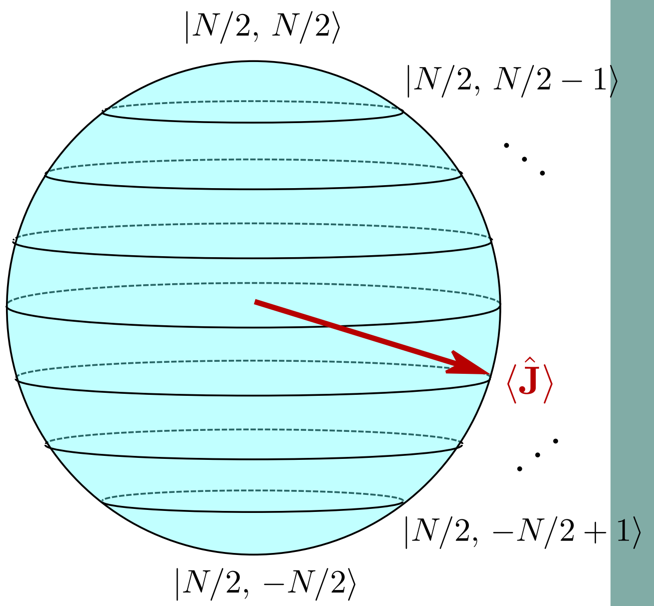
$$\hat{J}_z\,|J, M, \alpha\rangle = M\;|J, M, \alpha\rangle$$

$$\hat{J}_\pm\,|J,\,M,\,\alpha\rangle = \sqrt{J(J+1) - M(M \pm 1)}\,|J,\,M \pm 1,\,\alpha\rangle$$

Minimally excited state: only eigenvalue of $\hat{J}_z$ with eigenvalue $-N/2$

$$|J = N/2, M = -N/2\rangle = \bigotimes_{j=1}^{N} |g_j\rangle$$

The action of $\hat{J}_\pm, \hat{J}_z$ preserves the value of J →
Hamiltonian restricts evolution of the ground state to the
N+1 levels $|J = N/2,\, M\rangle$



Dicke symmetry

Tavis-Cummings Hamiltonian

Dissipation mechanisms: κ, Γ

$$\hat{\mathcal{L}}_\kappa(\hat{A}) = \kappa\left(2\hat{a}^\dagger\hat{A}\hat{a} - \hat{a}^\dagger\hat{a}\hat{A} - \hat{A}\hat{a}^\dagger\hat{a}\right)$$

$$\hat{\mathcal{L}}_\Gamma(\hat{A}) = \frac{\Gamma}{2}\sum_{j=1}^{N}\left(2\hat{\sigma}_+^{(j)}\hat{A}\hat{\sigma}_-^{(j)} - \hat{\sigma}_+^{(j)}\hat{\sigma}_-^{(j)}\hat{A} - \hat{A}\hat{\sigma}_+^{(j)}\hat{\sigma}_-^{(j)}\right)$$

→ breaks Dicke symmetry!

Pumping of cavity: non-Hermitian term

$$\hat{H}_\eta = i\eta\hat{a}^\dagger - i\eta^*\hat{a}$$

$$\hat{H} = \hat{H}_\eta + \hat{H}_{TC}$$

→ **Driven-dissipative system**: pumped cavity + atoms

Tavis-Cummings Hamiltonian

Time evolution of operators: $$\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} = \frac{i}{\hbar}\left[\hat{H}, \hat{A}\right] + \mathcal{L}_\Gamma(\hat{A}) + \mathcal{L}_\kappa(\hat{A})$$

$$\frac{\mathrm{d}\hat{\sigma}_-^{(j)}}{\mathrm{d}t} = \left(i\Delta_a - \frac{\Gamma}{2}\right)\hat{\sigma}_-^{(j)} + ig\,\underline{\hat{a}\,\hat{\sigma}_z^{(j)}}$$

Nonlinear equations

$$\frac{\mathrm{d}\hat{\sigma}_z^{(j)}}{\mathrm{d}t} = 2ig\,\underline{\left(\hat{a}^\dagger\,\hat{\sigma}_-^{(j)} - \hat{a}\,\hat{\sigma}_+^{(j)}\right)} - \Gamma\left(\hat{\mathbb{1}} + \hat{\sigma}_z^{(j)}\right)$$

$$\frac{\mathrm{d}\hat{a}}{\mathrm{d}t} = (i\Delta_c - \kappa)\,\hat{a} - ig\sum_{j=1}^{N}\sigma_-^{(j)} + \eta$$

Does not predict squeezing, quantum correlations

Let's look at **mean-field, stationary solution**: average of operators, neglecting correlations

$$\langle\hat{a}^\dagger\,\hat{\sigma}_-^{(j)}\rangle \simeq \langle\hat{a}^\dagger\rangle\langle\,\hat{\sigma}_-^{(j)}\rangle, \qquad \langle\hat{a}\,\hat{\sigma}_+^{(j)}\rangle \simeq \langle\hat{a}\rangle\langle\,\hat{\sigma}_+^{(j)}\rangle, \qquad \langle\hat{a}\,\hat{\sigma}_z^{(j)}\rangle \simeq \langle\hat{a}\rangle\langle\,\hat{\sigma}_z^{(j)}\rangle$$

$$\alpha = \langle\hat{a}\rangle$$

Tavis-Cummings Hamiltonian

$$\frac{\mathrm{d}\langle\hat{\sigma}_-^{(j)}\rangle}{\mathrm{d}t} = \frac{\mathrm{d}\langle\hat{\sigma}_z^{(j)}\rangle}{\mathrm{d}t} = \frac{\mathrm{d}\langle\hat{a}\rangle}{\mathrm{d}t} = 0 :$$

(i): $$\left(i\Delta_a - \frac{\Gamma}{2}\right)\langle\hat{\sigma}_-^{(j)}\rangle + ig\,\alpha\,\langle\hat{\sigma}_z^{(j)}\rangle = 0 \longrightarrow \boxed{\langle\hat{\sigma}_-^{(j)}\rangle = -\frac{ig\alpha}{i\Delta_a - \frac{\Gamma}{2}}\langle\hat{\sigma}_z^{(j)}\rangle = -\frac{U_\gamma}{g}\alpha\,\langle\hat{\sigma}_z^{(j)}\rangle}$$

$$\langle\hat{\sigma}_+^{(j)}\rangle = \langle\hat{\sigma}_-^{(j)}\rangle^* = -\frac{U_\gamma^*}{g}\alpha^*\,\langle\hat{\sigma}_z^{(j)}\rangle$$

$$U_\gamma \equiv \frac{g^2\left(\Delta_a - i\frac{\Gamma}{2}\right)}{\Delta_a^2 + \frac{\Gamma^2}{4}} \equiv U_0 - i\gamma_0$$

U_0 : single-atom frequency shift

γ_0 : single-atom broadening

(ii): $$2ig\left(\alpha^*\,\langle\hat{\sigma}_-^{(j)}\rangle - \alpha\,\langle\hat{\sigma}_+^{(j)}\rangle\right) - \Gamma\left(1 + \langle\hat{\sigma}_z^{(j)}\rangle\right) = 0$$

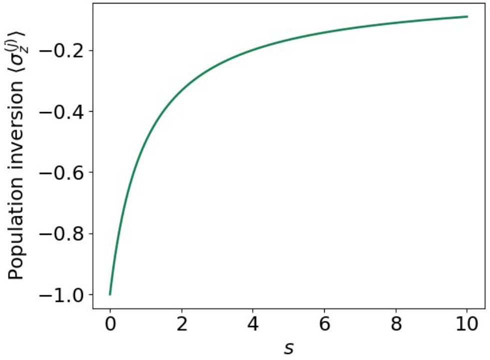
$$-2ig\left(\alpha^*\,\frac{U_\gamma}{g}\alpha - \alpha\,\frac{U_\gamma^*}{g}\alpha^*\right)\langle\hat{\sigma}_z^{(j)}\rangle - \Gamma\left(1 + \langle\hat{\sigma}_z^{(j)}\rangle\right) = 0$$

$$\boxed{\langle\hat{\sigma}_z^{(j)}\rangle = -\frac{1}{1+s}}$$

$$s \equiv \frac{2g^2\,|\alpha|^2}{\Delta_a^2 + \frac{\Gamma^2}{4}}$$: saturation parameter

Tavis-Cummings Hamiltonian

$$\boxed{\langle\hat{\sigma}_z^{(j)}\rangle = -\frac{1}{1+s}}$$



$\lim_{s\to 0}\langle\hat{\sigma}_z^{(j)}\rangle = -1$: linear regime.

All atoms in ground state

$\lim_{s\to\infty}\langle\hat{\sigma}_z^{(j)}\rangle = 0$: saturated atoms.

Equal populations for ground and excited state.

$$\langle\hat{\sigma}_-^{(j)}\rangle = \frac{i}{1+s}\,\frac{g\,\alpha}{i\Delta_a - \frac{\Gamma}{2}} = \frac{1}{1+s}\,\frac{U_\gamma}{g}\,\alpha$$

(iii): $$(i\Delta_c - \kappa)\,\alpha - ig\sum_{j=1}^{N}\langle\sigma_-^{(j)}\rangle + \eta = 0 \quad\longrightarrow\quad \eta - (i\Delta_c - \kappa)\,\alpha = \frac{iNU_\gamma\left(\Delta_a^2 + \frac{\Gamma^2}{4}\right)}{\Delta_a^2 + \frac{\Gamma^2}{4} + 2g^2\left|\alpha\right|^2}\,\alpha$$

$$\boxed{\alpha = \frac{\eta}{\kappa - i\Delta_c + i\frac{NU_\gamma}{1+s(\alpha)}}}$$

Not a solution! A 3rd order polynomial to be solved!

Tavis-Cummings Hamiltonian

In normalized detunings and pumping $\bar{\Delta}_c = \dfrac{\Delta_c}{\kappa}, \quad \bar{\eta} = \dfrac{\eta}{\kappa}, \quad \bar{\Delta}_a = \dfrac{2\Delta_a}{\Gamma}$:

$$\alpha = \frac{\bar{\eta}}{1 - i\bar{\Delta}_c + i\frac{N\Upsilon}{2}\frac{\bar{\Delta}_a - i}{1+\bar{\Delta}_a^2 + s_1|\alpha|^2}}$$

$s_1 = \dfrac{8g^2}{\Gamma^2}$: Saturation parameter of one intracavity photon

We can define a collective cooperative parameter $\Upsilon_N = N\Upsilon = N\dfrac{4g^2}{\kappa\Gamma}$

It determines the impact of the atoms on the amplitude of intracavity light

Tavis-Cummings Hamiltonian

Let's first consider two limits:

- Linear regime: $s \to 0$

$$\alpha = \frac{\eta}{\kappa - i\Delta_c + iNU_\gamma} = \frac{\eta}{\kappa + N\gamma_0 - i\left(\Delta_c - NU_0\right)}$$

$$\alpha = \frac{\eta}{\left[\kappa + \frac{Ng^2}{\Delta_a^2 + \frac{\Gamma^2}{4}}\frac{\Gamma}{2}\right] - i\left[\Delta_c - \frac{Ng^2}{\Delta_a^2 + \frac{\Gamma^2}{4}}\Delta_a\right]}$$

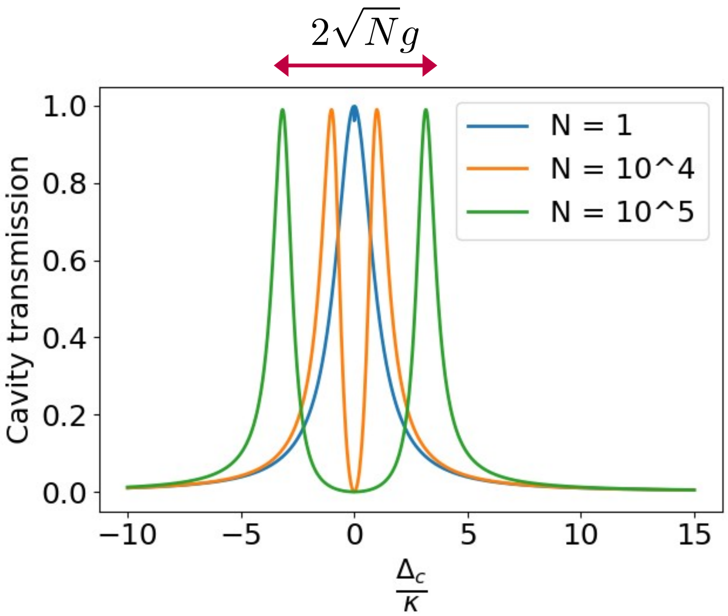
In normalized detunings,

$$\alpha = \frac{\bar{\eta}}{\left(1 + \frac{\Upsilon_N}{2}\frac{1}{1+\bar{\Delta}_a^2}\right) - i\left(\bar{\Delta}_c - \frac{\Upsilon_N}{2}\frac{\bar{\Delta}_a}{1+\bar{\Delta}_a^2}\right)}$$

$|\alpha|^2$

Tavis-Cummings Hamiltonian

For cavity-atom resonance, $\Delta_c = \Delta_a$, we find the normal mode splitting of the cavity resonance:



In bad cavity limit $\Gamma \ll \kappa$, $\alpha \simeq \dfrac{\eta}{i\left[\Delta_c - \frac{Ng^2}{\Delta_c}\right] - \kappa}$

Resonances: maxima of $|\alpha|^2$:

$$2\Delta_c = \pm\sqrt{N}g$$

Normal mode splitting indicates strong collective coupling regime

For the graphic above:

$$\Delta_c = \Delta_a \, , \; \frac{\Gamma}{\kappa} = 0.01 \, , \; \frac{g}{\kappa} = 0.01 \;\; \longrightarrow \; \Upsilon = 0.04$$

Tavis-Cummings Hamiltonian

- Saturated regime: $s \to \infty$

$$\alpha = \frac{\eta}{\kappa - i\Delta_c + i\frac{NU_\gamma}{1+s(\alpha)}} \quad \xrightarrow[s\to\infty]{} \quad \frac{\eta}{\kappa - i\Delta_c}$$

We recover back the empty cavity regime: the high intensity bleaches the atoms. They become transparent to light when fully saturated. Indeed,

$$\langle \hat{\sigma}_-^{(j)} \rangle = \frac{1}{1+s}\,\frac{U_\gamma}{g}\,\alpha \quad \xrightarrow[s\to\infty]{} \quad 0$$

The atomic dipole moment goes to zero in the limit of high saturation

$|\alpha|^2$

Tavis-Cummings Hamiltonian

The cavity resonances can be interpreted as a condition of zero phase accumulated by the light after a round trip:

$$\arg(\alpha) = \arg(\eta) \quad \longrightarrow \quad \Delta_c = \frac{NU_\gamma}{1+s(\alpha)} = \frac{Ng^2\Delta_a}{\Delta_a^2 + \frac{\Gamma^2}{4} + 2g^2|\alpha|^2}$$

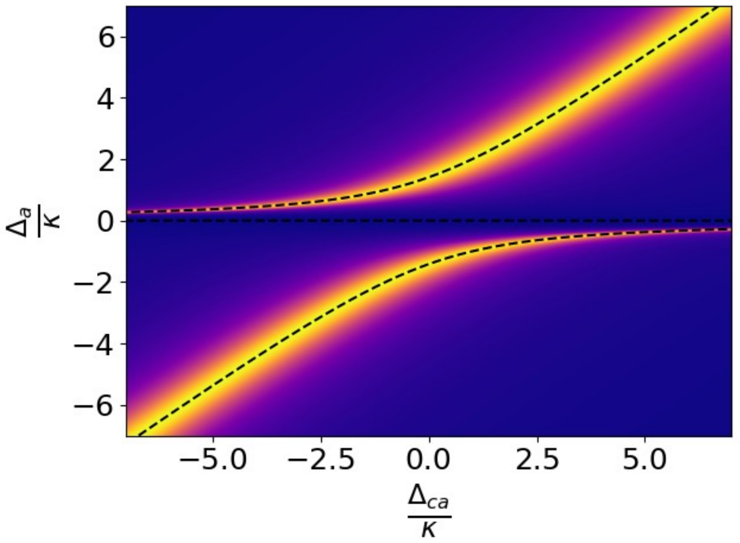
It is common to show this condition as a function of the cavity-atom detuning $\Delta_{ca} = \omega_c - \omega_a = \Delta_a - \Delta_c$:

$$\Delta_{ca} = \Delta_a \left[1 - \frac{Ng^2}{\Delta_a^2 + \frac{\Gamma^2}{4} + 2g^2|\alpha|^2}\right]$$

Three solutions, one not visible due to resonant absorption

$|\alpha|^2$ versus detunings

$|\alpha|^2 \to 0$



Graphic parameters:

$$\frac{\Gamma}{\kappa} = 0.01\,,\ \frac{g}{\kappa} = 0.01\,,\ N = 10^5 \ \longrightarrow\ \Upsilon = 0.04\,,\ \Upsilon_N = 4.10^3$$

Tavis-Cummings Hamiltonian

- Intermediate regime of saturation: Emergence of **bistability**

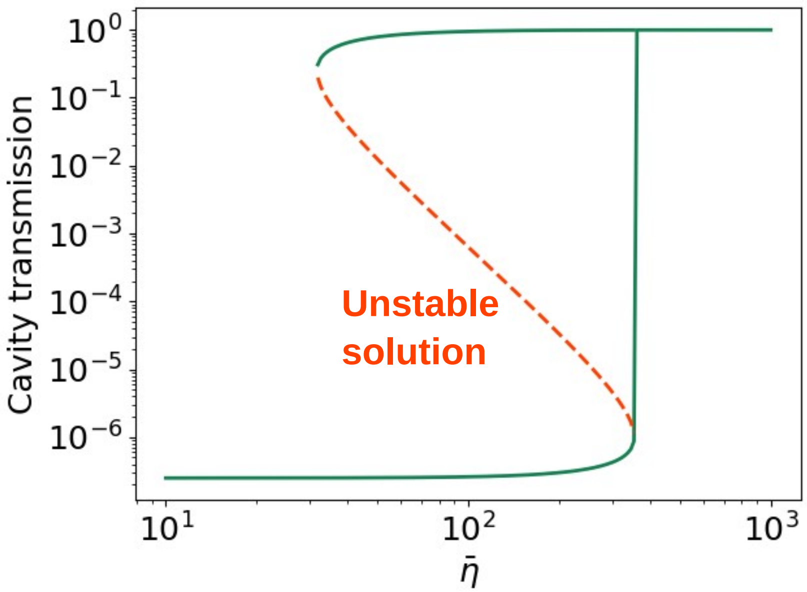
Up to 3 solutions of

$$\alpha = \frac{\bar{\eta}}{1 - i\bar{\Delta}_c + i\frac{N\Upsilon}{2}\frac{\bar{\Delta}_a - i}{1+\bar{\Delta}_a^2 + s_1|\alpha|^2}}$$

Transmission: $\frac{|\kappa\alpha|^2}{|\eta|^2}$

Saturation of atomic resonance entails bistability!

Resonant transmission



For the graphic,

$$\frac{\Gamma}{\kappa} = 0.01\,,\ \frac{g}{\kappa} = 0.01\,,\ N = 10^5$$

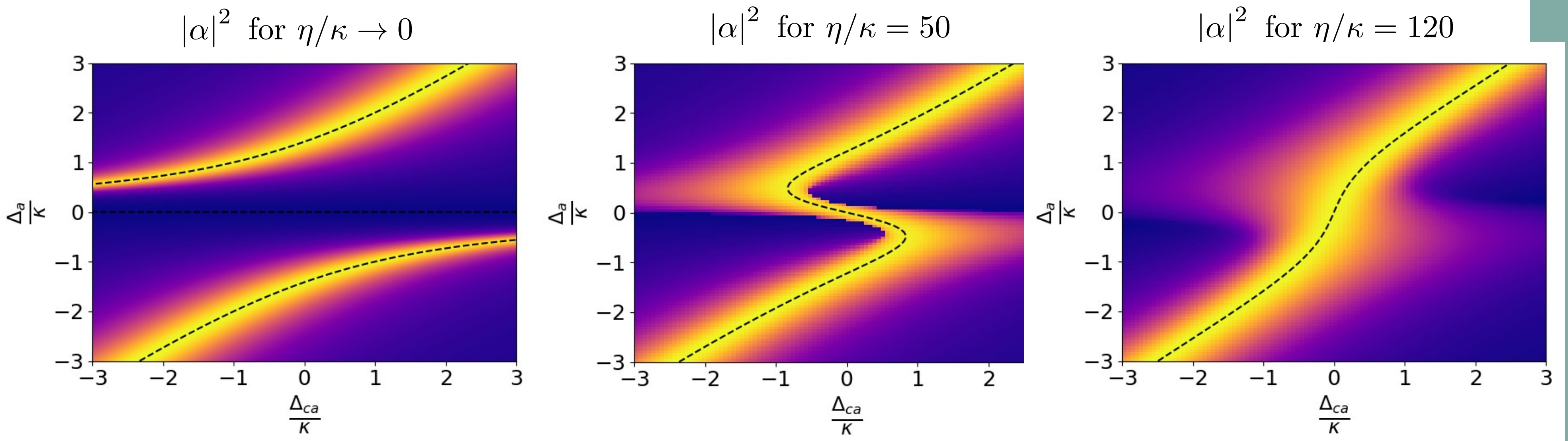
$$\Upsilon = 0.04\,,\ \Upsilon_N = 4.10^3$$

$$\Delta_a = \Delta_c = 0$$

$|\alpha|^2$

Tavis-Cummings Hamiltonian

The cavity resonances get modified by the saturation of the atomic medium



Graphic parameters:

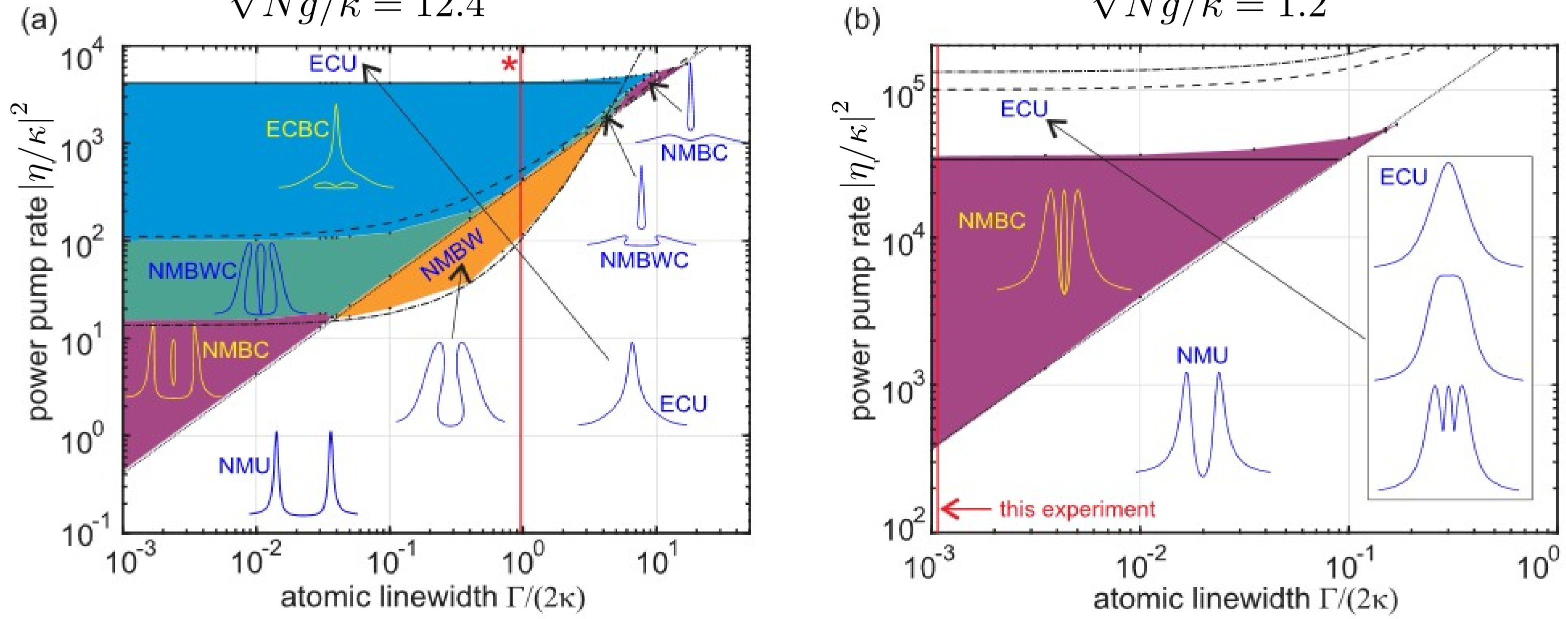
$$\frac{\Gamma}{\kappa} = 0.01\,,\ \frac{g}{\kappa} = 0.01\,,\ N = 10^5\ \longrightarrow\ \Upsilon = 0.04\,,\ \Upsilon_N = 4.10^3$$

Tavis-Cummings Hamiltonian

- Intermediate regime of saturation:

Phase diagram that depends on $\dfrac{\Gamma}{\kappa}, \dfrac{\sqrt{N}g}{\kappa}, \dfrac{\eta}{\kappa}$

$\sqrt{N}g/\kappa = 12.4$ $\qquad\qquad$ $\sqrt{N}g/\kappa = 1.2$



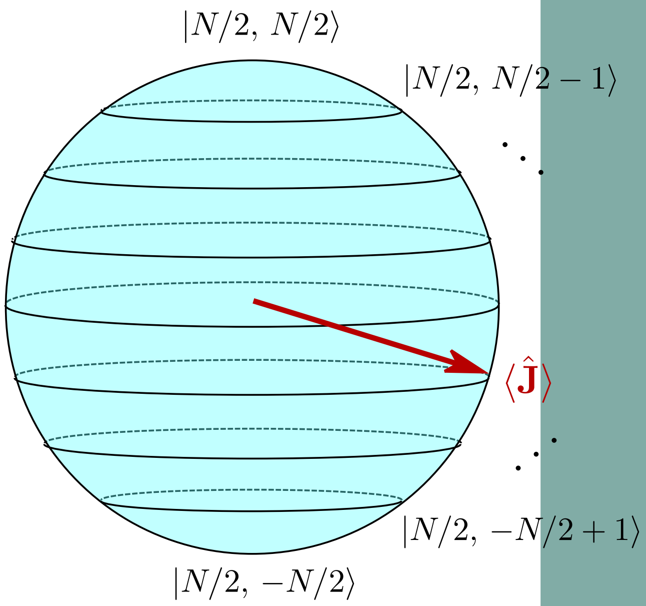
D Rivero et al., 2023. New J. Phys. 25, 093053

Tavis-Cummings Hamiltonian

If decay to free space can be neglected, then Dicke symmetry is preserved

$$\hat{H} = -\hbar\Delta_C\, \hat{a}^\dagger \hat{a} - \hbar\Delta_a \frac{\hat{J}_z}{2} + \hbar g \left(\hat{a}\hat{J}_+ + \hat{a}^\dagger \hat{J}_-\right) + i\hbar\left(\eta\hat{a}^\dagger - \eta^*\hat{a}\right)$$

$$\hat{\mathcal{L}}_\kappa(\hat{A}) = \kappa\left(2\hat{a}^\dagger\hat{A}\hat{a} - \hat{a}^\dagger\hat{a}\hat{A} - \hat{A}\hat{a}^\dagger\hat{a}\right)$$



Mean-field equations in resonance $(\Delta_a = \Delta_c = 0)$:

$$-\kappa\alpha - ig\langle\hat{J}_-\rangle + \eta = 0$$

$$2ig\left[\alpha^*\langle\hat{J}_-\rangle - \alpha\langle\hat{J}_+\rangle\right] = 0$$

$$ig\,\alpha\,\langle\hat{J}_z\rangle = 0$$

$$\downarrow$$

$$\alpha = 0 \text{ ou } \langle\hat{J}_z\rangle = 0$$

Tavis-Cummings Hamiltonian

$$\alpha = 0 : \quad \langle \hat{J}_- \rangle = -i\frac{\eta}{g}$$

Superradiant interaction with light mode from cavity

Atoms absorb incoming light, letting no transmission

Limit to the result: $\langle \hat{J}_- \rangle \leq \dfrac{N}{2} \quad \longrightarrow \quad \eta \leq \dfrac{gN}{2}$

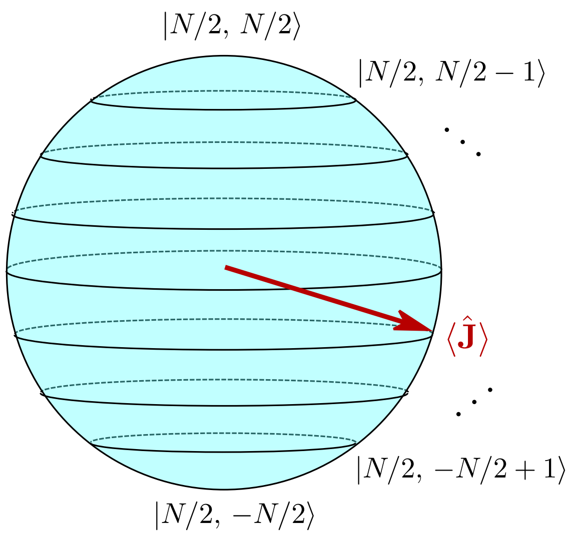
For higher pumping, intracavity field cannot be zero, atomic population saturated:

$$\langle \hat{J}_z \rangle = 0$$

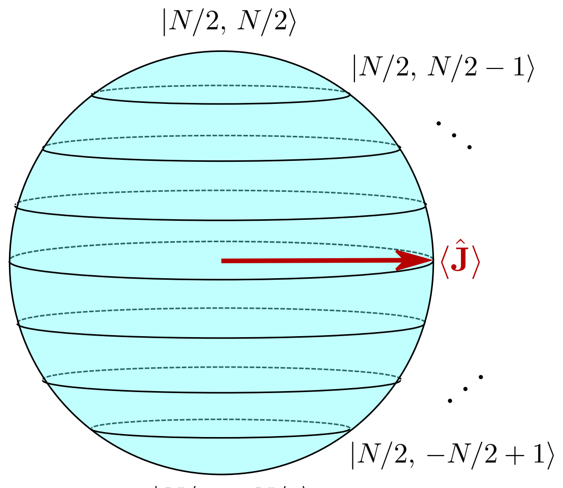
Atomic dipole in the equator of the Bloch sphere

No bistability

$\alpha = 0 :$



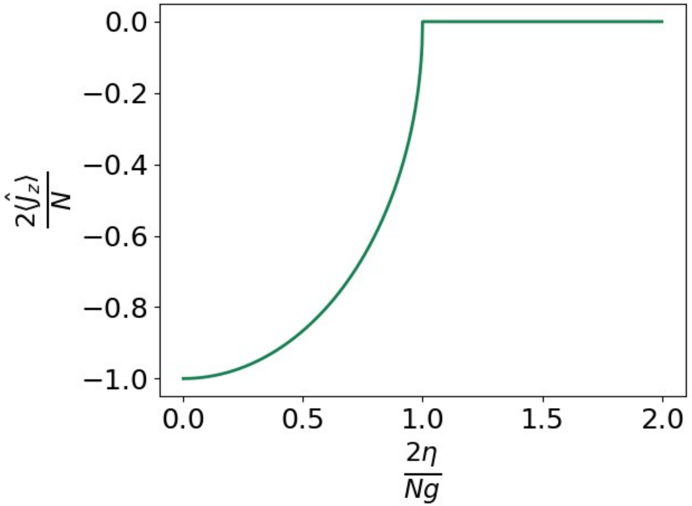
$\langle \hat{J}_z \rangle = 0 :$



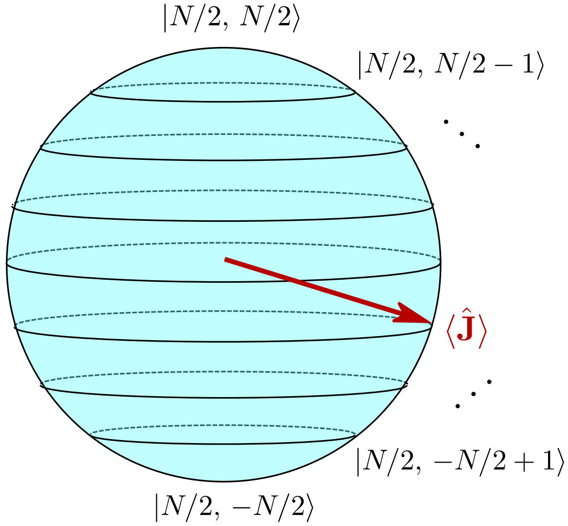
Tavis-Cummings Hamiltonian

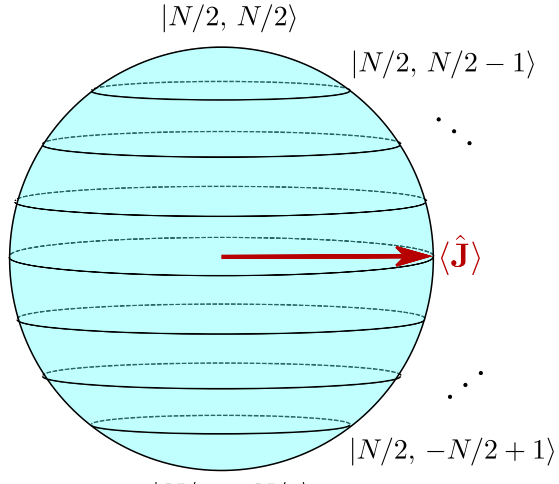
$$\langle \hat{J}_z \rangle^2 + \langle \hat{J}_- \rangle^2 = \frac{N}{2}$$

$$\langle \hat{J}_z \rangle = \begin{cases} -\frac{N}{2}\sqrt{1 - \left(\frac{2|\eta|}{Ng}\right)^2}, & \text{if } \eta \leq \frac{Ng}{2} \\ 0, & \text{if } \eta > \frac{Ng}{2} \end{cases}$$



Second-order transition

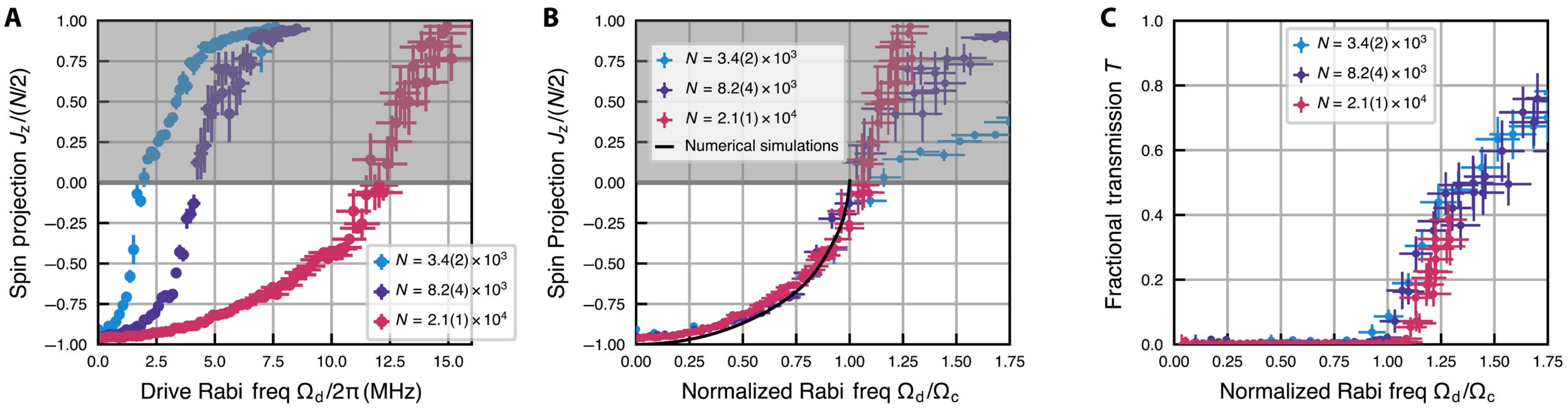




Tavis-Cummings Hamiltonian

$$\langle \hat{J}_z \rangle^2 + \langle \hat{J}_- \rangle^2 = \frac{N}{2}$$

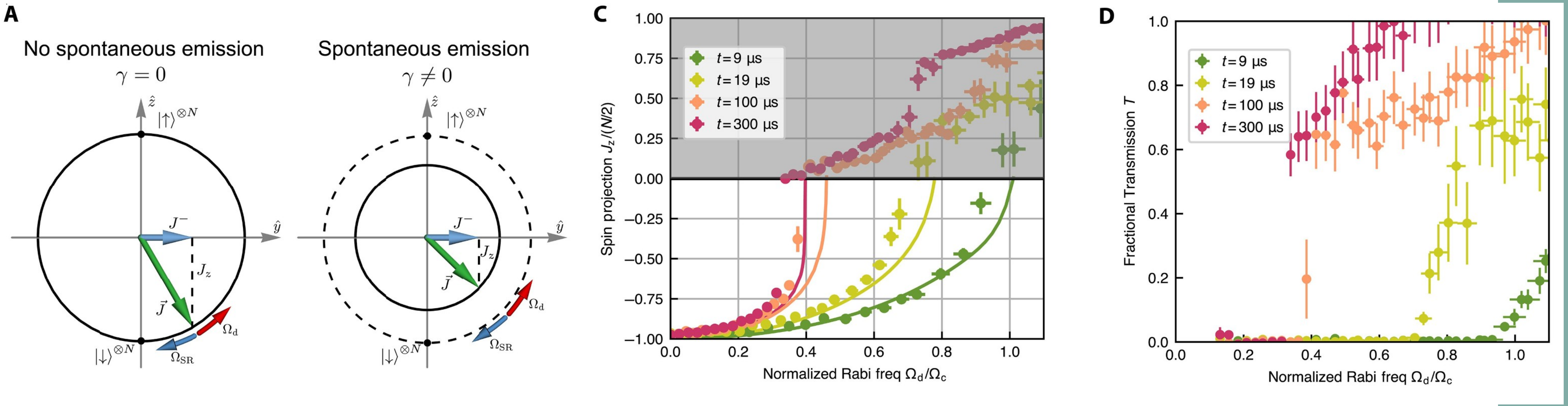
$$\langle \hat{J}_z \rangle = \begin{cases} -\frac{N}{2}\sqrt{1 - \left(\frac{2|\eta|}{Ng}\right)^2}, & \text{if } \eta \leq \frac{Ng}{2} \\ 0, & \text{if } \eta > \frac{Ng}{2} \end{cases}$$



Eric Yilun Song et al. A dissipation-induced superradiant transition in a strontium cavity-QED system.Sci. Adv.11,eadu5799 (2025).

Tavis-Cummings Hamiltonian

When spontaneous emission becomes important (slower timescales), second order transition changes to **first-order transition** + bistable behavior



Spontaneous emission breaks Dicke symmetry and reduces the total spin of the collective spin

Eric Yilun Song et al. A dissipation-induced superradiant transition in a strontium cavity-QED system.Sci. Adv.11,eadu5799 (2025).

Atomic spin manipulation

Dicke states: $|J = N/2, M\rangle \quad , \; m \in \left\{ -\frac{N}{2}, -\frac{N}{2}+1, \cdots, \frac{N}{2}-1, \frac{N}{2} \right\}$

$$|J = N/2, M = -N/2\rangle = \bigotimes_{j=1}^{N} |g_j\rangle$$

$$|N/2, -N/2+1\rangle = \frac{\hat{J}_+ |N/2, -N/2\rangle}{\sqrt{N}} = \bigotimes_{j=1}^{N} \frac{\hat{\sigma}_+^{(j)} |g_j\rangle}{\sqrt{N}} = \sum_{j=1}^{N} \frac{|g_1 \cdots e_j \cdots g_N\rangle}{\sqrt{N}}$$

And so far: $|N/2, M\rangle \propto \left(\hat{J}_+\right)^{M-N/2} |N/2, -N/2\rangle$

By symmetry, the Dicke states are the fully symmetrical ones:

$$|N/2, M\rangle \propto \left(\hat{J}_+\right)^{M-N/2} |N/2, -N/2\rangle = \frac{\hat{\mathcal{S}} \left| e_1 e_2 \cdots e_{(N/2+M)}\, g_{(N/2+M+1)} \cdots g_{N-1} g_N \right\rangle}{\left| \hat{\mathcal{S}} \left| e_1 e_2 \cdots e_{(N/2+M)}\, g_{(N/2+M+1)} \cdots g_{N-1} g_N \right\rangle \right|}$$

$\hat{\mathcal{S}} = \frac{1}{N} \sum \hat{\mathcal{P}}$: Symmetrization operator, summed over all permutations

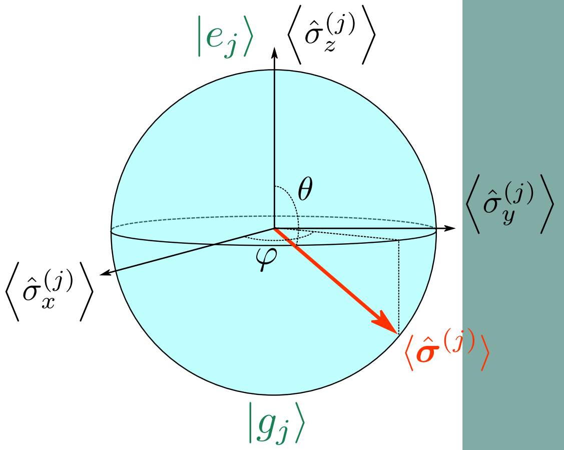
Atomic spin manipulation

Each 2-level state can be represented as a function of θ and φ angles over a sphere:

$$|\psi_j\rangle = \sin\left(\frac{\theta}{2}\right)|g_j\rangle + \mathrm{e}^{-i\varphi}\cos\left(\frac{\theta}{2}\right)|e_j\rangle$$

$$\left\langle \hat{\sigma}_z^{(j)} \right\rangle = \langle\psi_j|\,\hat{\sigma}_z^{(j)}\,|\psi_j\rangle = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) = \cos\theta$$

$$\left\langle \hat{\sigma}_x^{(j)} \right\rangle = \sin\theta\cos\varphi \qquad \left\langle \hat{\sigma}_y^{(j)} \right\rangle = \sin\theta\sin\varphi$$



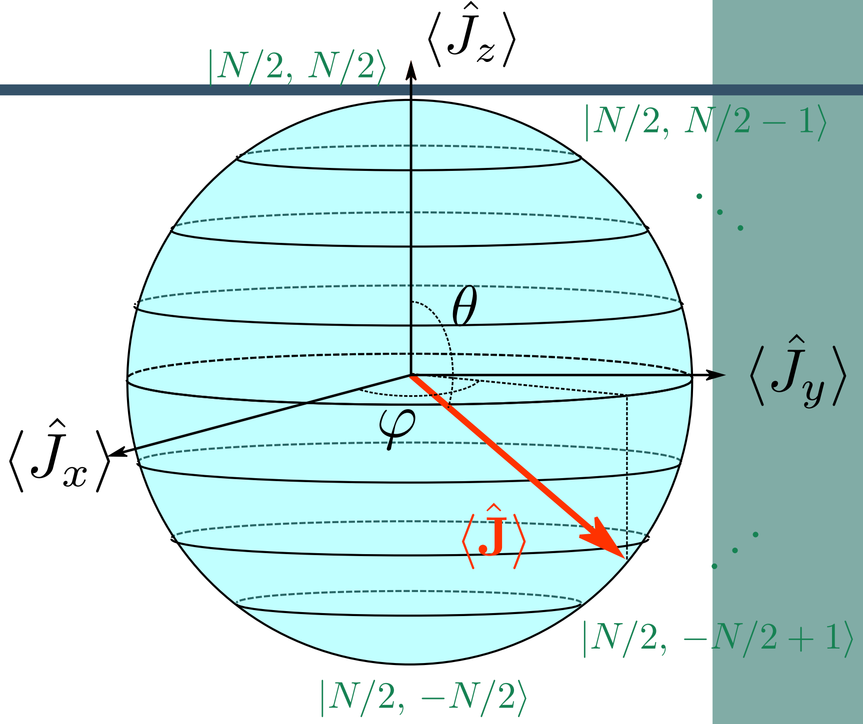
The separable state with all atoms in same state is a combination of Dicke states:

$$|\theta,\varphi\rangle = \bigotimes_j |\psi_j\rangle = \bigotimes_j \left[\sin\left(\frac{\theta}{2}\right)|g_j\rangle + \mathrm{e}^{-i\varphi}\cos\left(\frac{\theta}{2}\right)|e_j\rangle\right]$$

$$|\theta,\varphi\rangle = \sum_{j=0}^{N}\sqrt{\binom{N}{k}}\,\mathrm{e}^{-ik\varphi}\,\cos^k\left(\frac{\theta}{2}\right)\,\sin^{N-k}\left(\frac{\theta}{2}\right)\left|\frac{N}{2}, -\frac{N}{2}+k\right\rangle$$

→ A separable Dicke state can be represented in a Bloch sphere

Atomic spin manipulation



The sphere has radius N/2 :

$$\left\langle \hat{J}_z \right\rangle = \langle \theta,\varphi|\, \hat{J}_z\, |\theta,\varphi\rangle = \frac{1}{2}\sum_{j=1}^{N} \langle \psi_j|\, \hat{\sigma}_z^{(j)}\, |\psi_j\rangle = \frac{N}{2}\cos\theta$$

$$\left\langle \hat{J}_x \right\rangle = \frac{N}{2}\sin\theta\cos\varphi \qquad \left\langle \hat{J}_y \right\rangle = \frac{N}{2}\sin\theta\sin\varphi$$

$$\left\langle \hat{J}_x \right\rangle = \frac{N}{2}\sin\theta\cos\varphi$$

The separable states are also known as **coherent spin states.** They are eigenvalues of the spin operator pointing in direction (θ,φ) :

$$\hat{J}(\theta,\varphi) = \sin\theta\cos\varphi\hat{J}_x + \sin\theta\sin\varphi\hat{J}_y + \cos\theta\hat{J}_z$$

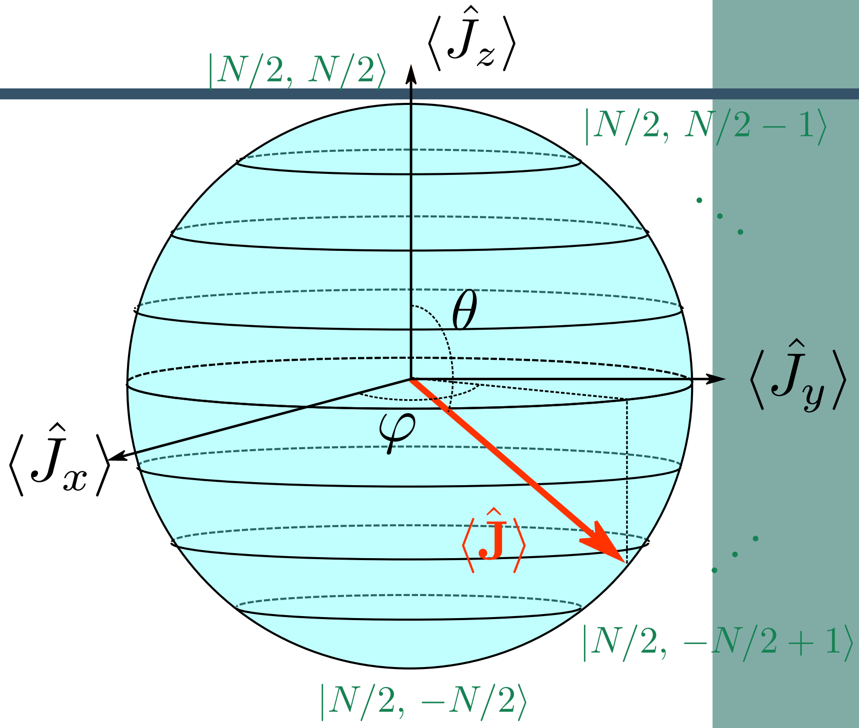
$$\hat{J}(\theta,\varphi)\,|\theta,\varphi\rangle = \frac{N}{2}\,|\theta,\varphi\rangle$$

The uncertainty on any spin direction perpendicular to (θ,φ) can be obtained:

For $\hat{J}_\perp = a\hat{J}_x + b\hat{J}_y + c\hat{J}_z$ such as $a\sin\theta\cos\varphi + b\sin\theta\sin\varphi + c\cos\theta = 0,$

$$\Delta\hat{J}_\perp = \sqrt{\langle\theta,\varphi|\,\hat{J}_\perp^2\,\langle\theta,\varphi| - \langle\theta,\varphi|\,\hat{J}_\perp\,\langle\theta,\varphi|^2} = \sqrt{\frac{N}{2}}$$

Atomic spin manipulation



Spontaneous emission makes individual atomic decays → states are not anymore fully symmetric

$$\left\langle \hat{J}_z \right\rangle = \langle \theta, \varphi | \, \hat{J}_z \, | \theta, \varphi \rangle = \frac{1}{2} \sum_{j=1}^{N} \langle \psi_j | \, \hat{\sigma}_z^{(j)} \, | \psi_j \rangle = \frac{N}{2} \cos\theta$$

$$\left\langle \hat{J}_x \right\rangle = \frac{N}{2} \sin\theta \cos\varphi \qquad \left\langle \hat{J}_y \right\rangle = \frac{N}{2} \sin\theta \sin\varphi$$

The separable states are also known as **coherent spin states.** They are eigenvalues of the spin operator pointing in direction (θ, φ) :

$$\hat{J}(\theta, \varphi) = \sin\theta \cos\varphi \hat{J}_x + \sin\theta \sin\varphi \hat{J}_y + \cos\theta \hat{J}_z$$

$$\hat{J}(\theta, \varphi) \, |\theta, \varphi\rangle = \frac{N}{2} \, |\theta, \varphi\rangle$$

The uncertainty on any spin direction perpendicular to (θ, φ) can be obtained:

$$\hat{J}_\perp = a\hat{J}_x + b\hat{J}_y + c\hat{J}_z \quad \text{such as} \quad a \sin\theta \cos\varphi + b \sin\theta \sin\varphi + c \cos\theta = 0$$

$$\Delta \hat{J}_\perp = \sqrt{\langle \theta, \varphi | \, \hat{J}_\perp^2 \, \langle \theta, \varphi | - \langle \theta, \varphi | \, \hat{J}_\perp \, \langle \theta, \varphi |^2} = \sqrt{\frac{N}{2}}$$

Atomic spin manipulation

A representation that allows to visualize the state uncertainty and quantum behavior is the **Wigner representation**

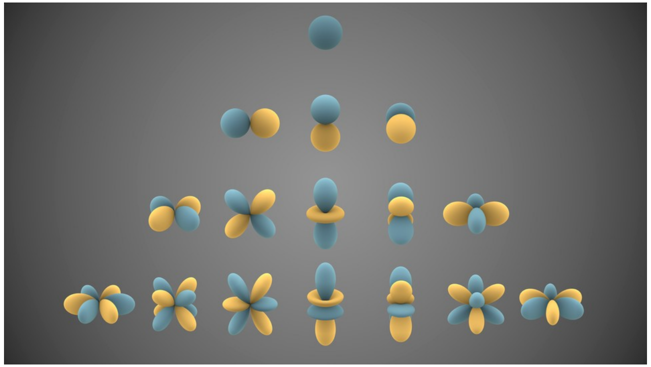
Wigner function in a nutshell:

Multipole operators: $\hat{\mathbf{T}}_{jm} = \sqrt{2j-1} \sum_{M,M'=-J}^{J} (-1)^{j-m} \begin{pmatrix} J & j & J \\ -M & m & M' \end{pmatrix} |J,M\rangle \langle J,M'|$

Wigner function: $W(\theta,\varphi) = \sum_{l=0}^{2J} \sum_{m=-l}^{l} \mathrm{Tr}\left[\hat{\rho}\,\hat{\mathbf{T}}_{lm}^{\dagger}\right] \mathrm{Y}_{lm}(\theta,\varphi)$

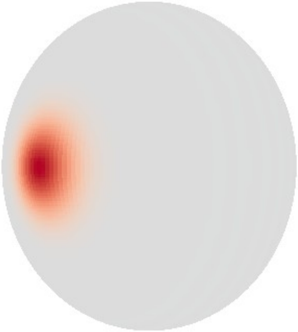
$\mathrm{Y}_{lm}(\theta,\varphi)$: Spherical harmonics

Dowling, Jonathan P., Girish S. Agarwal, and Wolfgang P. Schleich. "Wigner distribution of a general angular-momentum state: Applications to a collection of two-level atoms." Physical Review A 49.5 (1994): 4101.



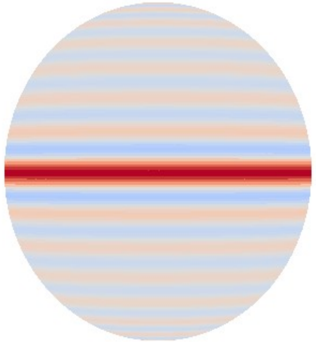
Atomic spin manipulation

Wigner representation of some states:



$N = 30, \quad \left|\theta = \frac{\pi}{2}, \varphi = 0\right\rangle$

$N = 30, \quad \left|\theta = \frac{9\pi}{10}, \varphi = 0\right\rangle$



$|J = 30/2, M = 0\rangle$

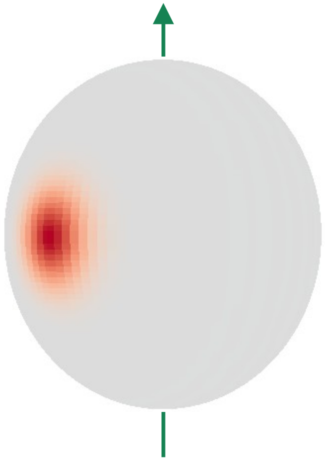
Coherent states: “most classical” ones

States with well-defined excitation number (Fock-like): negativity in the Wigner function

Atomic spin manipulation

Action of different Hamiltonian terms on a spin coherent state:

- Linear terms in $\hat{J}_x, \hat{J}_y, \hat{J}_z$: **rotation**



Ex: $\hat{H} = \hbar\Delta\hat{J}_z$

$$|\psi(t)\rangle = \mathrm{e}^{-i\hat{H}t/\hbar}\,|\psi(0)\rangle = \mathrm{e}^{-i\Delta t\hat{J}_z}\,|\theta,\varphi\rangle$$

$$|\psi(t)\rangle = \sum_{j=0}^{N}\sqrt{\binom{N}{k}}\,\mathrm{e}^{-ik\varphi}\,\cos^k\left(\frac{\theta}{2}\right)\,\sin^{N-k}\left(\frac{\theta}{2}\right)\,\mathrm{e}^{-i\Delta t\hat{J}_z}\left|\frac{N}{2},-\frac{N}{2}+k\right\rangle$$

$$|\psi(t)\rangle = \mathrm{e}^{iN\Delta\,t/2}\sum_{j=0}^{N}\sqrt{\binom{N}{k}}\,\mathrm{e}^{-ik(\varphi+\Delta\,t)}\,\cos^k\left(\frac{\theta}{2}\right)\,\sin^{N-k}\left(\frac{\theta}{2}\right)\left|\frac{N}{2},-\frac{N}{2}+k\right\rangle$$

$$|\psi(t)\rangle = \mathrm{e}^{iN\Delta\,t/2}\,|\theta,\varphi+\Delta\,t\rangle$$

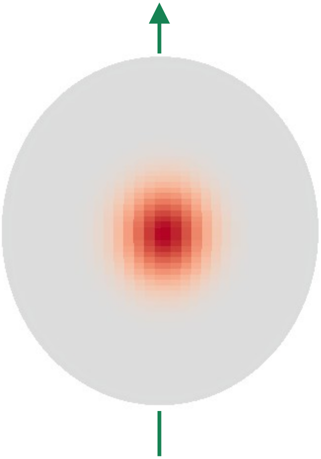
State remains coherent

Atomic spin manipulation

Action of different Hamiltonian terms on a spin coherent state:

- Quadratic terms: **squeezing**

Ex: $\hat{H} = \hbar\chi\hat{J}_z^2$



$$|\psi(t)\rangle = \mathrm{e}^{-i\hat{H}t/\hbar}|\psi(0)\rangle = \mathrm{e}^{-i\chi t\hat{J}_z^2}|\theta,\varphi\rangle$$

$$|\psi(t)\rangle = \mathrm{e}^{iN\Delta t/2}\sum_{j=0}^{N}\sqrt{\binom{N}{k}}\,\mathrm{e}^{-ik\varphi-i(k-N/2)^2t}\,\cos^k\left(\frac{\theta}{2}\right)\,\sin^{N-k}\left(\frac{\theta}{2}\right)\left|\frac{N}{2},-\frac{N}{2}+k\right\rangle$$

State remains in Dicke subspace, but is not coherent anymore

Differential rotation speed causes **squeezing** of state

Spin definition below shot noise limit → **quantum correlations** between spins
→ **Entanglement**

$$\hat{J}_z^2 = \left[\sum_{j=1}^{N}\hat{\sigma}_z^{(j)}\right]^2 = \sum_{j,k=1}^{N}\hat{\sigma}_z^{(j)}\hat{\sigma}_z^{(k)}$$

Atomic spin manipulation

Can we use the cavity to manipulate the spin state?

In the bad cavity limit, κ is the fastest decay rate → cavity dynamics adiabatic following atomic dynamics

Adiabatic elimination of cavity dynamics → Effective spin dynamics

$$\frac{\mathrm{d}\hat{a}}{\mathrm{d}t} = (i\Delta_c - \kappa)\,\hat{a} - ig\sum_{j=1}^{N}\sigma_-^{(j)} + \eta = 0 \quad \longrightarrow \quad \boxed{\hat{a} = \frac{g\hat{J}_- + i\eta}{\Delta_c + i\kappa}}$$

The effective Hamiltonian now contain nonlinear terms

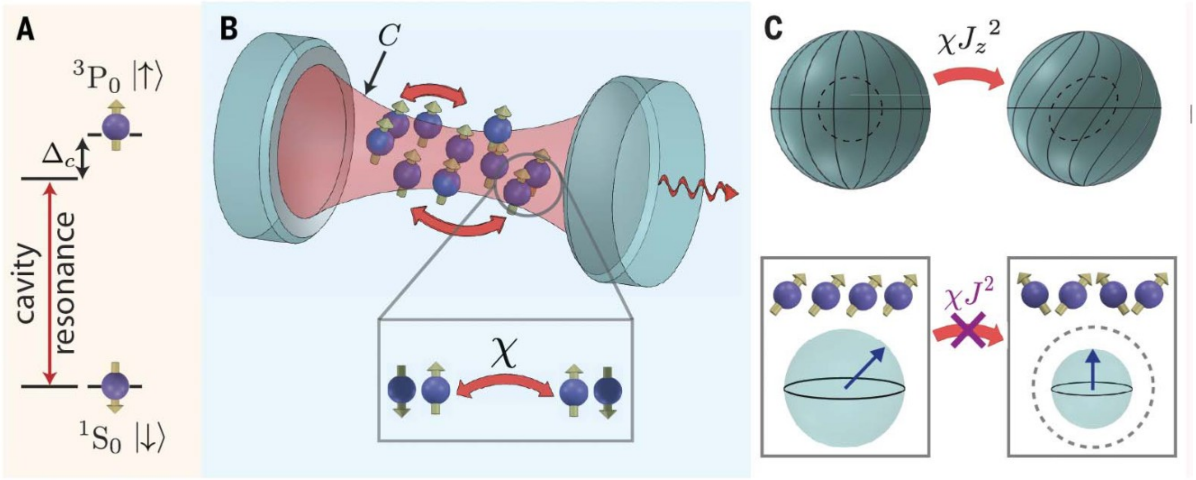
$$\hat{a}^\dagger\hat{a} \propto \hat{J}_+\hat{J}_-, \qquad \hat{a}^\dagger J_- \propto \hat{J}_+\hat{J}_-.$$

Through a rigorous approach via tracing over the degrees of freedom of the cavity, far from resonance we find an effective Hamiltonian which is approximately unitary, given by

$$\hat{H}_{JC} \longrightarrow \hat{H}_{eff} = -\hbar\Delta_a\hat{J}_z + \hbar\chi\hat{J}_+\hat{J}_- = \hbar\chi\left[\hat{J}^2 - \hat{J}_z^2\right] + \hbar\left(\chi - \Delta_a\right)\hat{J}_z \qquad \chi = \frac{g^2\Delta_c}{\Delta_c^2 + \kappa^2}$$

Agarwal, Girish S., Ravinder R. Puri, and R. P. Singh. "Atomic Schrödinger cat states." Physical Review A 56.3 (1997): 2249.

Atomic spin manipulation



^{87}Sr atoms, $^1S_0 \rightarrow ^3P_0$ transition

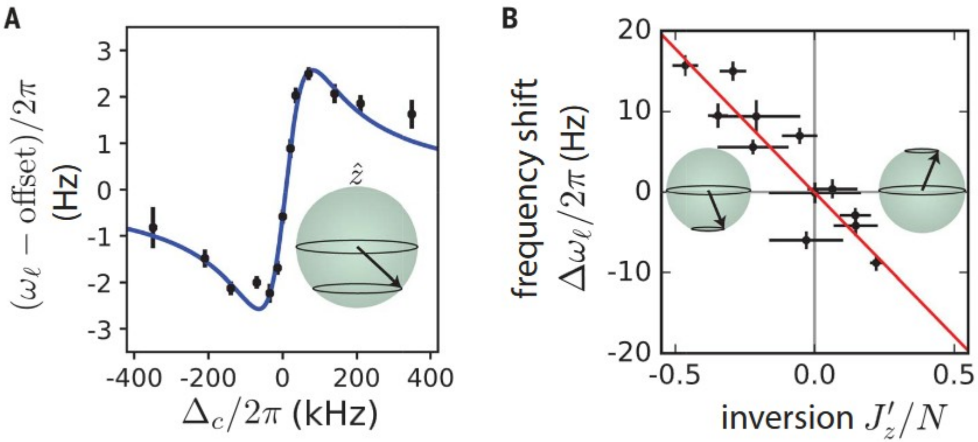
$\kappa = 2\pi \times 80\ \mathrm{kHz}$

$g = 2\pi \times 2\ \mathrm{kHz}$

$\Gamma \simeq 0$

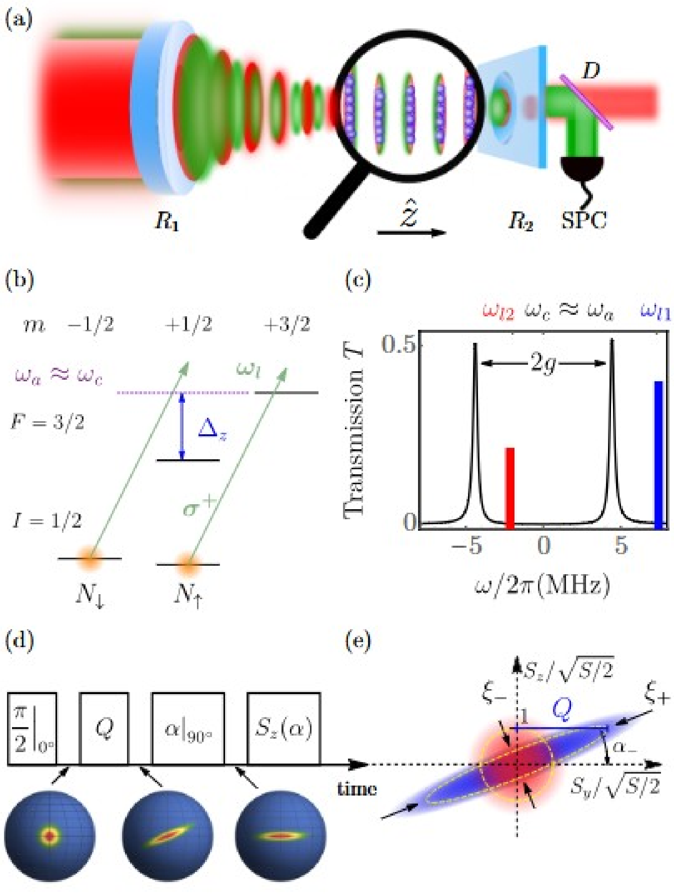
$N = 10^5$

Norcia, Matthew A., et al. "Cavity-mediated collective spin-exchange interactions in a strontium superradiant laser." Science361,259-262 (2018).



Indirect observation of OAT dynamics via the frequency shift of the light emitted by the cavity after atomic excitation

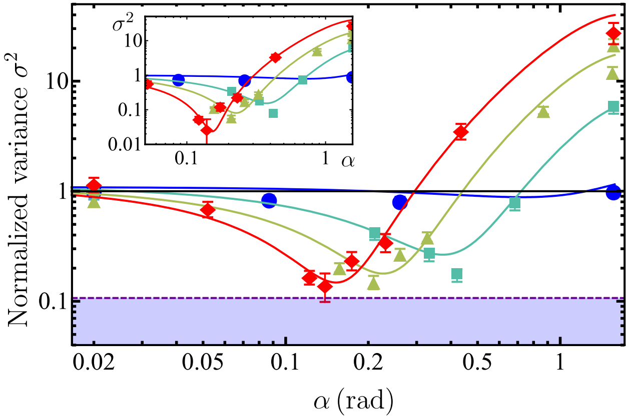
Atomic spin manipulation

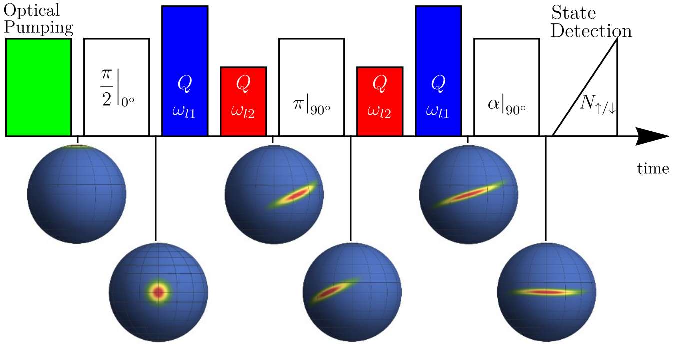


Braverman, Boris, et al. "Near-unitary spin squeezing in Yb 171." PRL 122, 223203 (2019).

^{171}Yb atoms

Squeezing done in optical transition but measured in hyperfine spin





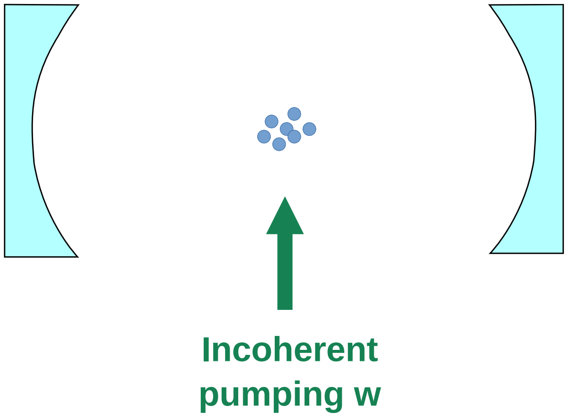
Superradiant laser

Bad cavity limit: coherence of the light state of the cavity governed by the atomic coherence

Possibility of **ultranarrow laser emission**

In order to calculate the width of light emission, one need to keep track of **correlations** between operators

A possibility: **cumulant expansion** of correlations.



$$\hat{H}_{TC} = -\hbar\Delta_C\, \hat{a}^\dagger \hat{a} - \hbar\Delta_a \sum_{j=1}^{N} \frac{\hat{\sigma}_z^{(j)}}{2} + \hbar g \sum_{j=1}^{N} \left(\hat{a}\hat{\sigma}_+^{(j)} + \hat{a}^\dagger \hat{\sigma}_-^{(j)}\right)$$

$$\hat{\mathcal{L}}_\kappa(\hat{A}) = \kappa \left(2\hat{a}^\dagger \hat{A}\hat{a} - \hat{a}^\dagger \hat{a}\hat{A} - \hat{A}\hat{a}^\dagger \hat{a}\right)$$

$$\hat{\mathcal{L}}_\Gamma(\hat{A}) = \frac{\Gamma}{2} \sum_{j=1}^{N} \left(2\hat{\sigma}_+^{(j)} \hat{A}\hat{\sigma}_-^{(j)} - \hat{\sigma}_+^{(j)}\hat{\sigma}_-^{(j)}\hat{A} - \hat{A}\hat{\sigma}_+^{(j)}\hat{\sigma}_-^{(j)}\right)$$

$$\hat{\mathcal{L}}_w(\hat{A}) = \frac{w}{2} \sum_{j=1}^{N} \left(2\hat{\sigma}_-^{(j)} \hat{A}\hat{\sigma}_+^{(j)} - \hat{\sigma}_-^{(j)}\hat{\sigma}_+^{(j)}\hat{A} - \hat{A}\hat{\sigma}_-^{(j)}\hat{\sigma}_+^{(j)}\right)$$

$$\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} = \frac{i}{\hbar}\left[\hat{H}_{JC}, \hat{A}\right] + \mathcal{L}_\Gamma(\hat{A}) + \mathcal{L}_\kappa(\hat{A}) + \mathcal{L}_w(\hat{A})$$

Superradiant laser

Beyond the mean field approximation and with no pumping, all phases are equally probable for $\langle \hat{a} \rangle$, $\langle \hat{a}^\dagger \rangle$, $\langle \hat{\sigma_\pm}^{(j)} \rangle$

→ $\langle \hat{a} \rangle = \langle \hat{a}^\dagger \rangle = \langle \hat{\sigma_\pm}^{(j)} \rangle = 0$

$$\frac{\mathrm{d}\hat{\sigma}_z^{(j)}}{\mathrm{d}t} = 2ig\left(\hat{a}^\dagger\,\hat{\sigma}_-^{(j)} - \hat{a}\,\hat{\sigma}_+^{(j)}\right) - \Gamma\left(\hat{\mathbb{1}} + \hat{\sigma}_z^{(j)}\right) + w\left(\hat{\mathbb{1}} - \hat{\sigma}_z^{(j)}\right)$$

$$\Rightarrow \quad \frac{\mathrm{d}\langle\hat{\sigma}_z^{(j)}\rangle}{\mathrm{d}t} = 2ig\left(\langle\hat{a}^\dagger\,\hat{\sigma}_-^{(j)}\rangle - \langle\hat{a}\,\hat{\sigma}_+^{(j)}\rangle\right) - (w+\Gamma)\langle\hat{\sigma}_z^{(j)}\rangle + w - \Gamma$$

Those average values depend on products of operators, whose average we must also calculate:

$$\frac{\mathrm{d}\left(\hat{a}^\dagger\hat{\sigma}_-^{(j)}\right)}{\mathrm{d}t} = \frac{i}{\hbar}\left[\hat{H}_{JC}, \hat{a}^\dagger\hat{\sigma}_-^{(j)}\right] + \mathcal{L}_\Gamma(\hat{a}\hat{\sigma}_-^{(j)}) + \mathcal{L}_\kappa(\hat{a}^\dagger\hat{\sigma}_-^{(j)}) + \mathcal{L}_w(\hat{a}^\dagger\hat{\sigma}_-^{(j)})$$

$$\frac{\mathrm{d}\left(\hat{a}\hat{\sigma}_+^{(j)}\right)}{\mathrm{d}t} = \frac{i}{\hbar}\left[\hat{H}_{JC}, \hat{a}\hat{\sigma}_+^{(j)}\right] + \mathcal{L}_\Gamma(\hat{a}\hat{\sigma}_+^{(j)}) + \mathcal{L}_\kappa(\hat{a}\hat{\sigma}_+^{(j)}) + \mathcal{L}_w(\hat{a}\hat{\sigma}_+^{(j)})$$

We obtain the equations and the averages, to arrive to

Superradiant laser

$$\frac{\mathrm{d}\langle\hat{a}^\dagger\hat{\sigma}_-^{(j)}\rangle}{\mathrm{d}t} = -\left(\frac{w+\Gamma+\kappa}{2} - i\Delta_{ca}\right)\langle\hat{a}^\dagger\hat{\sigma}_-^{(j)}\rangle + ig\left[\cancel{\underline{\langle\hat{a}^\dagger\hat{a}\hat{\sigma}_z^{(j)}\rangle}} + \langle\hat{a}^\dagger\hat{a}\rangle\langle\hat{\sigma}_z^{(j)}\rangle + \frac{\langle\hat{\sigma}_z^{(j)}\rangle+1}{2} + \sum_{k\neq j}^{N}\langle\hat{\sigma}_z^{(j)}\hat{\sigma}_z^{(k)}\rangle\right]$$

3rd order cumulant - truncated

In the same way, we can obtain the equations for the time evolution of $\langle\hat{a}^\dagger\hat{a}\rangle, \langle\hat{\sigma}_z^{(j)}\hat{\sigma}_z^{(k)}\rangle$, eliminating already the 3rd order terms:

$$\frac{\mathrm{d}\langle\hat{\sigma}_+^{(j)}\hat{\sigma}_-^{(k)}\rangle}{\mathrm{d}t} = -(w+\Gamma)\langle\hat{\sigma}_+^{(j)}\hat{\sigma}_-^{(k)}\rangle - ig\langle\hat{\sigma}_z^{(j)}\rangle\left[\langle\hat{a}^\dagger\hat{\sigma}_-^{(j)}\rangle - \langle\hat{\sigma}_+^{(j)}\hat{a}\rangle\right]$$

$$\frac{\mathrm{d}\langle\hat{a}^\dagger\hat{a}\rangle}{\mathrm{d}t} = -\kappa\langle\hat{a}^\dagger\hat{a}\rangle - ig\sum_{j=1}^{N}\langle\hat{\sigma}_z^{(j)}\rangle\left[\langle\hat{a}^\dagger\hat{\sigma}_-^{(j)}\rangle - \langle\hat{\sigma}_+^{(j)}\hat{a}\rangle\right]$$

The symmetry of the atomic coupling to light imposes that $\langle\hat{a}^\dagger\hat{\sigma}_z^{(j)}\rangle = \langle\hat{a}^\dagger\hat{\sigma}_z^{(1)}\rangle$, $\langle\hat{\sigma}_z^{(j)}\hat{\sigma}_z^{(k)}\rangle = \langle\hat{\sigma}_z^{(1)}\hat{\sigma}_z^{(2)}\rangle$

Superradiant laser

In the limit $\Gamma, w \ll \kappa$ (bad cavity limit), we obtain the steady-state equation for the atomic correlations

$$\langle \hat{\sigma}_+^{(j)} \hat{\sigma}_-^{(k)} \rangle \left[-(w+\Gamma) + N\Upsilon\Gamma \frac{w-\Gamma}{w+\Gamma} - 2\frac{N^2\Upsilon^2\Gamma^2}{w+\Gamma} \langle \hat{\sigma}_+^{(j)} \hat{\sigma}_-^{(k)} \rangle \right] = 0$$

The threshold for the development of positive atomic correlations $\left(\langle \hat{\sigma}_+^{(j)} \hat{\sigma}_-^{(k)} \rangle > 0 \right)$ in the limit $N \gg 1$ is given by

$$w > \Gamma$$

Above the threshold,

$$\langle \hat{\sigma}_+^{(j)} \hat{\sigma}_-^{(k)} \rangle = \frac{w-\Gamma}{2N\Upsilon\Gamma} - \frac{1}{2} \left(\frac{w+\Gamma}{N\Upsilon\Gamma} \right)^2$$

The onset of correlations among scatterers represent a **collective response** of the atomic cloud

All other equations can also be solved, to obtain the stationary solution for the correlators

Superradiant laser

To obtain the linewidth of the atomic laser, one needs to compute the $g^{(1)}(\tau)$ of light:

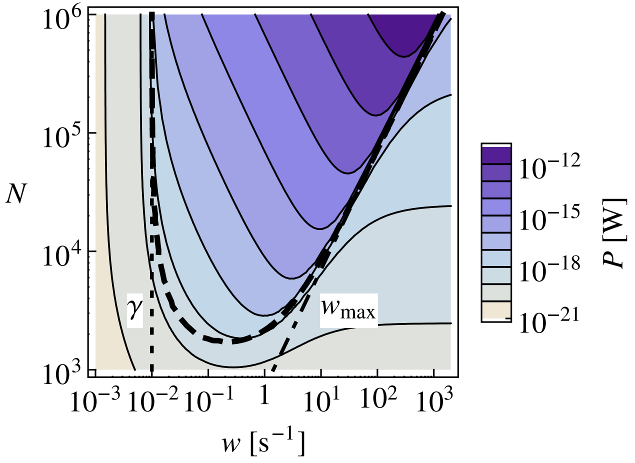
$$g^{(1)}(\tau) = \frac{\langle a^\dagger(\tau) a(0)\rangle}{\sqrt{\langle a^\dagger(\tau) a(\tau)\rangle \langle a^\dagger(0) a(0)\rangle}}$$

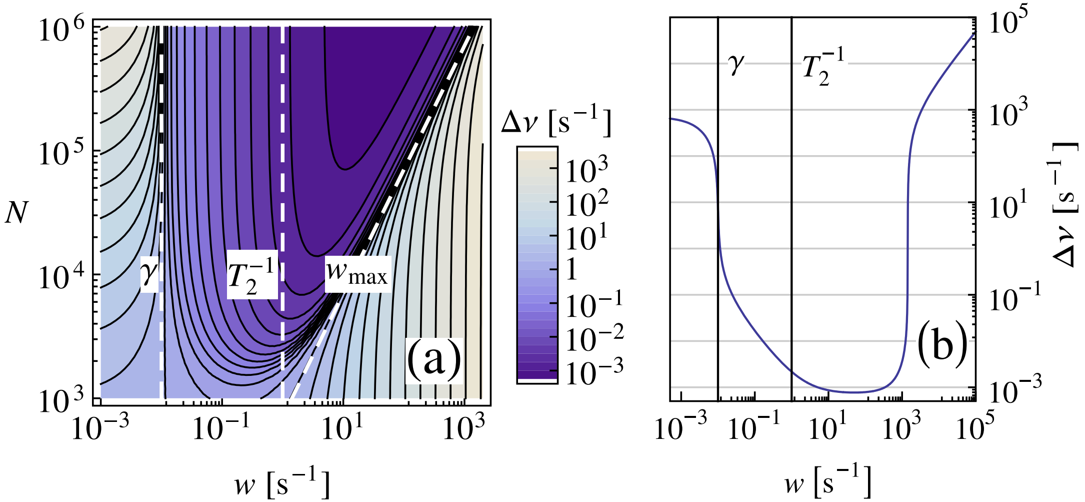
Such that the power spectral density is

$$S(\omega) = \int_{\mathbb{R}} \mathrm{d}\tau \, \mathrm{e}^{-\omega\tau} \, g^{(1)}(\tau)$$

For computing it, one needs a two-time correlator $\langle a^\dagger(\tau) a(0)\rangle$ that can be calculated using the **quantum regression theorem**

Superradiant laser





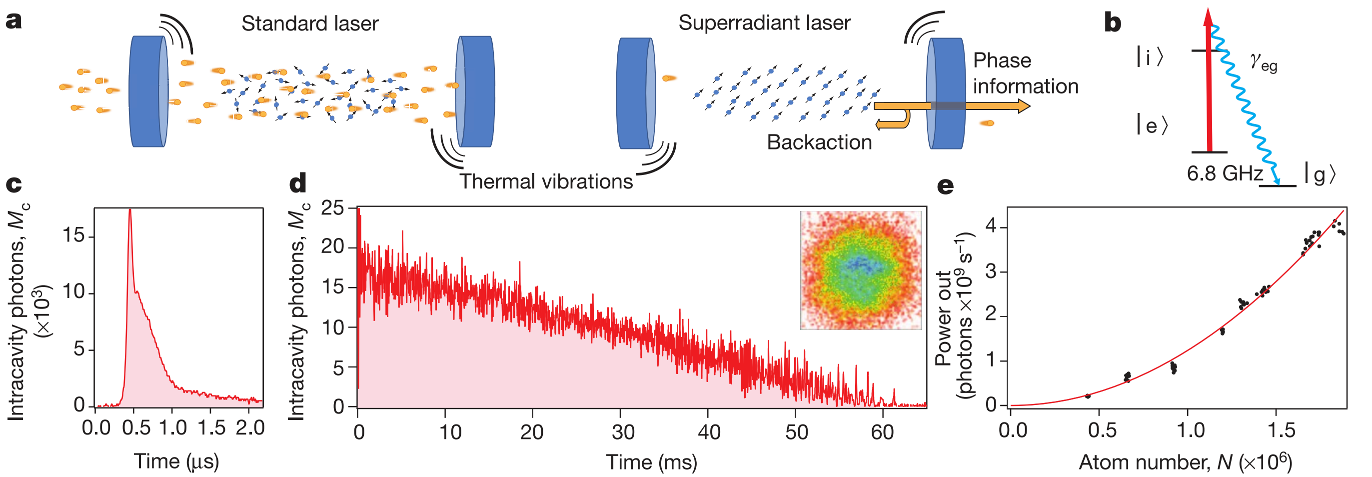
Meiser, Dominic, et al. "Prospects for a millihertz-linewidth laser." Physical review letters 102.16 (2009): 163601.

- Threshold behavior for emitted power
- Loss of laser emission for too much pumping (destruction of correlations)
- Linewidth much smaller than κ
-
- Cavity pulling in bad cavity limit: $P \simeq \dfrac{\Gamma}{\kappa}$

In peak of emission, $\Delta\omega = \dfrac{\Upsilon}{2}\Gamma$

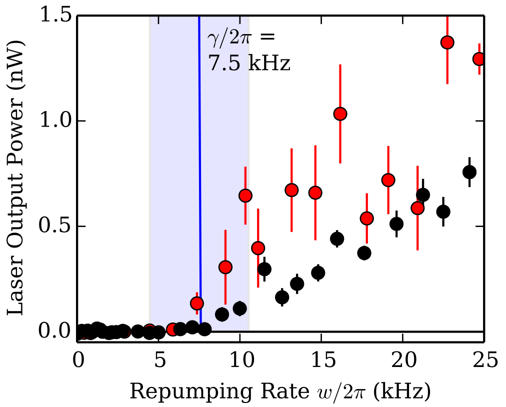
Superradiant laser

- Quadratic dependance of emitted power with N (superradiance):



Bohnet, Justin G., et al. "A steady-state superradiant laser with less than one intracavity photon." Nature 484.7392 (2012): 78-81.

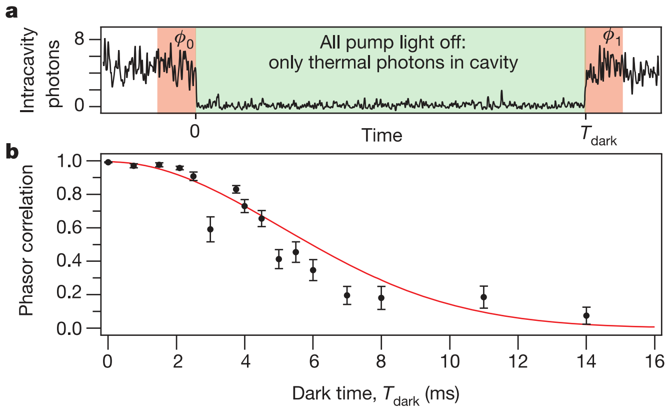
- Threshold behavior:



Norcia, Matthew A., and James K. Thompson. "Cold-strontium laser in the superradiant crossover regime." Physical Review X 6.1 (2016): 011025.

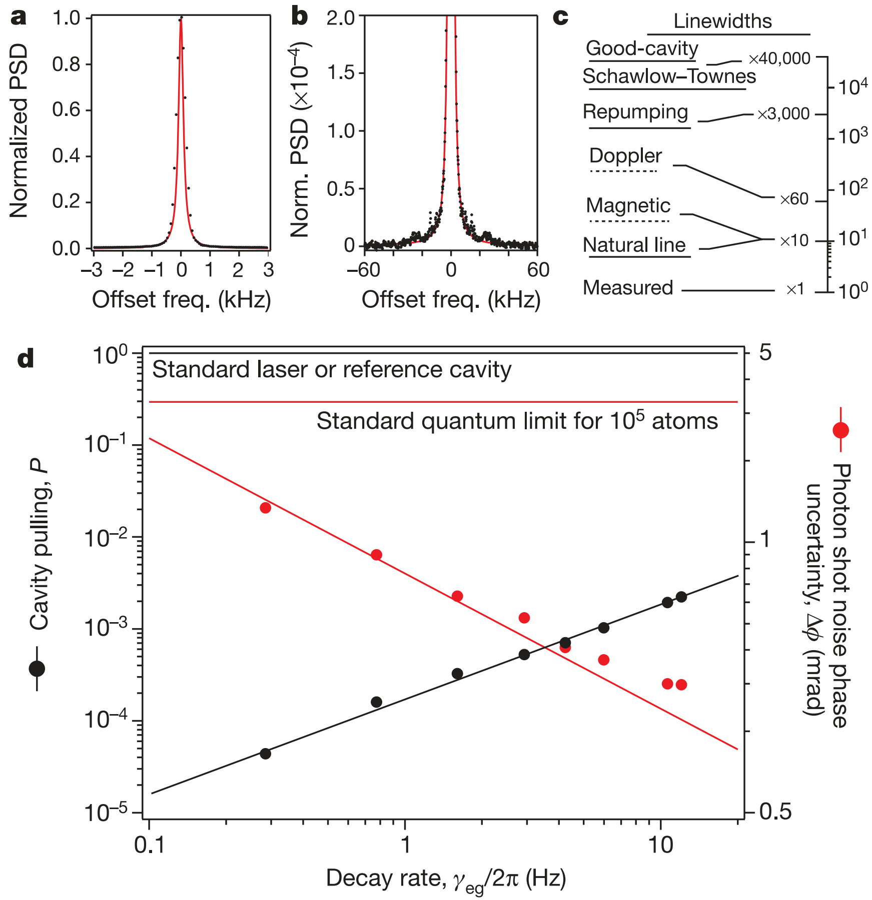
Superradiant laser

- Storage of excitation on the atoms:



- Linewidth:

- Cavity pulling:



Bohnet, Justin G., et al. "A steady-state superradiant laser with less than one intracavity photon." Nature 484.7392 (2012): 78-81.

Tavis-Cummings Hamiltonian

What is not described by the mean-field model?

- Correlations, including quantum
- Linewidth of lasers
- Squeezing

In order to grasp that, one need to keep track of correlations

A possibility: **cumulant expansion** of correlations