Experiments on cold atoms and optical cavities
3 – Coupling atomic momentum to light

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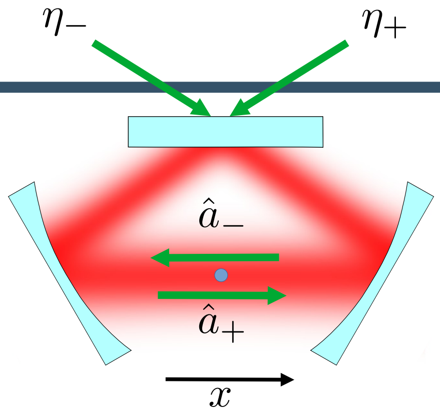
Atoms in ring cavities



Ring cavities present two counterpropagating modes to couple to atoms:

Hamiltonian for homogeneous coupling between 1 atom and 2 cavity modes: $\hat{a}_+, \hat{a}_-$

$$\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} = \frac{i}{\hbar}\left[\hat{H}, \hat{A}\right] + \mathcal{L}_\Gamma(\hat{A}) + \mathcal{L}_\kappa(\hat{A})$$

$$\hat{H}_{AC} = -\hbar\Delta_C\left(\hat{a}_+^\dagger\hat{a}_+ + \hat{a}_-^\dagger\hat{a}_-\right) - \hbar\Delta_a\frac{\hat{\sigma}^z}{2} + \frac{\hat{p}^2}{2m} + \hbar g\left[\left(\hat{a}_+\mathrm{e}^{ik\hat{x}} + \hat{a}_-\mathrm{e}^{-ik\hat{x}}\right)\hat{\sigma}^+ + \left(\hat{a}_+^\dagger\mathrm{e}^{-ik\hat{x}} + \hat{a}_-^\dagger\mathrm{e}^{ik\hat{x}}\right)\hat{\sigma}^-\right]$$

$$\hat{\mathcal{L}}_\kappa(\hat{A}) = \kappa\left(2\hat{a}^\dagger\hat{A}\hat{a} - \hat{a}^\dagger\hat{a}\hat{A} - \hat{A}\hat{a}^\dagger\hat{a}\right)$$

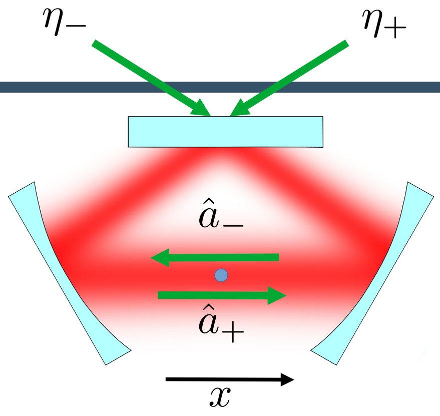
$$\hat{H}_\eta = i\eta_+\hat{a}_+^\dagger - i\eta_+^*\hat{a}_+ + i\eta_-\hat{a}_-^\dagger - i\eta_-^*\hat{a}_-$$

$$\hat{\mathcal{L}}_\Gamma(\hat{A}) = \frac{\Gamma}{2}\left(2\hat{\sigma}_+\hat{A}\hat{\sigma}_- - \hat{\sigma}_+\hat{\sigma}_-\hat{A} - \hat{A}\hat{\sigma}_+\hat{\sigma}_-\right)$$

$$\hat{H} = \hat{H}_{AC} + \hat{H}_\eta$$

- Phase of light modes is not fixed by mirrors, but is a free parameter of the system
- Momentum of atom can couple to the relative phase of counterpropagating modes

Atoms in ring cavities



Evolution equations for the operators:

$$\frac{\mathrm{d}\hat{a}_\pm}{\mathrm{d}t} = (i\Delta_c - \kappa)\,\hat{a}_\pm - ig\,\mathrm{e}^{\mp ik\hat{x}}\hat{\sigma}^- + \eta_\pm$$

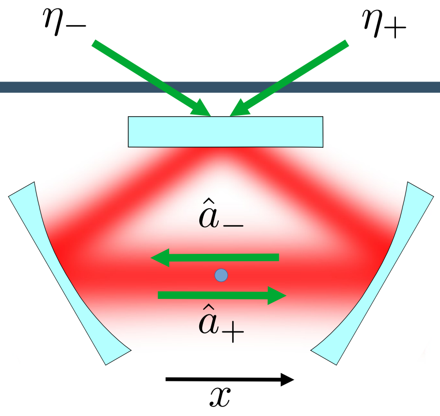
$$\frac{\mathrm{d}\hat{\sigma}^-}{\mathrm{d}t} = \left(i\Delta_a - \frac{\Gamma}{2}\right)\hat{\sigma}^- + ig\,\left(\hat{a}_-\mathrm{e}^{-ik\hat{x}} + \hat{a}_+\mathrm{e}^{ik\hat{x}}\right)\,\hat{\sigma}^z$$

$$\frac{\mathrm{d}\hat{\sigma}^z}{\mathrm{d}t} = 2ig\,\left[\left(\hat{a}_-^\dagger\mathrm{e}^{ik\hat{x}} + \hat{a}_+^\dagger\mathrm{e}^{-ik\hat{x}}\right)\,\hat{\sigma}^- - \left(\hat{a}_-\mathrm{e}^{-ik\hat{x}} + \hat{a}_+\mathrm{e}^{ik\hat{x}}\right)\,\hat{\sigma}^+\right] - \Gamma\left(\hat{\mathbb{1}} + \hat{\sigma}^z\right)$$

$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = \frac{\hat{p}}{m}$$

$$\frac{\mathrm{d}\hat{p}}{\mathrm{d}t} = ig\hbar k\,\left[\left(\hat{a}_+^\dagger\mathrm{e}^{-ik\hat{x}} - \hat{a}_-^\dagger\mathrm{e}^{ik\hat{x}}\right)\,\hat{\sigma}^- - \left(\hat{a}_+\mathrm{e}^{ik\hat{x}} - \hat{a}_-\mathrm{e}^{-ik\hat{x}}\right)\,\hat{\sigma}^+\right]$$

Atoms in ring cavities



Let's examine the conservation of momentum for the cavity:

Momentum of light: $\hat{P}_\pm = \pm\hbar k \; \hat{a}_\pm^\dagger \hat{a}_\pm$

$$\frac{\mathrm{d}\hat{P}_\pm}{\mathrm{d}t} = \pm\hbar k \; \frac{\mathrm{d}\left(\hat{a}_\pm^\dagger \hat{a}_\pm\right)}{\mathrm{d}t}$$

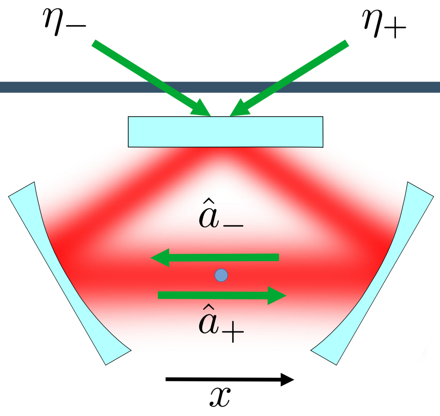
$$\frac{\mathrm{d}\hat{P}_\pm}{\mathrm{d}t} = \pm\hbar k \; \left[-2\kappa \hat{a}_\pm^\dagger \hat{a}_\pm + \left(\eta_\pm - \underline{ig\,\mathrm{e}^{\mp ik\hat{x}}\hat{\sigma}^-}\right)\hat{a}_\pm^\dagger + \left(\eta_\pm + \underline{ig\mathrm{e}^{\pm ik\hat{x}}\hat{\sigma}^+}\right)\hat{a}_\pm\right]$$

Exchange of momentum with atoms

We obtain for the total momentum

$$\frac{\mathrm{d}\hat{p}}{\mathrm{d}t} + \frac{\mathrm{d}\hat{P}_+}{\mathrm{d}t} + \frac{\mathrm{d}\hat{P}_-}{\mathrm{d}t} = \underbrace{-2\kappa\hat{P}_+ - 2\kappa\hat{P}_-}_{\text{Cavity losses}} + \underbrace{\hbar k\left(\eta_+\hat{a}_+^\dagger + \eta_+^*\hat{a}_+ - \eta_-\hat{a}_-^\dagger - \eta_-^*\hat{a}_-\right)}_{\text{Two-side pumping}}$$

Atoms in ring cavities



We obtain the dynamical equations for the average values in the mean-field limit:

$$\langle \hat{a}_\pm \rangle = \alpha_\pm, \quad \langle \hat{a}_\pm^\dagger \rangle = \alpha_\pm^*, \quad \langle \hat{x} \rangle = x, \quad \langle \hat{p} \rangle = p$$

$$\frac{\mathrm{d}\alpha_\pm}{\mathrm{d}t} = (i\Delta_c - \kappa)\,\alpha_\pm - ig\,\mathrm{e}^{\mp ikx}\langle \hat{\sigma} \rangle^- + \eta_\pm$$

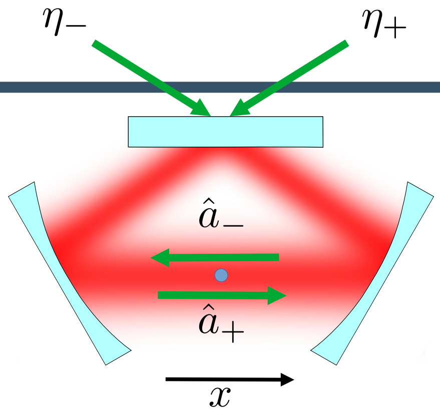
$$\frac{\mathrm{d}\langle \hat{\sigma}^- \rangle}{\mathrm{d}t} = \left(i\Delta_a - \frac{\Gamma}{2}\right)\langle \hat{\sigma}^- \rangle + ig\,\left(\alpha_- \mathrm{e}^{-ikx} + \alpha_+ \mathrm{e}^{ikx}\right)\,\langle \hat{\sigma}^z \rangle$$

$$\frac{\mathrm{d}\langle \hat{\sigma}^z \rangle}{\mathrm{d}t} = 2ig\,\left[\left(\alpha_-^* \mathrm{e}^{ikx} + \alpha_+^* \mathrm{e}^{-ikx}\right)\,\langle \hat{\sigma}^- \rangle - \left(\alpha_- \mathrm{e}^{-ikx} + \alpha_+ \mathrm{e}^{ikx}\right)\,\langle \hat{\sigma}^+ \rangle\right] - \Gamma\,(1 + \langle \hat{\sigma}^z \rangle)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{p}{m}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = ig\hbar k\,\left[\left(\alpha_+^* \mathrm{e}^{-ikx} - \alpha_-^* \mathrm{e}^{ikx}\right)\,\langle \hat{\sigma}^- \rangle - \left(\alpha_+ \mathrm{e}^{ikx} - \alpha_- \mathrm{e}^{-ikx}\right)\,\langle \hat{\sigma}^+ \rangle\right]$$

Atoms in ring cavities



The Rabi frequency of the superposition of light modes is

$$\Omega(x) = 2g\left(\alpha_- \mathrm{e}^{-ikx} + \alpha_+ \mathrm{e}^{ikx}\right)$$

For large Γ, the internal state of the atom evolves in a timescale much faster than the external state → adiabatic elimination of $\langle\hat{\sigma}^-\rangle,\ \langle\hat{\sigma}^z\rangle$

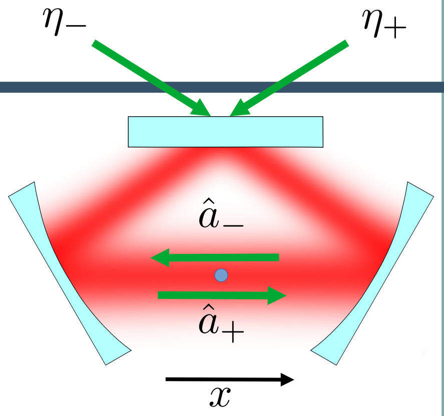
Steady-state solution for $\langle\hat{\sigma}^-\rangle,\ \langle\hat{\sigma}^z\rangle$: $\quad \langle\hat{\sigma}^z\rangle = -\dfrac{1}{1+s(x)} \qquad s \equiv \dfrac{\frac{|\Omega(x)|^2}{2}}{\Delta_a^2 + \frac{\Gamma^2}{4}}$

$$\langle\hat{\sigma}^-\rangle = \frac{i}{1+s(x)}\,\frac{\frac{\Omega(x)}{2}}{i\Delta_a - \frac{\Gamma}{2}}$$

Replacing at the other dynamical equations, we get

$$\frac{\mathrm{d}\alpha_\pm}{\mathrm{d}t} = (i\Delta_c - \kappa)\,\alpha_\pm + \eta_\pm + \frac{1}{1+s}\,\frac{g^2}{i\Delta_a - \frac{\Gamma}{2}}\,\mathrm{e}^{\mp ikx}\left(\alpha_+ \mathrm{e}^{ikx} + \alpha_- \mathrm{e}^{-ikx}\right)$$

Atoms in ring cavities



$$\frac{\mathrm{d}\alpha_\pm}{\mathrm{d}t} = \eta_\pm - \kappa\left(1 - i\bar{\Delta}_c + \underbrace{\frac{1}{1+s(z)}\frac{\Upsilon}{2}\frac{1}{1-i\bar{\Delta}_a}}_{\text{Nonlinear impact of atomic cloud on cavity}}\right)\alpha_\pm - \underbrace{\kappa\frac{\mathrm{e}^{\mp 2ikx}}{1+s(z)}\frac{\Upsilon}{2}\frac{1}{1-i\bar{\Delta}_a}\alpha_\mp}_{\text{(Nonlinear) coupling between light modes}}$$

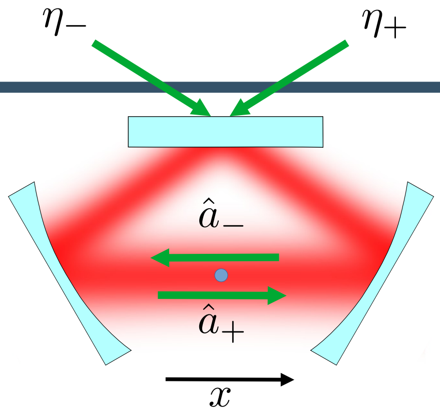
For the momentum,

$$\frac{\mathrm{d}p}{\mathrm{d}t} = ig\hbar k\left[\left(\alpha_+^*\mathrm{e}^{-ikx} - \alpha_-^*\mathrm{e}^{ikx}\right)\frac{1}{1+s}\frac{ig(\alpha_+\mathrm{e}^{ikx} + \alpha_-\mathrm{e}^{-ikx})}{i\Delta_a - \frac{\Gamma}{2}} - \mathrm{c.c.}\right]$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{g^2\hbar k}{1+s(z)}\left[\underbrace{\frac{\Gamma\left(|\alpha_+|^2 - |\alpha_-|^2\right)}{\Delta_a^2 + \frac{\Gamma^2}{4}}}_{\text{Radiation pressure}} + \underbrace{2|\alpha_+||\alpha_-|\frac{2\Delta_a}{\Delta_a^2 + \frac{\Gamma^2}{4}}\sin\left(2kx + \arg\left(\alpha_+ - \alpha_-\right)\right)}_{\text{Dipolar force}}\right]$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \kappa\Upsilon\frac{\hbar k}{1+s(z)}\left[\frac{|\alpha_+|^2 - |\alpha_-|^2}{1+\bar{\Delta}_a^2} + \frac{\bar{\Delta}_a}{1+\bar{\Delta}_a^2}|\alpha_+||\alpha_-|\sin\left(2kx + \arg\left(\alpha_+ - \alpha_-\right)\right)\right]$$

Atoms in ring cavities



In linear regime $(s(z) \ll 1)$,

$$\frac{\mathrm{d}\alpha_\pm}{\mathrm{d}t} = \eta_\pm - \kappa\left(1 - i\bar{\Delta}_c + \frac{\Upsilon}{2}\frac{1}{1-i\bar{\Delta}_a}\right)\alpha_\pm - \kappa \mathrm{e}^{\mp 2ikx}\frac{\Upsilon}{2}\frac{1}{1-i\bar{\Delta}_a}\alpha_\mp$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \kappa\Upsilon\,\hbar k\left[\frac{|\alpha_+|^2 - |\alpha_-|^2}{1+\bar{\Delta}_a^2} + \frac{\bar{\Delta}_a}{1+\bar{\Delta}_a^2}|\alpha_+|\,|\alpha_-|\,\sin\left(2kx + \arg\left(\alpha_+ - \alpha_-\right)\right)\right]$$

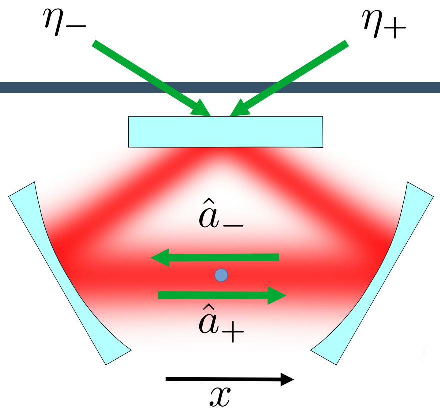
Coupled equations for two light modes + atomic external degrees of freedom

The two components of the force present very different behavior:

- **Radiation pressure**: dissipative force. The fluctuations of the force create heating (scattered photons)

$$F_{rp} = \Gamma_{ph}^{+}\hbar k + \Gamma_{ph}^{+}(-\hbar k) = \frac{\kappa\Upsilon}{1+\bar{\Delta}_a^2}\hbar k\left(|\alpha_-|^2 - |\alpha_-|^2\right)$$

Atoms in ring cavities



- **Dipolar force**: Conservative force. It is derived from a potential proportional to the light intensity

$$I \propto \left|\alpha_+ e^{ikx} + \alpha_- e^{-ikx}\right|^2 = |\alpha_-|^2 + |\alpha_-|^2 + 2\,|\alpha_-|\,|\alpha_+|\cos(2kx + \arg(\alpha_+ - \alpha_-))$$

$$U_d \propto I$$

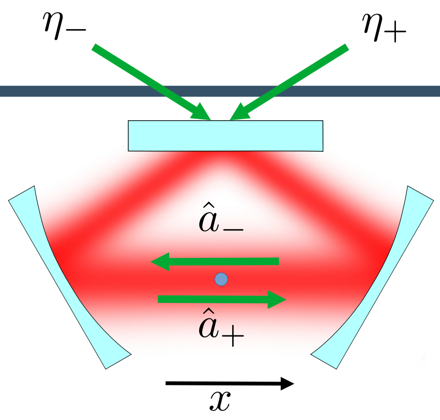
$$F_d = -\frac{\mathrm{d}U_d}{\mathrm{d}x} \propto 4k\,|\alpha_-|\,|\alpha_+|\sin(2kx + \arg(\alpha_+ - \alpha_-))$$

For high enough light detuning $(\bar{\Delta}_a \gg 1)$, one can eliminate the radiation pressure and keep only a **coherent** exchange between atomic and light degrees of freedom

$$\frac{\mathrm{d}\alpha_\pm}{\mathrm{d}t} = \eta_\pm - \kappa\left(1 - i\bar{\Delta}_c + \frac{\Upsilon}{2}\frac{1}{1 - i\bar{\Delta}_a}\right)\alpha_\pm - \kappa e^{\mp 2ikx}\frac{\Upsilon}{2}\frac{1}{1 - i\bar{\Delta}_a}\alpha_\mp$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\kappa\Upsilon\,\hbar k}{\bar{\Delta}_a}\,|\alpha_+|\,|\alpha_-|\,\sin\left(2kx + \arg\left(\alpha_+ - \alpha_-\right)\right)$$

Collective Atomic Recoil Laser (CARL)



We consider now that pumping happens from a single side + with steady-state amplitude

$$\alpha_+ = \frac{\eta_+}{\kappa}\,, \quad |\alpha_+| \gg |\alpha_-|$$

The new dynamics reads

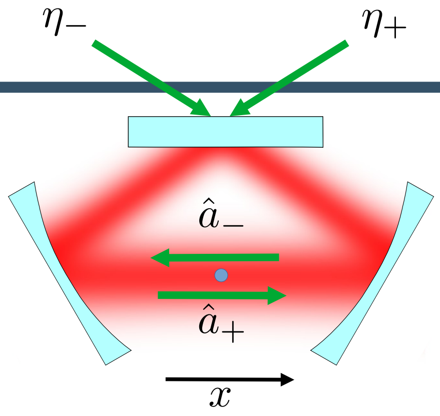
$$\frac{\mathrm{d}\alpha_-}{\mathrm{d}t} = -\kappa\left(1 - i\bar{\Delta}_c\right)\alpha_- - i\eta_+ \mathrm{e}^{2ikx}\frac{\Upsilon}{2\bar{\Delta}_a}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{|\eta_+|\,\Upsilon\,\hbar k}{\bar{\Delta}_a}\,|\alpha_-|\,\sin\left(2kx + \arg\left(\eta_+ - \alpha_-\right)\right)$$

The atomic position couples to the phase of the light mode α_- , creating positive or negative feedback on the light-momentum coupling

For atoms equally distributed within the cavity mode, coupling averages to zero!

Collective Atomic Recoil Laser (CARL)



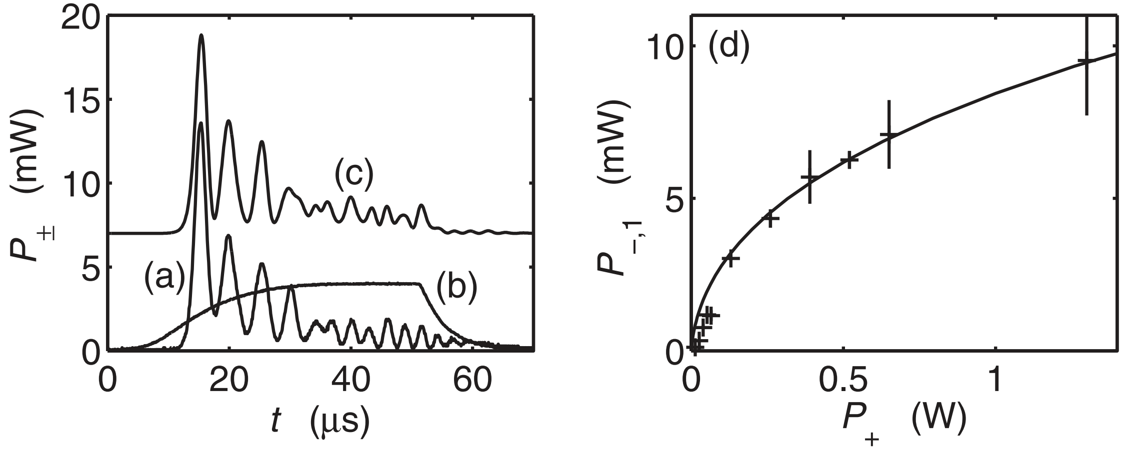
We suppose now N in the cavity, bunched atoms, with positions

$$x_j = \frac{\pi}{2} + \frac{n\pi}{k} - \frac{\arg(\eta_+)}{2k}\ ,\ n \in \mathbb{N}$$

$$\frac{\mathrm{d}\alpha_-}{\mathrm{d}t} = -\left(\kappa - i\Delta_c\right)\alpha_- + |\eta_+|\,\frac{N\Upsilon}{2\bar{\Delta}_a}$$

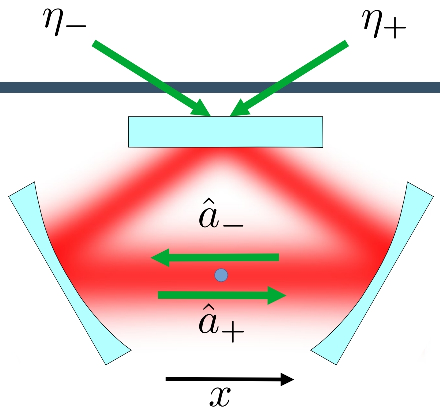
$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{|\eta_+|\,\Upsilon\,\hbar k}{\bar{\Delta}_a}\,\Re[\alpha_-]$$

Positive feedback: Atomic acceleration and exponential light emission!



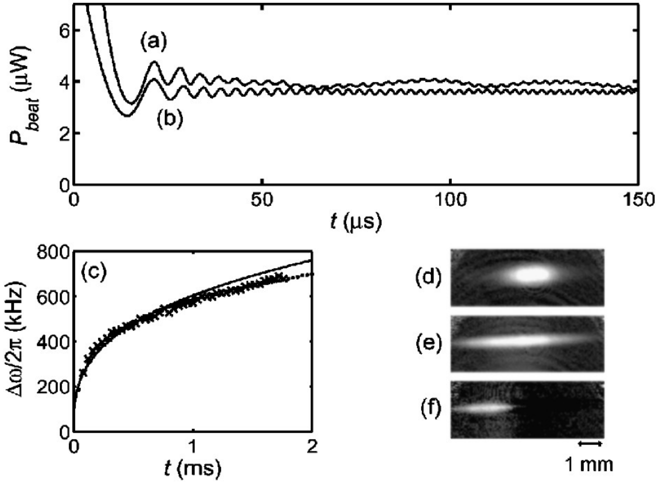
Slama, S., et al. "Superradiant Rayleigh scattering and collective atomic recoil lasing in a ring cavity." *Physical review letters* 98.5 (2007): 053603.

Collective Atomic Recoil Laser (CARL)



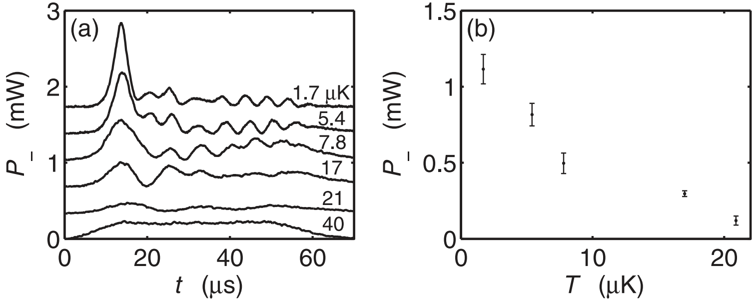
Doppler effect: Atomic movement shifts the resonance:

- Shifted frequency of light emission



Kruse, D., et al. "Observation of lasing mediated by collective atomic recoil." Physical review letters 91.18 (2003): 183601.

- Impact of temperature in reducing CARL effect



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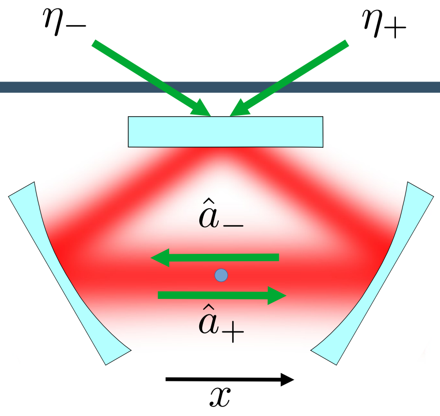
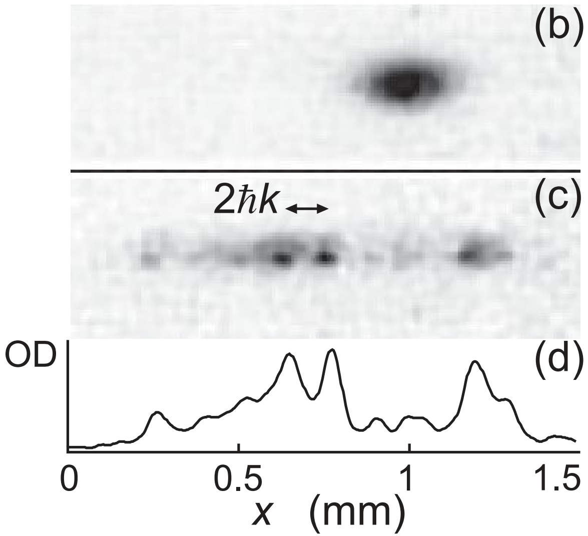


For cold enough atoms: Observation of quantization of momentum kicks (not included in our pure classical model)



Conservative interaction with light implies redistribution of photons from one mode to the other – momentum exchange in packets of $2\hbar k$

Slama, S., et al. "Superradiant Rayleigh scattering and collective atomic recoil lasing in a ring cavity." *Physical review letters* 98.5 (2007): 053603.