IFT-Perimeter-SAIFR
Journeys into Theoretical Physics 2026
Saturday Exam

• Write your name on each page

• Number each page used to solve a given problem as $1/n, 2/n, \ldots, n/n$ where n is the number of pages used to solve that problem

• Do not solve more than one problem per page – these exams will be split apart and graded by different people.

• Problem 1 (A flat-spacetime theory of gravity): 25%

• Problem 2 (Determinants from Bosons and Fermions): 25%

• Problem 3 (Electrostatics on a sphere): 25%

• Problem 4 (Bohr-Sommerfeld quantization condition via the WKB scheme): 25%

Suggestion: Try to first do the easiest parts of each exercise, and then try to do the harder parts on as many exercises as possible. This is a difficult exam, so do not be discouraged if you get stuck on an exercise.

1 A flat-spacetime theory of gravity

Is it sensible to have gravitational fields in special relativity? In this problem, we will explore this possibility in the case of a scalar gravitational field $\Phi(x)$ in flat spacetime, with Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$. We will use units where the speed of light and gravitational constant are $c = G = 1$ throughout.

We consider a test particle of mass m in the scalar gravitational field Φ. Its path through spacetime is recorded as $z^\mu(\lambda) = (z^0(\lambda), z^1(\lambda), z^2(\lambda), z^3(\lambda))$, where λ is just a bookkeeping parameter telling you where you are along the worldline. The proper time τ of the test particle is defined by:

$$d\tau = \left(-\eta_{\mu\nu}\frac{dz^\mu}{d\lambda}\frac{dz^\nu}{d\lambda}\right)^{1/2} d\lambda.$$

The (normalized) four-velocity of the particle along its worldline is denoted as

$$u^\mu = \frac{dz^\mu}{d\tau} = (\gamma, \gamma\mathbf{v}) , \qquad u^\mu u^\nu \eta_{\mu\nu} = -1 .$$

where γ is the Lorentz factor for dilations.

1. a) **[5pt]** First, take $\Phi(x)$ to be a fixed background gravitational field. Derive the relativistic differential equations governing the particle's motion in that field,

$$\frac{du^\mu}{d\tau} = -\left(\eta^{\mu\nu} + u^\mu u^\nu\right) \left.\frac{\partial\Phi}{\partial x^\nu}\right|_{x^\nu = z^\nu(\tau)} , \tag{1}$$

by extremizing the action S below (say, using the Euler–Lagrange equations):

$$S = -m \int e^{\Phi(z(\lambda))} \left(-\eta_{\mu\nu}\frac{dz^\mu}{d\lambda}\frac{dz^\nu}{d\lambda}\right)^{1/2} d\lambda . \tag{2}$$

   b) **[3pt]** In the limit where $\Phi(x)$ is static and the particle is moving very slowly, derive the non-relativistic limit of the equations (1). Why does it make sense to call Φ a Newtonian gravitational potential?

2. a) **[4pt]** Now, treat the particle moving along $z^\mu(\tau)$ as a source point of mass m. This source will generate a scalar gravitational field $\Phi(x)$, which we now treat as dynamical. Obtain the field equation for $\Phi(x)$, by extremizing the action S_Φ below:

$$S_\Phi = \int \mathcal{L}\, d^4x , \qquad \mathcal{L} = -\frac{1}{8\pi G}\eta^{\mu\nu}\frac{\partial\Phi}{\partial x^\mu}\frac{\partial\Phi}{\partial x^\nu} - \int m e^{\Phi(x)} \delta^4[x - z(\tau)]\, d\tau. \tag{3}$$

   Hint: Your equation of motion should contain both derivatives and integrals. It may be convenient to use the d'Alembert operator $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = -\partial_t^2 + \nabla^2$.

   b) **[1pt]** Show that the second term in S_Φ (the integral over the delta function) is really just the action S in (2).

c) **[4pt]** Assume that the source particle is at rest and the gravitational field is static. If you further assume the weak field approximation $e^{\Phi(x)} \simeq 1$, show that your field equation for $\Phi(x)$ reduces to the Poisson equation:

$$\nabla^2 \Phi = 4\pi G\, m\, \delta^{(3)}(\mathbf{x})\,. \tag{4}$$

Give a nontrivial solution $\Phi(x)$ to this equation, with boundary condition $\Phi \to 0$ at infinity.

d) **[2pt]** Is your weak field approximation a good one at the surface of the Earth?
Hint: Restoring G and c for a moment, the following data may be useful:
Earth mass: $M_{\text{Earth}} \simeq 6 \times 10^{24}$ kg
Earth radius: $R_{\text{Earth}} \simeq 6 \times 10^{6}$ m
Gravitational constant: $G \simeq 7 \times 10^{-11}$ m^3 kg^{-1} s^{-2}
Speed of light: $c \simeq 3 \times 10^{8}$ m s^{-1}

3. a) **[3pt]** You will now study the limit of a zero rest-mass particle in the equations of motion in part 1. Do this by using a parameter λ different from proper time, chosen so that

$$k^\mu = \frac{dz^\mu}{d\lambda} \tag{5}$$

is the energy-momentum vector, and by taking the limit $m \to 0$ with $k^0 = \gamma m = E$ remaining finite (so in particular $u^0 = \gamma \to \infty$). Use the equations of motion to show that the quantities

$$k^\mu e^\Phi \tag{6}$$

are constants of motion.

b) **[3pt]** Deduce the trajectory of a light ray as a function of coordinate time by evaluating the following integral:

$$\int_{t_0}^{t} \frac{dz^i}{dt'}\, dt' = \ldots\,. \tag{7}$$

Do massive objects bend light in our flat-spacetime theory of gravity?

2 Determinants from Bosons and Fermions

Consider a system of M complex bosons $(x_j, \bar{x}^j)$ for $j = 1$ to M, and N complex fermions $(\psi_r, \bar{\psi}^r)$ for $r = 1$ to N. Any boson commutes with any other boson and with any fermion, e.g. $x_i x_j = x_j x_i$ and $x_i \psi_r = \psi_r x_i$. And any fermion anticommutes with any fermion, e.g. $\psi_r \psi_s = -\psi_s \psi_r$.

Define the norm $C = \sum_{j=1}^{M} x_j \bar{x}^j + \sum_{r=1}^{N} \psi_r \bar{\psi}^r = \sum_{j=1}^{M} \bar{x}^j x_j - \sum_{r=1}^{N} \bar{\psi}^r \psi_r$.

1. (a) **[1pt]** Show that $C = C^\dagger$ where $\dagger$ is the Hermitian conjugate defined such that $(AB)^\dagger \equiv B^\dagger A^\dagger$.

   (b) **[1pt]** Show that C is preserved by infinitesimal transformations

$$\delta x_j = \sum_{k=1}^{M} F_j^k x_k, \quad \delta \bar{x}^j = -\sum_{k=1}^{M} F_k^j \bar{x}^k, \quad \delta\psi_r = \delta\bar{\psi}^r = 0$$

   where F_j^k is an anti-Hermitian matrix.

   (c) **[1pt]** Find the condition on the matrix P_j^k such that C is preserved by the finite transformation

$$x_j \to \sum_{k=1}^{M} P_j^k x_k, \quad \psi_r \to \psi_r.$$

   (d) **[3pt]** Find the conditions on the matrices (G_j^r, H_r^j, I_r^s) such that C is preserved by the infinitesimal transformations

$$\delta x_j = \sum_{k=1}^{M} F_j^k x_k + \sum_{s=1}^{N} G_j^s \psi_s$$

$$\delta \psi_r = \sum_{k=1}^{M} H_r^k x_k + \sum_{s=1}^{N} I_r^s \psi_s$$

   Note that the elements of F_j^k and I_r^s are bosons and the elements of G_j^s and H_r^k are fermions.

   (e) **[3pt]** Find the conditions on the matrices $(P_j^k, Q_j^r, R_r^j, S_r^s)$ such that C is preserved by the finite transformation

$$x_j \to \sum_{k=1}^{M} P_j^k x_k + \sum_{s=1}^{N} Q_j^s \psi_s$$

$$\psi_r \to \sum_{k=1}^{M} R_r^k x_k + \sum_{s=1}^{N} S_r^s \psi_s$$

2. (a) **[3pt]** Suppose P_j^k is a Hermitian matrix with non-negative eigenvalues. Show that

$$(\det P)^{-1} = (\pi)^{-M} \prod_{j=1}^{M} \int d\bar{x}^j dx_j \exp\left(-\sum_{j,k=1}^{M} \bar{x}^k P_k^j x_j \right)$$

   where the integral $\int d\bar{x}^j dx_j$ is over the complex plane for each j.
   Hint: $\int d\bar{x} dx e^{-\bar{x}x} = 2\pi \int_0^\infty dr r e^{-r^2} = -\pi e^{-r^2}|_{r=0}^{r=\infty} = \pi$.

(b) **[3pt]** Show that

$$(\det S) = \prod_{r=1}^{N} \frac{d}{d\bar{\psi}^r} \frac{d}{d\psi_r} \exp\left(\sum_{r,s=1}^{N} \bar{\psi}^s S^r_s \psi_r \right)$$

where S^r_s is a Hermitian matrix.

(c) **[7pt]** Show that

$$(\pi)^{-M} \prod_{j=1}^{M} \int d\bar{x}^j dx_j \prod_{r=1}^{N} \frac{d}{d\bar{\psi}^r} \frac{d}{d\psi_r}$$

$$\exp\left(- \sum_{j,k=1}^{M} \bar{x}^k P^j_k x_j + \sum_{r=1}^{N} \sum_{j=1}^{M} \bar{x}^j Q^r_j \psi_r + \sum_{j=1}^{M} \sum_{r=1}^{N} \bar{\psi}^r \bar{Q}^j_r x_j + \sum_{r,s=1}^{N} \bar{\psi}^s S^r_s \psi_r \right)$$

$$= (\det P)^{-1} (\det T)$$

where T is an $M \times M$ matrix defined by

$$T^s_r \equiv S^s_r + \sum_{j,k=1}^{M} \bar{Q}^k_r (P^{-1})^j_k Q^s_j.$$

Hint: Complete the square.

(d) **[3pt]** Show that $(\det P)^{-1}(\det T) = (\det U)^{-1}(\det S)$ where U is an $N \times N$ matrix defined by

$$U^k_j \equiv P^k_j + \sum_{r,s=1}^{N} Q^r_j (S^{-1})^s_r \bar{Q}^k_s.$$

Hint: Complete the square in a different way.

3 Electrostatics on a Three-Sphere

In this problem the space-time is $\mathbb{R} \times S^3$ (the first factor is time) instead of the Minkowski $\mathbb{R} \times \mathbb{R}^3$.

We will consider electrostatics of N static point-like charges $q_1, \ldots, q_N$ at positions $\vec{n}_1, \ldots, \vec{n}_N$ on the sphere; that is $\vec{n}$ is four component vector with $\vec{n}^2 = R^2$. So time will not play a role there and instead we have the electrostatic potential energy

$$U = \sum_{j<k}^{N} q_j q_k G(\vec{n}_j, \vec{n}_k) \tag{8}$$

where the sphere green's function G obeys

$$\Delta G(\vec{n}, \vec{n'}) = -\delta_{S^3}(\vec{n}, \vec{n}') + \mathbb{C} \tag{9}$$

As we will review below, the constant $\mathbb{C}$ is not there in Minkowski but it is present here because of the compactness of the space.

Throughout this problem $\delta_{\mathcal{M}}(\vec{r}, \vec{r}')$ is the Dirac delta function on $\mathcal{M}$ defined as the distribution such that for any smooth function on $\mathcal{M}$, $f(\vec{r}) = \int_{\mathcal{M}} \delta_{\mathcal{M}}(\vec{r}, \vec{r}') f(\vec{r}')$. For $\mathcal{M} = \mathbb{R}^3$ this is the usual three-dimensional delta function.

1. **[2pt]** In flat space $\Delta G(\vec{r}, \vec{r'}) = -\delta_{R^3}(\vec{r} - \vec{r}')$ is given by

$$G(\vec{r}, \vec{r}') = \frac{1}{4\pi |\vec{r} - \vec{r}'|} \,. \tag{10}$$

   Derive the Coulomb force between two charged particles from the variation of their potential energy.

2. In a compact space $\mathcal{M}$ such as S^3, the equation

$$\Delta G(\vec{r}, \vec{r}') = -\delta_{\mathcal{M}}(\vec{r}, \vec{r}') \,, \qquad \text{naive equation} \,. \tag{11}$$

   does not make sense.

   (a) **[2pt]** Why is (11) not a reasonable equation? (Hint: What operation can we do on both sides of this equation to reach a contradiction?)

   (b) **[2pt]** If we add a constant to the right hand side of this equation as in (9) then

$$\Delta G(\vec{r}, \vec{r}') = -\delta_{\mathcal{M}}(\vec{r}, \vec{r}') + \mathbb{C}$$

   can make sense. What value of $\mathbb{C}$ – simply related to the volume of this compact space – makes this equation well defined?

   (c) **[2pt]** Consider the full electrostatic potential for a bunch of charges $\phi(\vec{n}) = \sum_{j=1}^{N} q_j G(\vec{n}, \vec{n}_j)$ which should obey

$$\Delta \phi = -\sum_j q_j \delta_{\mathcal{M}}(\vec{n}, \vec{n}_j) \tag{12}$$

   without any constant on the right hand side. What happened to the constant $\mathcal{C}$? What condition n the charges do we need to impose? Why is this a physical condition to impose in a compact space but one we can relax in infinite space?

3. For the sphere we can parametrize our vectors as

$$\vec{n} = R(\cos(\chi), \sin(\chi)\cos(\phi), \sin(\chi)\sin(\phi)\cos(\theta), \sin(\chi)\sin(\phi)\sin(\theta)) \tag{13}$$

where $0 < \chi < \pi$.

(a) **[1pt]** Show that this is indeed a point on S^3.

(b) **[1pt]** Explain why the geodesic distance d between two points on the sphere is given by

$$\cos(d/R) = \vec{n} \cdot \vec{n}' \tag{14}$$

The Green's function $G(\vec{n}, \vec{v}')$ is a simple function of this distance, $G = G(d/R)$.

(c) **[1pt]** To compute this green's function between two points, explain why – without loss of generality – we can always put one point at the north pole (with $\chi = 0$). Show that the distance is then simply given by $d/R = \chi$ where χ is the coordinate of the second point.

(d) **[6pt]** The Laplacian acting on the sphere acting on functions of χ alone acts as

$$\Delta G(\chi) = \frac{1}{R^2}(G'' + 2\cot(\chi)G') \,. \tag{15}$$

Show that

$$G(\chi) = \frac{1}{4\pi^2 R}(\pi - \chi)\cot(\chi) \tag{16}$$

Is a good sphere green's function.

Hint: You only need to check that it behaves as expected for $\chi \to 0$ and that the Laplacian of this function is what it should be for other χ's. If you plug (16) into the right hand side of (15) you get $1/(2\pi^2 R^3)$.

(e) **[3pt]** Consider two opposite charges on the sphere. Show that the force between them has magnitude

$$|F| = \frac{q^2}{4\pi^2 R^2}(\cot(\chi) + (\pi - \chi)\csc^2(\chi)) \tag{17}$$

Sketch its direction on a cartoon of a sphere with two particles on it.

(f) **[5pt]** Discuss what happens for $\chi \to 0$ and for $\chi \to \pi$ and why this is precisely the expected bahavior. Can we have two opposite charged particles on the sphere in a static equilibrium configuration? Is that equilibrium configuration stable or unstable?

4 Bohr-Sommerfeld quantization condition via the WKB scheme

The Wentzel–Kramers–Brillouin (WKB) method is an approximation scheme to obtain the wavefunctions and the energies of the Schrödinger equation, expressed as *formal* power series in powers of $\hbar$, viewed as a small parameter, around the point $\hbar = 0$.

Consider a one-dimensional particle, having mass m, subjected to the potential $V(x)$. We will assume that $V(x)$ satisfies the condition

$$V(x) \to \infty \text{ if } |x| \to \infty, \tag{18}$$

which, in turns, implies that the spectrum is discrete with eigenvalues E_n, $n = 0, 1, 2, \ldots$. Let us rewrite the Schrödinger equation in the following equivalent form

$$\hbar^2 \psi''(x) + p^2(x)\psi(x) = 0, \qquad p(x) \equiv \sqrt{2m\,(E - V(x))}. \tag{19}$$

1. **(2 points)** Consider the quantity $p(x)$ in Eq. (19). What is the classical interpretation of the positions x such that $2m\,(E - V(x)) \geq 0$?

2. **(5 points)** Consider the following Ansatz for the solution $\psi(x)$

$$\psi(x) = \exp\left[\frac{i}{\hbar}\int^x Y(x')\mathrm{d}x'\right]. \tag{20}$$

Show that the Schrödinger equation, Eq. (19), gets rewritten in terms of $Y(x)$ as

$$Y^2(x) - i\hbar Y'(x) = p^2(x). \tag{21}$$

3. **(7 points)** By writing $Y(x)$ as a formal power series in $\hbar$

$$Y(x) = \sum_{k=0}^{\infty} Y_k(x)\hbar^k, \tag{22}$$

Show that the function $Y_k(x)$ obey the following recursion relations

$$\begin{aligned} Y_0(x) &= \pm p(x), \\ Y_1(x) &= \frac{i}{2}\frac{Y_0'(x)}{Y_0(x)}, \\ Y_{n+1}(x) &= \frac{1}{2Y_0(x)}\left(iY_n'(x) - \sum_{k=1}^{n} Y_k(x)Y_{n+1-k}(x)\right) \qquad n \geq 1. \end{aligned} \tag{23}$$

In particular, show that $Y_2(x)$ can be written as

$$Y_2(x) = -\frac{1}{8}\frac{(Y_0'(x))^2}{Y_0^3(x)} + \text{total derivative}. \tag{24}$$

4. **(6 points)** To proceed, let us split the series expansion of $Y(x)$ into even and odd (in powers of $\hbar$) parts

$$Y(x) = P(x) + Y_{\text{odd}}(x) \Rightarrow P(x) = \sum_{k=0}^{\infty} Y_{2k}(x)\hbar^{2k}, \; Y_{\text{odd}}(x) = \sum_{k=0}^{\infty} Y_{2k+1}(x)\hbar^{2k+1}. \tag{25}$$

Show that, accordingly, Eq. (21) splits into two equations

$$\begin{aligned} \text{even}: &\quad Y_{\text{odd}}^2(x) + P^2(x) - i\hbar Y'_{\text{odd}}(x) = p^2(x), \\ \text{odd}: &\quad 2Y_{\text{odd}}(x)P(x) - i\hbar P'(x) = 0. \end{aligned} \tag{26}$$

Moreover, show that the second equation can be solved to give

$$Y_{\text{odd}}(x) = \frac{i\hbar}{2}\frac{\mathrm{d}}{\mathrm{d}x}\log P(x), \tag{27}$$

and use this result to obtain the following expression for the wavefunction

$$\psi(x) = \frac{1}{\sqrt{P(x)}}\exp\left[\frac{i}{\hbar}\int^x P(x)\mathrm{d}x\right]. \tag{28}$$

5. **(5 points)** Consider the specific case of the Hamiltonian

$$H = \frac{p^2}{2m} + gx^4. \tag{29}$$

By denoting with $x_\pm$ the points on the real line such that $p(x) = 0$, the Bohr-Sommerfeld quantization condition reads

$$2\int_{x_-}^{x_+} p(x)\mathrm{d}x = 2\pi\hbar\left(n + \frac{1}{2}\right), \tag{30}$$

which can be viewed as a semiclassical approximation for the energy levels of a given Hamiltonian. Given the identification (we are taking the positive solution in Eq. (23)) $Y_0(x) = p(x)$, Eq. (30) can be promoted to the following expression

$$\oint_{\mathcal{A}} P(x)\mathrm{d}x = 2\pi\hbar\left(n + \frac{1}{2}\right), \tag{31}$$

with $\oint_{\mathcal{A}}$ being an appropriate contour integral, encircling $x_\pm$, which allows extending the Bohr-Sommerfeld conditions to the entire WKB Ansatz. Compute the Bohr-Sommerfeld approximation to the energy levels E_n, Eq. (30), for the Hamiltonian in Eq. (29). Express your answer in terms of the quantity

$$\int_{-1}^{1}\sqrt{1-x^4}\mathrm{d}x = \frac{\Gamma(1/4)^2}{3\sqrt{2\pi}} \equiv b_0, \tag{32}$$