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1980 J. Phys. B: At. Mol. Phys. 13 3551

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Analytical and numerical results for the steady state in cooperative resonance fluorescence†

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Received 1 February 1980

Abstract. The cooperative resonance fluorescence steady state is discussed within the context of an operator master equation which conserves total pseudospin. Emphasis throughout is on quantum fluctuations and their significance in relation to a background of factorised dynamics. Atom–atom correlations are shown to play a fundamental role for systems driven beyond the linear régime. Use of the atomic coherent state representation yields a Fokker–Planck description closely allied to the dynamics for a classical angular momentum oscillator. For intense incident fields the quantum-mechanical steady state is understood in terms of diffusion both around and between classical trajectories on the Bloch sphere. In the limit of infinite systems simple closed-form expressions for steady-state features are derived. Coherent and incoherent fluorescent intensities are obtained together with the second-order correlation function for fluorescent light. Specific features are illustrated by numerical results for systems of from two to fifty atoms.

1. Introduction

Cooperative effects arising in the coupling of a collection of two-level atoms to a common radiation field have been known and discussed for a number of years. In particular, superradiance and superfluorescence have both seen extensive development since Dicke first addressed the question of cooperative spontaneous emission (Dicke 1954, Dillard and Robl 1969, Agarwal 1970, 1971, 1974, Dialetis 1970, Lemberg 1970, Bonifacio *et al* 1971, Eberly 1972, Glauber and Haake 1972, 1976, Stenholm 1972, Friedberg and Hartmann, 1974a, b, Bonifacio and Lugiato 1975). On the other hand, cooperative resonance fluorescence has only attracted wide attention more recently. The current interest follows success in the study of single-atom resonance fluorescence (Mollow 1969, 1975, Carmichael and Walls 1976, Hartig *et al* 1976, Kimble and Mandel 1976, Cohen-Tannoudji and Reynaud 1977, Dagenais and Mandel 1978) and is a further expression of the general interest in cooperative phenomena far from thermal equilibrium (Haken 1977, Nicolis and Prigogine 1977).

It is useful at the outset to contrast cooperative resonance fluorescence briefly with the more familiar superradiant effect. For a system of N atoms, the hallmark of superradiant emission is an intensity proportional to N^2 . Independent emission of spontaneous photons would give a linear dependence on the number of atoms. Cooperative emission from a Dicke superradiant state is in fact an essentially classical effect, understandable as radiation from a classical dipole prepared by an initial phasing

† Work supported in part by a grant from the Robert A Welch Foundation.

of the atomic state. A more profound consequence of cooperativity is revealed in superfluorescence, where, in an initially inverted population, such a dipole moment arises spontaneously from quantum fluctuations. Cooperative resonance fluorescence is an extension of the Dicke model for superradiance. Here N two-level atoms are confined to a volume less than a resonant wavelength cubed, and coherently driven, on resonance, by intense laser light. We are no longer dealing with a transient effect and long-time evolution takes the atomic system to a steady state (Senitzky 1972, Drummond and Carmichael 1978, Narducci *et al* 1978a, Puri and Lawande 1979). Introduction of a coherent driving field imposes a further major distinguishing feature; the continued presence of an external agent inducing a phased dipole moment. Classical radiation from this collective dipole offers itself as a simple explanation for fluorescent intensities proportional to N^2 .

Superficially then, an N^2 dependence in cooperative resonance fluorescence appears to be trivially imposed rather than a consequence of communication via radiation reaction. However, this is surely not the full story. As is the case in single-atom resonance fluorescence, an N -atom system is saturated by sufficiently high laser intensities (Senitzky 1972, Narducci *et al* 1978a, Puri and Lawande 1979). In this limit the mean dipole moment vanishes (Drummond and Carmichael 1978). The persistence of a fluorescent intensity proportional to N^2 calls for an understanding of quantum fluctuations. Moreover, atom-atom correlations generated via radiation reaction appear to be fundamental in restricting the dephasing tendency of fluctuations. Destruction of such correlations allows evolution between states of different cooperation number and yields a dramatically different behaviour. Features analogous to absorptive optical bistability have been predicted (Carmichael and Walls 1977, Walls *et al* 1978) where for strong incident fields fluorescent intensities are proportional to N rather than N^2 (Carmichael and Walls 1977, Bonifacio and Lugiato 1978, Narducci *et al* 1978b, Lugiato 1979). We might suggest that cooperative resonance fluorescence operates at two levels. When incident intensities are low, fluorescence is described simply as classical radiation from an induced collective dipole. On the other hand, for intense incident light cooperativity must mold collective dynamics around quantum fluctuations. The comparison is not unlike that between superradiance and superfluorescence.

A number of contributions have recently appeared discussing the spectrum for cooperative resonance fluorescence (Agarwal *et al* 1977, 1979, Mollow 1977, Amin and Cordes 1978, Mavroyannis 1978, Senitzky 1978, Carmichael 1979, Drummond and Hassan 1979, Kilin 1979, Freedhoff 1979). In the present article we are not concerned directly with the fluorescent light but focus attention on the atomic steady state. Various authors have dealt with aspects of this long-time behaviour in earlier work. Senitzky (1972) has developed a perspective contrasting classical and quantum-mechanical treatments for a macroscopic angular momentum oscillator. He makes pertinent comment on the relationship between the saturated steady state and the corresponding oscillatory classical motion. More recently, both analytical (Drummond and Carmichael 1978) and numerical techniques (Narducci *et al* 1978a) have been used to establish the distinction between cooperative resonance fluorescence and absorptive optical bistability. In the asymptotic limit $\Omega \rightarrow \infty$, $N \rightarrow \infty$, Ω/Ω_0 constant ($\Omega_0 = \frac{1}{4}N\gamma$), where 2Ω is the Rabi frequency and γ the Einstein A coefficient, cooperative resonance fluorescence displays behaviour resembling a second-order phase transition (Drummond and Carmichael 1978, Drummond 1979). This contrasts the first-order transition in absorptive optical bistability (Bonifacio and Lugiato 1976). The transition

threshold at $\Omega/\Omega_0 = 1$ marks the onset of saturation in the collective dipole moment and the conspicuous intrusion of fluctuations into the underlying dynamics.

The present publication reports numerical results illustrating steady-state features for large numbers of atoms and demonstrating the trend towards a second-order phase transition. Emphasis is placed on quantum fluctuations and correlations with explicit comparisons given between exact averages for operator products and those derived from factorisation assumptions. The second-order correlation function for fluorescent light is obtained. Numerical analyses have been reported previously for values of N ranging up to $N = 15$ (Agarwal *et al* 1977, Narducci *et al* 1978a). We complement these earlier studies by highlighting correlations and extending results to $N = 50$.

Recently an exact steady-state solution has been derived for arbitrary values of N (Puri and Lawande 1979, Kilin 1979). This does not diminish the value of the results presented here however. The analytic solution expresses matrix elements as a complicated series and itself offers little insight into steady-state features. Numerical computation of operator averages is still required other than in limiting cases.

As Senitzky has realised (Senitzky 1972), operator averages alone reveal little of the physics underlying the cooperative resonance fluorescence steady state. Recently an asymptotic analysis has been developed using a Fokker–Planck equation derived in the atomic coherent state representation (Drummond and Carmichael 1978, Drummond 1979). This formulation is valuable as a route to a more physical and intuitive understanding. Here quantum fluctuations above threshold enter as phase diffusion around classical trajectories on the Bloch sphere and as diffusion between these trajectories. It is useful to outline this Fokker–Planck treatment in some detail prior to the presentation of numerical results. Asymptotic expressions in closed form are derived for operator averages, including those corresponding to coherent and incoherent fluorescent intensities and the second-order correlation function. Only single-operator averages have previously been published (Drummond and Carmichael 1978). An asymptotic distribution on the Bloch sphere is obtained above threshold and shown to agree with the density matrix derived by Puri and Lawande (1979).

In § 2 a master equation formulation for cooperative resonance fluorescence is outlined and two factorisation schemes from the recent literature are briefly reviewed. The Fokker–Planck treatment utilising directed angular momentum states is described in § 3. Section 4 discusses the numerical results and makes comparisons with asymptotic expressions derived from the Fokker–Planck equation. Features of quantum correlations are discussed in relation to operator factorisation schemes. Conclusions are summarised in § 5. Some detailed comments on classical trajectories and details in the comparison between our asymptotic steady state above threshold and the exact steady-state density matrix have been relegated to appendices.

2. Formulation and factorisation

We consider a collection of N two-level atoms, each with resonant frequency ω_0 , confined to a volume less than a resonant wavelength cubed. The entire population is driven continuously, on resonance, by a classical radiation field

$$E(\mathbf{r}, t) = \hat{\mathbf{e}}_0 \mathcal{E}_0 \exp[-i(\omega_0 t - \mathbf{k}_0 \cdot \mathbf{r})] + \text{cc.} \quad (2.1)$$

Here $\hat{\mathbf{e}}_0$ is a unit polarisation vector, $\mathbf{k}_0$ is the incident wavevector and $\mathcal{E}_0$ is a complex amplitude. Radiative decay to the vacuum is introduced through the Dicke

Hamiltonian and with the added coherent interaction we write

$$H = \sum_{j=1}^N \hbar \omega_0 \sigma_z^j + \sum_{j=1}^N \hbar (\Omega e^{-i\omega_0 t} \sigma_+^j + \Omega^* e^{i\omega_0 t} \sigma_-^j) \\ + \sum_{k,\lambda} \hbar \omega_k a_{k,\lambda}^\dagger a_{k,\lambda} + \sum_{k,\lambda} \sum_{j=1}^N \hbar [\kappa_{k,\lambda} a_{k,\lambda} (\sigma_+^j + \sigma_-^j) + \text{HC}] \quad (2.2)$$

where

$$\Omega = -(\hat{\epsilon}_0 \cdot \boldsymbol{\mu}) \exp(i\mathbf{k}_0 \cdot \mathbf{r}_a) \hbar^{-1} \mathcal{E}_0 \quad (2.3)$$

and

$$\kappa_{k,\lambda} = -i \left(\frac{\omega_k}{2\hbar \epsilon_0 V} \right)^{1/2} \hat{\epsilon}_{k,\lambda} \cdot \boldsymbol{\mu} \exp(i\mathbf{k} \cdot \mathbf{r}_a). \quad (2.4)$$

The atomic pseudospin operators σ_z^j and $\sigma_\pm^j = \sigma_x^j \pm i\sigma_y^j$ obey commutation relations $[\sigma_+^j, \sigma_-^k] = 2\sigma_z^j \delta_{j,k}$ and $[\sigma_\pm^j, \sigma_z^k] = \mp \sigma_\pm^j \delta_{j,k}$. Each atom has a dipole moment $\boldsymbol{\mu}$ and $\mathbf{r}_a$ marks the centre of the atomic distribution. Creation and annihilation operators $a_{k,\lambda}^\dagger$ and $a_{k,\lambda}$ describe the vacuum field mode with wavevector $\mathbf{k}$, frequency ω_k and polarisation $\hat{\epsilon}_{k,\lambda}$. V is the quantisation volume and ϵ_0 is the permittivity of free space. MKS units will be used throughout.

This form for the Hamiltonian suggests that the summation over j be absorbed in the definition of a collective spin $\mathbf{S}$ with

$$S_z = \sum_{j=1}^N \sigma_z^j \quad S_\pm = S_x \pm iS_y = \sum_{j=1}^N \sigma_\pm^j. \quad (2.5)$$

The collective operators obey commutation relations $[S_+, S_-] = 2S_z$ and $[S_\pm, S_z] = \mp S_\pm$. Here $\mathbf{S}^2$ commutes with each component of spin and therefore with the prescribed Hamiltonian. However, it is well known that $\mathbf{S}^2$ conservation follows only because the Dicke Hamiltonian fails to distinguish phase factors $\exp(i\mathbf{k} \cdot \mathbf{r}_j)$ at different atomic sites. Dipole-dipole contributions which break the $\mathbf{S}^2$ symmetry have been omitted and are divergent in a small volume. Within the context of superradiance these additional terms are well studied (Friedberg *et al* 1972, Friedberg and Hartmann 1974a, b), although for cooperative resonance fluorescence work in this direction is just beginning (Freedhoff 1979, Kilin 1979). The present paper limits attention to the idealised Hamiltonian and only tentatively addresses the possible consequences of $\mathbf{S}^2$ -breaking interactions.

A master equation formulation follows from the foregoing Hamiltonian as described by Agarwal (1974) and others (Lehmberg 1970, Bonifacio and Lugiato 1975). In a frame rotating at the frequency ω_0 we may write

$$\frac{d\rho}{dt} = -i\Omega[S_+ + S_-, \rho] + \frac{1}{2}\gamma(2S_- \rho S_+ - \rho S_+ S_- - S_+ S_- \rho) \quad (2.6)$$

where ρ is the reduced density operator in the rotating frame and γ is the Einstein A coefficient. Frequency shifts have been neglected, and with an appropriate choice of phase, Ω has been taken real and positive; 2Ω is the Rabi frequency. Corresponding to $\mathbf{S}^2$ conservation in the Hamiltonian, it is readily shown that the master equation conserves $\langle \mathbf{S}^2 \rangle$. Matrix elements in the Dicke representation $|S, M\rangle$ are coupled only by M -changing interactions. For convenience we confine our attention to the manifold with maximum Dicke cooperation number $S = \frac{1}{2}N$. This corresponds for example to

evolution from the ground state. Then in addition to S_z , S_+ and S_- it is useful to define

$$s_z = (2/N)S_z \quad s_{\pm} = s_x \pm is_y = (2/N)S_{\pm}. \quad (2.7)$$

Puri and Lawande (1979) have obtained a steady-state solution to this master equation. In fact, there exists a very simple route to their result. Writing $\hat{S}_{\pm} = S_{\pm} \mp i2\Omega\gamma^{-1}$, equation (2.6) becomes

$$\frac{d\rho}{dt} = \frac{1}{2}\gamma(2\hat{S}_-\rho\hat{S}_+ - \rho\hat{S}_+\hat{S}_- - \hat{S}_+\hat{S}_-\rho). \quad (2.8)$$

The steady-state solution follows by inspection in the form†

$$\rho_{ss} = \hat{S}_-^{-1} \hat{S}_+^{-1} = \mathcal{N} \sum_{n=0}^N \sum_{m=0}^N \left(-i \frac{2\Omega}{\gamma}\right)^{N-n} \left(i \frac{2\Omega}{\gamma}\right)^{N-m} S_-^n S_+^m \quad (2.9)$$

where for the normalisation $\mathcal{N}$ we have (Puri and Lawande 1979)

$$\mathcal{N}^{-1} = \sum_{n=0}^N \sum_{m=0}^n \frac{(N-n+m)!}{(n-m)!(N-n)!} \left(\frac{2\Omega}{\gamma}\right)^{2(N-m)}. \quad (2.10)$$

We shall return to this exact solution further on. To conclude the present section, we review briefly two factorisation schemes which illustrate dramatically the importance of fluctuations and atom-atom correlations in this system.

The master equation is equivalent to a finite hierarchy for operator averages. At the lowest level we have

$$\frac{d\langle S_+ \rangle}{dt} = -2i\Omega\langle S_z \rangle + \gamma\langle S_+ S_z \rangle \quad (2.11)$$

$$\frac{d\langle S_z \rangle}{dt} = -i\Omega(\langle S_+ \rangle - \langle S_- \rangle) - \gamma\langle S_+ S_- \rangle. \quad (2.12)$$

Since correlations between like atoms are determined algebraically (commutation and anticommutation relations), we might factor only the unlike atoms (Carmichael and Walls 1977) and write

$$\frac{d\langle S_+ \rangle}{dt} = -2i\bar{\Omega}^*\langle S_z \rangle - \frac{1}{2}\gamma\langle S_+ \rangle \quad (2.13)$$

$$\frac{d\langle S_z \rangle}{dt} = -i(\bar{\Omega}\langle S_+ \rangle - \bar{\Omega}^*\langle S_- \rangle) - \gamma(\frac{1}{2}N + \langle S_z \rangle) \quad (2.14)$$

where $\bar{\Omega} = \Omega - \frac{1}{2}i\gamma(1 - 1/N)\langle S_- \rangle$ is an effective driving term incorporating the mean radiation reaction field from $N-1$ neighbours. Defining $\Omega_0 = \frac{1}{4}N\gamma$ ($N \gg 1$), in the steady state, we find

$$\langle S_x \rangle_{ss} = 0 \quad \langle S_z \rangle_{ss} = -\frac{1}{N} \frac{\langle S_y \rangle_{ss}}{(\Omega/\Omega_0) - \langle S_y \rangle_{ss}} \quad (2.15)$$

and

$$\left(\langle S_y \rangle_{ss} - \frac{\Omega}{\Omega_0}\right) \left(\langle S_y \rangle_{ss}^2 - \frac{\Omega}{\Omega_0} \langle S_y \rangle_{ss} + \frac{2}{N}\right) + \frac{2}{N^2} \langle S_y \rangle_{ss} = 0. \quad (2.16)$$

† This simple method was first described to the author by Dr P D Drummond (see also Killin 1979).

As $N \rightarrow \infty$ the role of quantum fluctuations should decrease, and indeed terms of order N^{-1} and N^{-2} become vanishingly small. However, they retain their profound influence on the form of the solutions. A first-order phase transition is predicted (figure 1(a)) with bistability in the range $2\sqrt{2/N} < \Omega/\Omega_0 < 1/\sqrt{2}$. For an S^2 -conserving system it is firmly established that this is an incorrect result[†] (Drummond and Carmichael 1978, Narducci *et al* 1978a). A second branch of solutions arises only through violation of the S^2 symmetry. Since single-atom contributions to the fluctuations have been included exactly only atom-atom correlations can rectify the situation. We return to this feature in § 4.

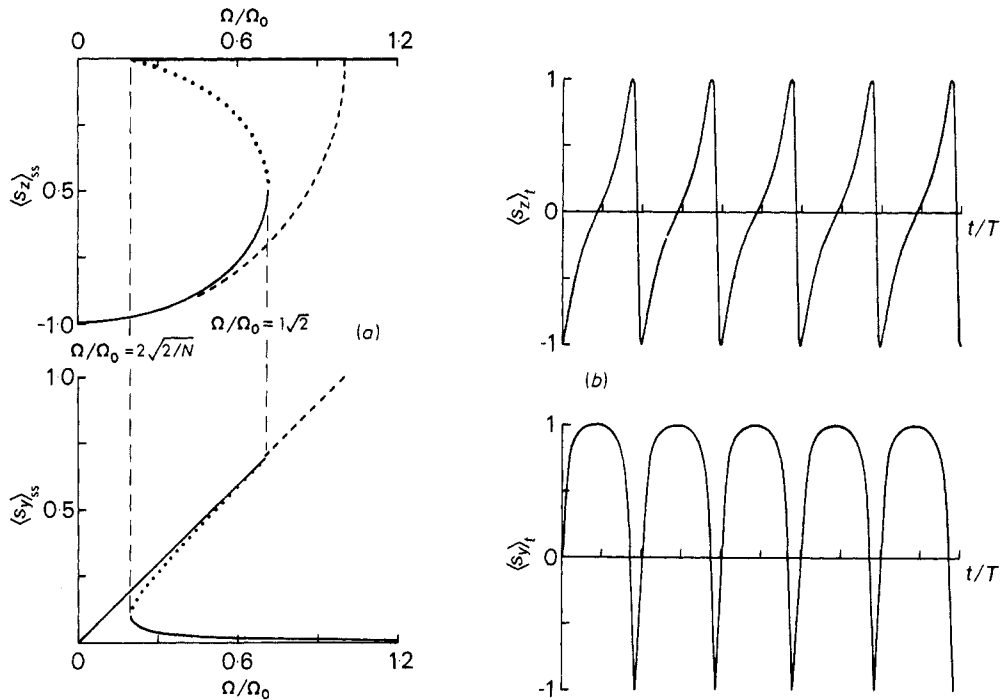


Figure 1. Long-time solutions for the two factorisation schemes. (a) Full and dotted curves represent respectively the stable and unstable steady states given by (2.15) and (2.16) ($N = 200$). Broken curves correspond to (2.20). (b) Periodic solutions (2.21) and (2.22) for $\Omega/\Omega_0 = 1.1$.

An alternative procedure factors both like and unlike atoms. In place of (2.11) and (2.12) we find

$$\frac{d\langle s_x \rangle}{dt} = -2\Omega_0 \langle s_x \rangle \langle s_z \rangle \quad (2.17)$$

$$\frac{d\langle s_y \rangle}{dt} = -2\Omega_0 \left(\frac{\Omega}{\Omega_0} - \langle s_y \rangle \right) \langle s_z \rangle \quad (2.18)$$

[†] This factorisation does, however, suggest a potential for bistability in the presence of S^2 -breaking interactions. Such an occurrence might lead to bistable devices with dimensions less than a wavelength. Bowden and Sung (1979) also describe a small-volume model which predicts bistability from imposed S^2 -breaking features.

$$\frac{d\langle s_z \rangle}{dt} = 2\Omega_0 \left(\frac{\Omega}{\Omega_0} - \langle s_y \rangle \right) \langle s_y \rangle - 2\Omega_0 \langle s_x \rangle^2. \quad (2.19)$$

Solutions lie on the unit sphere since here $\langle S^2 \rangle$ is conserved (compare $\langle S^2 \rangle$ conservation in the master equation). Senitzky (1972) has considered this set of equations in weak- and strong-field limits without giving general solutions. More recently full time-dependent solutions have been obtained (Drummond and Carmichael 1978, Kilin 1978). These are recorded for arbitrary initial conditions in appendix 1. In particular, for $\Omega/\Omega_0 \leq 1$ a steady state develops with the mean polarisation cancelling the driving field (figure 1(a)). Here

$$\langle s_x \rangle_{ss} = 0 \quad \langle s_y \rangle_{ss} = \Omega/\Omega_0 \quad \langle s_z \rangle_{ss} = -[1 - (\Omega/\Omega_0)^2]^{1/2}. \quad (2.20)$$

For more intense incident light the polarisation is constrained by the requirement $\langle s_y \rangle^2 \leq 1$, and oscillatory behaviour sets in corresponding to closed trajectories on the unit sphere. For example, an initial ground state evolves with $\langle s_x \rangle_t = 0$ and

$$\langle s_y \rangle_t = \frac{\Omega}{\Omega_0} \left(1 + \frac{(\Omega/\Omega_0) - (\Omega_0/\Omega)}{\cos(2\pi t/T + \phi) - (\Omega/\Omega_0)} \right) \quad (2.21)$$

$$\langle s_z \rangle_t = \frac{\Omega}{\Omega_0} \left[1 - \left(\frac{\Omega_0}{\Omega} \right)^2 \right]^{1/2} \frac{\sin(2\pi t/T + \phi)}{\cos(2\pi t/T + \phi) - (\Omega/\Omega_0)} \quad (2.22)$$

where $\phi = \cos^{-1} \Omega_0/\Omega$ and $T = \pi/(\Omega^2 - \Omega_0^2)^{1/2}$. The anharmonic character of these oscillations (figure 1(b)) is manifest as additional sidebands in the fluorescent spectrum (Senitzky 1978, Carmichael 1979, Drummond and Hassan 1979). Oscillatory solutions are in sharp contrast to the steady state given by (2.9). However, as we show in the following section, these features are fully reconciled in the presence of quantum fluctuations.

3. Fokker-Planck description in the atomic coherent state representation

In this section we present an expanded discussion of the Fokker-Planck formalism introduced in an earlier paper (Drummond and Carmichael 1978). This Fokker-Planck approach seems to merit such attention since it offers valuable insight into the dynamics underlying the steady state.

The atomic coherent state representation is specifically tailored for the treatment of S^2 -conserving systems and has been used by a number of authors in the context of superradiance (Narducci *et al* 1974, Narducci and Bluemel 1975, Glauber and Haake 1976). Formal aspects of this representation are adequately treated in the literature (Radcliffe 1971, Arecchi *et al* 1972), and we shall reproduce here only those features which are essential as background to our discussion. We begin with the familiar Dicke states $|S, M\rangle$, for which

$$\begin{aligned} S^2 |S, M\rangle &= S(S+1) |S, M\rangle \\ S_z |S, M\rangle &= M |S, M\rangle \\ S_{\pm} |S, M\rangle &= [(S \mp M)(S \pm M + 1)]^{1/2} |S, M \pm 1\rangle. \end{aligned} \quad (3.1)$$

The atomic coherent states, or Bloch states, $|S, \theta, \phi\rangle$ $S = (0, \frac{1}{2}), \dots, \frac{1}{2}N$ are minimum uncertainty states defined on the subspace spanned by $|S, M\rangle$ $M = -S, \dots, S$. In

particular, for $S = \frac{1}{2}N$ we have

$$|\frac{1}{2}N, \theta, \phi\rangle = \sum_{M=-N/2}^{N/2} \langle \frac{1}{2}N, M | \frac{1}{2}N, \theta, \phi \rangle |\frac{1}{2}N, M\rangle \quad (3.2)$$

where

$$\langle \frac{1}{2}N, M | \frac{1}{2}N, \theta, \phi \rangle = \binom{N}{\frac{1}{2}N + M}^{1/2} \cos(\frac{1}{2}\theta)^{\frac{1}{2}N+M} \sin(\frac{1}{2}\theta)^{\frac{1}{2}N-M} \exp[-i(\frac{1}{2}N + M)\phi]. \quad (3.3)$$

It is natural to represent Bloch states by points on the unit sphere (figure 2). Projections on the x , y and z axes then correspond respectively to $\langle s_x \rangle$, $\langle s_y \rangle$ and $\langle s_z \rangle$. An alternative parametrisation is also commonly encountered and is often convenient from a computational standpoint. Here $|\frac{1}{2}N, \theta, \phi\rangle$ is equivalently denoted $|\frac{1}{2}N, z\rangle$ with the complex variable z defined by

$$z = \tan(\frac{1}{2}\theta) e^{i\phi}. \quad (3.4)$$

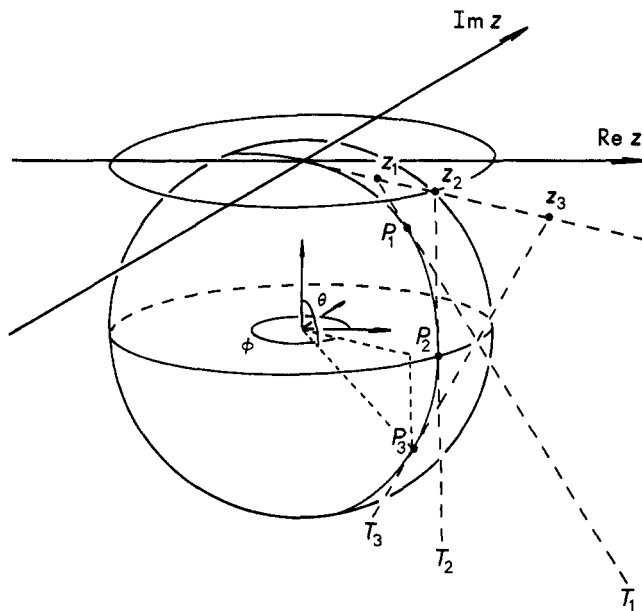


Figure 2. Geometrical representation of atomic coherent states on both the unit sphere (Bloch sphere) and the complex plane.

A geometric interpretation for this mapping is illustrated in figure 2. The complex plane is tangential to the north pole of the unit sphere (Bloch sphere). A point (θ, ϕ) is mapped to the point z by a tangent in the plane containing (θ, ϕ) and the north-south axis of the sphere. With this construction the equator maps to the unit circle, and points in the northern and southern hemispheres are respectively mapped inside and outside this circle. We shall formulate our Fokker-Planck description on the complex plane but refer regularly to dynamics on the Bloch sphere in our discussion.

Atomic coherent states form an overcomplete set, and the density operator $\rho(t)$ for a system confined to the $\frac{1}{2}N$ manifold has a diagonal representation (Arecchi *et al* 1972)

$$\rho(t) = \int d^2z P(z, z^*, t) |\frac{1}{2}N, z\rangle \langle \frac{1}{2}N, z| \quad (3.5)$$

where

$$\text{Tr } \rho(t) = \int d^2z P(z, z^*, t) = 1. \quad (3.6)$$

Then, while atomic coherent states may themselves be represented by points in the complex plane, more general states are represented by a quasi-distribution $P(z, z^*)$. The master equation (2.6) is transformed to a Fokker-Planck equation for $P(z, z^*, t)$. We find (for details see Glauber and Haake 1976)

$$\frac{dP}{d\tau}(z, z^*, \tau) = \left[-\frac{\partial}{\partial z} \left(i \frac{\Omega}{\Omega_0} (z^2 - 1) + 2z \right) + \frac{2}{N} \left(\frac{\partial}{\partial z} z^2 \frac{\partial}{\partial z} + \frac{\partial^2}{\partial z \partial z^*} \right) + \text{cc} \right] P(z, z^*, \tau) \quad (3.7)$$

where we have defined $\tau = \Omega_0 t$. Operator averages are evaluated according to the general prescription

$$\langle S_+^p S_z^q S_-^q \rangle_\tau = \int d^2z P(z, z^*, \tau) (1 + |z|^2)^{-N} \frac{\partial^{p+q}}{\partial z^p \partial z^{*q}} \left(\frac{N}{2} - z^* \frac{\partial}{\partial z^*} \right)^r (1 + |z|^2)^N. \quad (3.8)$$

3.1. Dynamics due to drift

The diffusion term in (3.7) is of the order N^{-1} and would appear to have little consequence for the limit $N \rightarrow \infty$. In fact appearances here are misleading, as we see further on. However, for the present we shall include only the drift and consider the evolution of an atomic state $|\frac{1}{2}N, z(\tau)\rangle$ with

$$\frac{dz}{d\tau} = i \frac{\Omega}{\Omega_0} (z^2 - 1) + 2z = i \frac{\Omega}{\Omega_0} (z - r_+)(z - r_-) \quad (3.9)$$

$$r_{\pm} = i\Omega_0/\Omega \pm [1 - (\Omega_0/\Omega)^2]^{1/2}. \quad (3.10)$$

It is interesting to note the connection between each of the factorisation schemes from the previous section and this assumption of zero width. For an atomic coherent state $| \rangle$ it may be shown that

$$\begin{aligned} \langle S_+ S_z \rangle &= -\frac{1}{2} \langle S_+ | + [1 - (1/N)] \langle S_+ | \rangle \langle S_z | \rangle \\ \langle S_+ S_- \rangle &= \frac{1}{2} N + \langle S_z | \rangle + [1 - (1/N)] \langle S_+ | \rangle \langle S_- | \rangle. \end{aligned} \quad (3.11)$$

This is precisely the factorisation employed by (2.13) and (2.14), and if evolution truly preserves an atomic coherent state a prediction of bistability must follow. However, it is inconsistent to include quantum fluctuations in (3.11) while terms of a similar order are neglected in the Fokker-Planck equation. Equations (2.17)–(2.19) correspond to a consistent approximation to lowest order ($\langle S_+ S_- \rangle = \langle S_+ | \rangle \langle S_- | \rangle$ etc). Here the designation of an atomic coherent state is restricted to the form of single-operator averages. When from (3.9) we write (here P is a δ function)

$$\langle s_+ \rangle = 2z/(1 + |z|^2) \quad \langle s_z \rangle = (1 - |z|^2)/(1 + |z|^2) \quad (3.12)$$

equations (2.17)–(2.19) are transformed to (3.9).

The transformation (3.12) in fact provides a convenient route to solutions for these classical equations. Defining c and ψ with

$$\sqrt{c} e^{i\psi} = (z(0) - r_+) / (z(0) - r_-) \quad (3.13)$$

we readily show (Drummond 1979)

$$z(\tau) = \frac{r_- \sqrt{c} e^{i\psi} e^{-\bar{\Omega}\tau} - r_+}{\sqrt{c} e^{i\psi} e^{-\bar{\Omega}\tau} - 1} \quad \frac{\Omega}{\Omega_0} < 1 \quad (3.14)$$

$$z(\tau) = \frac{z(0)(1 + \tau) - i\tau}{(1 - \tau) - iz(0)\tau} \quad \frac{\Omega}{\Omega_0} = 1 \quad (3.15)$$

$$z(\tau) = \frac{r_- \sqrt{c} \exp[i(\bar{\Omega}\tau + \psi)]}{\sqrt{c} \exp[i(\bar{\Omega}\tau + \psi)] - 1} \quad \frac{\Omega}{\Omega_0} > 1 \quad (3.16)$$

where

$$\bar{\Omega} = 2|1 - (\Omega/\Omega_0)^2|^{1/2}. \quad (3.17)$$

The detailed relationship to corresponding trajectories on the Bloch sphere is given in appendix 1. Below the threshold $\Omega/\Omega_0 = 1$ long-time solutions approach the steady state (2.20), with

$$z = r_+ = i \frac{\Omega_0}{\Omega} \left\{ 1 + \left[1 - \left(\frac{\Omega}{\Omega_0} \right)^2 \right]^{1/2} \right\} \quad (\theta = \pi - \sin^{-1}(\Omega/\Omega_0) \quad \phi = \frac{1}{2}\pi).$$

Above threshold cyclic behaviour in the complex plane corresponds to closed trajectories on the Bloch sphere. Here

$$\left| \frac{z(\tau) - r_+}{r(\tau) - r_-} \right|^2 = \left| \frac{z(0) - r_+}{z(0) - r_-} \right|^2 = c \quad (3.18)$$

and c is identified as a constant of motion which parametrises a family of cycles for each Ω/Ω_0 . The angle ψ specifies an initial phase.

3.2. Fokker-Planck equation in c, ψ variables above threshold†

The inclusion of diffusion terms will bring corrections of order N^{-1} to the steady state below threshold. However, above threshold no steady state is evidenced by the deterministic equations and the role of quantum fluctuations needs careful consideration. In the cyclic solutions (3.16) both the trajectory and phase are determined by initial conditions alone. It seems then that phase diffusion around cycles and diffusion between cycles may proceed unrestrained in the presence of fluctuations. To formulate this suggestion precisely we adopt c and ψ as variables, and from (3.7) derive a new Fokker-Planck equation via the transformation

$$z = \frac{r_- \sqrt{c} e^{i\psi} - r_+}{\sqrt{c} e^{i\psi} - 1}. \quad (3.19)$$

† The approach detailed here was first formulated by Dr P D Drummond and is the basis for results reported by Drummond and Carmichael (1978). An equivalent treatment is given by Drummond (1979) within the context of an Ito stochastic differential equation.

If $\bar{P}(c, \psi, \tau)$ denotes the new quasi-distribution, after much tedious algebra we find

$$\frac{d\bar{P}}{d\tau} = \left[-\left(\frac{\partial}{\partial \psi} A_\psi + \frac{\partial}{\partial c} A_c \right) + \frac{1}{2} \left(\frac{\partial^2}{\partial \psi^2} B_{\psi, \psi} + \frac{\partial^2}{\partial c^2} B_{c, c} + \frac{\partial^2}{\partial \psi \partial c} B_{\psi, c} + \frac{\partial^2}{\partial c \partial \psi} B_{c, \psi} \right) \right] \bar{P} \quad (3.20)$$

where

$$A_\psi = \bar{\Omega} - \frac{1}{N} \frac{2}{(r_+ - r_-)^2} \frac{1}{c} \operatorname{Im} \left(\Phi(c, \psi) \frac{r_+ + r_- \sqrt{c} e^{i\psi}}{r_+ - r_- \sqrt{c} e^{i\psi}} \right) \quad (3.21)$$

$$A_c = \frac{1}{N} \frac{4}{(r_+ - r_-)^2} \left[\Psi(c, \psi) - 2 \operatorname{Re} \left(\Phi(c, \psi) \frac{r_- \sqrt{c} e^{i\psi}}{r_+ - r_- \sqrt{c} e^{i\psi}} \right) \right] \quad (3.22)$$

and

$$B_{\psi, \psi} = \frac{1}{N} \frac{2}{(r_+ - r_-)^2} \frac{1}{c} (\Psi(c, \psi) - \operatorname{Re} \Phi(c, \psi)) \quad (3.23)$$

$$B_{c, c} = \frac{1}{N} \frac{8c}{(r_+ - r_-)^2} (\Psi(c, \psi) + \operatorname{Re} \Phi(c, \psi)) \quad (3.24)$$

$$B_{\psi, c} = B_{c, \psi} = \frac{1}{N} \frac{4}{(r_+ - r_-)^2} \operatorname{Im} \Phi(c, \psi) \quad (3.25)$$

with

$$\Phi(c, \psi) = [e^{-i\psi} (1 - \sqrt{c} e^{i\psi}) (r_+ - r_- \sqrt{c} e^{i\psi})]^2 \quad (3.26)$$

$$\Psi(c, \psi) = [(1 - \sqrt{c} e^{i\psi}) (1 - \sqrt{c} e^{-i\psi})]^2. \quad (3.27)$$

In accordance with the cyclic behaviour determined above, if in (3.20) we neglect terms of order N^{-1} only the drift in ψ remains to give an advancing phase $\psi(\tau) = \bar{\Omega}\tau$. The additional terms contribute both drift and diffusion in ψ and c . Except in the immediate vicinity of threshold, for large N we have $\bar{\Omega} \gg N^{-1}$. Then time scales characterising the classical cycles and the drift and diffusion from quantum noise are widely separated. The latter will be unable to follow rapid oscillations in drift and diffusion coefficients. Transforming (3.20) to a frame which follows the deterministic phase ($\psi \rightarrow \bar{\Omega}\tau + \psi$) we then average these coefficients over a period to obtain

$$\begin{aligned} \frac{\partial \bar{P}}{\partial \bar{\tau}} = & \left[1 - \left(\frac{\Omega_0}{\Omega} \right)^2 \right]^{-1} \left\{ \frac{\Omega_0}{\Omega} \left[1 - \left(\frac{\Omega_0}{\Omega} \right)^2 \right]^{1/2} \frac{\partial}{\partial \psi} - \frac{1}{2} \frac{\partial}{\partial c} \left[(1+c)^2 - 4c \left(\frac{\Omega_0}{\Omega} \right)^2 \right] \right. \\ & \left. + \frac{1}{8c} \left\{ (1+c)^2 + 4c \left[1 + \left(\frac{\Omega_0}{\Omega} \right)^2 \right] \right\} \frac{\partial^2}{\partial \psi^2} + \frac{1}{2} \frac{\partial}{\partial c^2} c \left[(1+c)^2 - 4c \left(\frac{\Omega_0}{\Omega} \right)^2 \right] \right\} \bar{P}. \end{aligned} \quad (3.28)$$

Here we have introduced a new normalised time $\bar{\tau} = (2/N)\tau = \frac{1}{2}\gamma t$. Above threshold it is this separation of time scales which reflects the small size of quantum fluctuations for large N . On the time scale characterising deterministic dynamics quantum fluctuations are indeed negligible. However, over many cycles, and in an absolute time of approximately γ^{-1} , quantum effects destroy the well determined phase and trajectory on the Bloch sphere. We might note that the first term in (3.28) merely contributes a correction of order N^{-1} to the frequency $\bar{\Omega}$. This term may be absorbed by a redefinition of Ω_0 with $\Omega_0 \rightarrow \Omega_0 [1 + (1/N)]$.

3.3. Steady state above threshold

For long times the phase diffusion in (3.28) leads to a quasi-distribution which is uniform in the variable ψ . Within the rest frame this removes any explicit time dependence associated with the underlying periodic dynamics. Senitzky (1972) identified essentially the same mechanism as the basis for steady-state and saturation features. However, in our more detailed formulation we also find diffusion between classical trajectories. A broadened steady-state quasi-distribution develops in both c and ψ . From (3.28) we readily find (Drummond and Carmichael 1978)

$$\bar{P}_{ss}(c, \psi) = \frac{1}{2\pi} \frac{(\Omega_0/\Omega)[1 - (\Omega_0/\Omega)^2]^{1/2}}{\sin^{-1}(\Omega_0/\Omega)} [(1+c)^2 - 4c(\Omega_0/\Omega)^2]^{-1}. \quad (3.29)$$

It is useful to derive the corresponding distribution on the Bloch sphere. Via the transformation (3.4) we define $Q(\theta, \psi, t)$ with normalisation

$$\int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta Q(\theta, \phi, t) = 1. \quad (3.30)$$

Then from (3.29) we find

$$Q_{ss}(\theta, \phi) = \frac{1}{4\pi} \frac{(\Omega_0/\Omega)[1 - (\Omega_0/\Omega)^2]^{1/2}}{\sin^{-1}(\Omega_0/\Omega)} \left[1 - 2 \frac{\Omega_0}{\Omega} \sin \theta \sin \phi + \left(\frac{\Omega_0}{\Omega} \right)^2 \sin^2 \theta \right]^{-1}. \quad (3.31)$$

The significance of quantum fluctuations is most dramatically illustrated by the limit $\Omega \rightarrow \infty$. Here $Q_{ss}(\theta, \phi) \rightarrow (4\pi)^{-1}$ and the quasi-distribution is spread uniformly over the entire Bloch sphere. If we make use of (3.3) the matrix elements

$$\rho_{M,M'}^{ss} = \langle \frac{1}{2}N, M | \rho_{ss} | \frac{1}{2}N, M' \rangle$$

are given by

$$\begin{aligned} \rho_{M,M'}^{ss} &= \left(\frac{N}{\frac{1}{2}N + M} \right)^{1/2} \left(\frac{N}{\frac{1}{2}N + M'} \right)^{1/2} \int_0^{2\pi} d\phi \exp[-i(M - M')\phi] \int_0^\pi d\theta \sin \theta \\ &\quad \times Q_{ss}(\theta, \phi) \cos(\tfrac{1}{2}\theta)^{N+M+M'} \sin(\tfrac{1}{2}\theta)^{N-M-M'}. \end{aligned} \quad (3.32)$$

On substituting for $Q_{ss}(\theta, \phi)$ and carrying through the integration (see appendix 2) we have ($M \geq M'$)

$$\begin{aligned} \rho_{M,M'}^{ss} &= \frac{(\Omega_0/\Omega)[1 - (\Omega_0/\Omega)^2]^{1/2}}{\sin^{-1}(\Omega_0/\Omega)} \left(\frac{N}{\frac{1}{2}N + M} \right)^{1/2} \left(\frac{N}{\frac{1}{2}N + M'} \right)^{1/2} \\ &\quad \times \exp[-i(M - M')\pi/2] (2\Omega_0/\Omega)^{M-M'} \\ &\quad \times \sum_{k=0}^{\infty} \left(2 \frac{\Omega_0}{\Omega} \right)^{2k} B(\tfrac{1}{2}N + M + k + 1, \tfrac{1}{2}N - M' + k + 1) \end{aligned} \quad (3.33)$$

where $B(\alpha, \beta)$ is the beta function. This may be compared with the exact solution (2.9) taken in the asymptotic limit $\Omega \rightarrow \infty$, $N \rightarrow \infty$, Ω/Ω_0 constant. Due to the nature of the limiting procedure a simple judgment which applies uniformly to all matrix elements cannot be made. However, agreement can be asserted in a more restricted sense; for example, where M and M' are fixed while $N \rightarrow \infty$. Details are given in appendix 2.

3.4. Operator averages in the asymptotic limit

In the asymptotic limit $\Omega \rightarrow \infty$, $N \rightarrow \infty$, Ω/Ω_0 constant, averages below threshold may be factored and expressed in terms of the deterministic steady state (2.20). Above threshold, full implementation of (3.8) is required, where to zeroth order in N^{-1} we find

$$\langle s_+^p s_z^r s_-^q \rangle = \int d^2z P_{ss}(z, z^*) O_{p,q}^r(z, z^*) \quad (3.34)$$

with

$$O_{p,q}^r(z, z^*) = \left(\frac{2z}{1+|z|^2} \right)^p \left(\frac{1-|z|^2}{1+|z|^2} \right)^r \left(\frac{2z^*}{1+|z|^2} \right)^q. \quad (3.35)$$

A more convenient formula is obtained if we adopt c , ψ variables and replace integration over ψ by contour integration around each cycle. With

$$dz = [i(z - r_+)(z - r_-)/(r_+ - r_-)] d\psi$$

we have

$$\langle s_+^p s_z^r s_-^q \rangle = (r_+ - r_-) \int_0^\infty dc \bar{P}_{ss}(c) \frac{1}{2\pi i} \oint_c dz (z - r_+)(z - r_-) \bar{O}_{q,p}^r(z, c) \quad (3.36)$$

where $\bar{P}_{ss}(c)$ is derived from (3.29) by integration over ψ and $\bar{O}_{p,q}^r(z, c)$ is a meromorphic function chosen to agree with $O_{p,q}^r(z, z^*)$ on the cycle c . In an earlier publication (Drummond and Carmichael 1978, see also Drummond 1979) this result was applied to single-operator averages. Here we also include asymptotic expressions for quantum correlations. We have ($\Omega/\Omega_0 > 1$)

$$\langle s_+ \rangle_{ss} = i \frac{\Omega}{\Omega_0} \left(1 - \frac{(\Omega_0/\Omega)[1 - (\Omega_0/\Omega)^2]^{1/2}}{\sin^{-1}(\Omega_0/\Omega)} \right) \quad (3.37)$$

$$\langle s_z \rangle_{ss} = 0 \quad (3.38)$$

$$\langle s_+ s_- \rangle_{ss} = \left(\frac{\Omega}{\Omega_0} \right)^2 \left(1 - \frac{(\Omega_0/\Omega)[1 - (\Omega_0/\Omega)^2]^{1/2}}{\sin^{-1}(\Omega_0/\Omega)} \right) \quad (3.39)$$

$$\langle s_+ s_z \rangle = 0 \quad (3.40)$$

$$\langle s_+^2 s_-^2 \rangle_{ss} = \left(\frac{\Omega}{\Omega_0} \right)^4 \left\{ 1 - \frac{\Omega_0}{\Omega} \left[1 - \left(\frac{\Omega_0}{\Omega} \right)^2 \right]^{1/2} \frac{1 + \frac{2}{3}(\Omega_0/\Omega)^2}{\sin^{-1}(\Omega_0/\Omega)} \right\}. \quad (3.41)$$

Further discussion of these expressions is left to the following section where we can compare numerical results for finite N .

4. Numerical results for finite systems

In this section we illustrate various specific steady-state features for cooperative resonance fluorescence and compare the behaviour in finite systems to predictions for the asymptotic limit. The relationship to deterministic dynamics is discussed in further detail and questions pertaining to atom-atom correlations are again addressed.

The steady-state density matrix (2.9) may be employed to compute arbitrary operator averages. However, our own numerical calculations were completed before

we became aware of this analytic solution. The results reported here were obtained using direct numerical techniques to solve the master equation (2.6) in the Dicke representation. The $(N+1)^2$ equations for matrix elements in this representation separate into two dynamically independent schemes: a system of $\frac{1}{2}N(N+1)$ equations coupling only off-diagonal elements, and a system of $(\frac{1}{2}N+1)(N+1)$ equations which couple both diagonal and off-diagonal elements. Providing the steady state is unique the former can have only the trivial solution, and from the outset we find

$$\rho_{M,M'}^{ss} = \begin{cases} \text{Re } \rho_{M,M'}^{ss} & |M-M'| \text{ even} \\ i \text{Im } \rho_{M,M'}^{ss} & |M-M'| \text{ odd.} \end{cases} \quad (4.1)$$

Dynamics within the second system of equations are governed by a matrix having at most six non-zero entries in any row. For the majority of cases considered we were able to employ an efficient numerical algorithm tailored to the specific form of this sparse matrix and yielding steady-state matrix elements directly. Unfortunately, for large N and small Ω/Ω_0 this method could not be applied as the algorithm became increasingly ill conditioned. Here the steady-state density matrix was developed from an initial ground state by direct integration. All such cases were restricted to the region below threshold and therefore avoided any problems with a rapid oscillatory approach to the steady state.

4.1. Single operator averages

As a direct consequence of (4.1) $\langle s_+ \rangle_{ss}$ is pure imaginary and $\langle s_x \rangle_{ss} = 0$ for all values of Ω/Ω_0 . The behaviour for $\langle s_y \rangle_{ss}$ and $\langle s_z \rangle_{ss}$ is shown in figure 3. In both instances results for ten, twenty and fifty atoms support strongly an approach towards the asymptotic expressions from the previous section. In the region above threshold saturation features are molded from the underlying oscillatory dynamics and are understood simply from the perspective of § 3.4. Equation (3.36) accomplishes two averages; the first by integration around trajectories on the Bloch sphere and the second by integration over a distribution of such trajectories. Saturation of the population

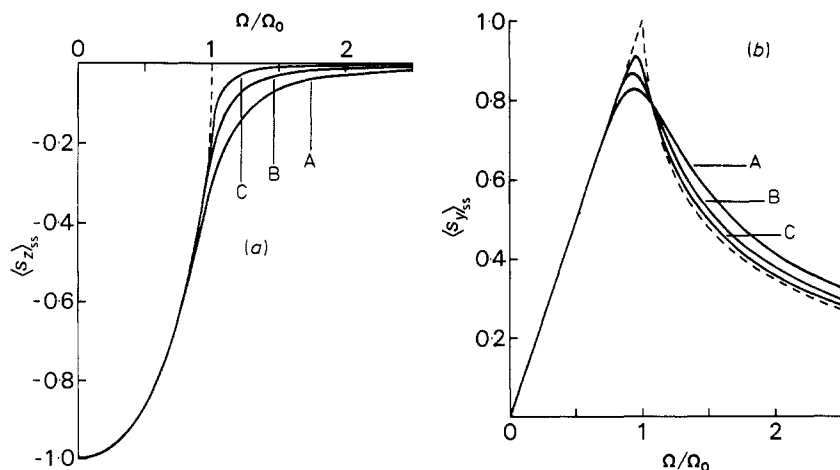


Figure 3. Steady-state averages (a) $\langle s_z \rangle_{ss}$ and (b) $\langle s_y \rangle_{ss}$ for: A, $N = 10$; B, $N = 20$ and C, $N = 50$. The broken curve gives the asymptotic results.

difference $\langle s_z \rangle_{ss}$ follows directly from the symmetry of the deterministic trajectories themselves. On each cycle, $\langle s_z \rangle_\tau$ (see (A.12)) averages to zero over a period for all values of $\Omega/\Omega_0 > 1$ (also see figure 1(b)). In contrast, the condition $\langle s_x \rangle_{ss} = 0$ is preserved above threshold by the symmetry of the distribution $\bar{P}_{ss}(c)$. Since $\bar{P}_{ss}(c) dc = -\bar{P}_{ss}(c^{-1}) d(c^{-1})$, any trajectory in the eastern hemisphere of the Bloch sphere ($0 \leq c < 1$, $\langle s_x \rangle_\tau > 0$) cancels its mirror image ($c \rightarrow 1/c$, $\langle s_x \rangle_\tau \rightarrow -\langle s_x \rangle_\tau$) in the western hemisphere (see appendix 1). Saturation in the polarisation $\langle s_y \rangle_{ss}$ finds its onset at threshold and increases gradually for larger Ω/Ω_0 . Both averages contribute to the quantitative predictions of (3.37). However, it is again the phase average which leads to saturation. As evidenced by (A.11), $\langle s_y \rangle_\tau$ does not average to zero over a period so long as its oscillations remain non-harmonic (see also figure 1(b)). An approach towards harmonic oscillations for large Ω/Ω_0 brings increased cancellation around each cycle, and, as a consequence, saturation of $\langle s_y \rangle_{ss}$.

4.2. Quantum correlations above threshold

We noted in an earlier section that the master equation (2.8) conserves $\langle \mathbf{S}^2 \rangle = \frac{1}{2}N(\frac{1}{2}N + 1)$ while the deterministic equations (2.17)–(2.19) replace this symmetry by conservation of $\langle \mathbf{S} \rangle^2 = (\frac{1}{2}N)^2$. Exact solutions show saturation in both $\langle s_y \rangle_{ss}$ and $\langle s_z \rangle_{ss}$ and $\langle \mathbf{S} \rangle^2$ conservation actually fails dramatically for $\Omega/\Omega_0 > 1$ (figure 4(a)). For $\langle \mathbf{S}^2 \rangle$ to itself remain constant quantum correlations must play an increasingly important role above threshold. To illustrate this feature we have plotted $\langle s_z^2 \rangle_{ss}$ as a function of Ω/Ω_0 in figure 4(b). While $\langle s_z \rangle_{ss}$ vanishes above threshold $\langle s_z^2 \rangle_{ss}$ increases with increasing Ω/Ω_0 . Complementary behaviour is exhibited by $\langle s_+ s_- \rangle_{ss}$ giving $\langle s_+ s_- \rangle_{ss} + \langle s_z^2 \rangle_{ss} = 1$ for the asymptotic limit. Again, figure 4 shows good agreement with asymptotic expressions.

The role of fluctuations and correlations is directly evidenced by features in the spectrum of the fluorescent light. Above threshold coherent and incoherent intensities are obtained from (3.34) and (3.39) with

$$I_{ss}^{\text{COH}} \propto \frac{1}{4}N^2(\Omega/\Omega_0)^2[1 - \Lambda(\Omega_0/\Omega)]^2 \quad (4.2)$$

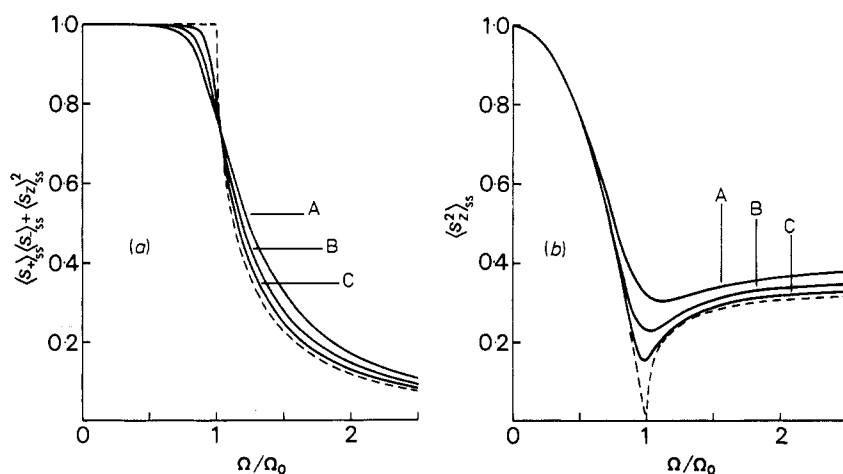


Figure 4. Steady-state averages (a) $\langle s \rangle_{ss}^2$ and (b) $\langle s_z^2 \rangle_{ss}$ for: A, $N = 10$; B, $N = 20$ and C, $N = 50$. The broken curves give the asymptotic results.

$$I_{ss}^{\text{INC}} \propto \frac{1}{4} N^2 (\Omega/\Omega_0)^2 \Lambda(\Omega_0/\Omega) [1 - \Lambda(\Omega_0/\Omega)] \quad (4.3)$$

where

$$\Lambda(\Omega_0/\Omega) = \frac{(\Omega_0/\Omega) [1 - (\Omega_0/\Omega)^2]^{1/2}}{\sin^{-1} \Omega_0/\Omega}. \quad (4.4)$$

As the coherent intensity decays with the saturation of $\langle s_y \rangle_{ss}$ incoherent scattering increases to preserve a net fluorescent intensity proportional to N^2 . The responsibility for this feature lies with atom-atom correlations. Single-atom fluctuations merely give a correction of order N^{-1} to the curves of figure 4(a). In the asymptotic limit, for $j \neq k$ we have

$$\langle \sigma_+^j \sigma_-^k \rangle_{ss} - \langle \sigma_+^j \rangle_{ss} \langle \sigma_-^k \rangle_{ss} = \frac{1}{4} \left(\frac{\Omega}{\Omega_0} \right)^2 \Lambda \left(\frac{\Omega_0}{\Omega} \right) \left[1 - \Lambda \left(\frac{\Omega_0}{\Omega} \right) \right] \quad (4.5)$$

$$\langle \sigma_z^j \sigma_z^k \rangle_{ss} - \langle \sigma_z^j \rangle_{ss} \langle \sigma_z^k \rangle_{ss} = \frac{1}{4} - \frac{1}{4} \left(\frac{\Omega}{\Omega_0} \right)^2 \left[1 - \Lambda \left(\frac{\Omega_0}{\Omega} \right) \right]. \quad (4.6)$$

Corresponding correlations for like atoms are given by

$$\langle \sigma_+^j \sigma_-^j \rangle_{ss} - \langle \sigma_+^j \rangle_{ss} \langle \sigma_-^j \rangle_{ss} = \frac{1}{2} - \frac{1}{4} \left(\frac{\Omega}{\Omega_0} \right)^2 \left[1 - \Lambda \left(\frac{\Omega_0}{\Omega} \right) \right]^2 \quad (4.7)$$

$$\langle \sigma_z^j \rangle_{ss} - \langle \sigma_z^j \rangle_{ss}^2 = \frac{1}{4}. \quad (4.8)$$

Near threshold both (4.5) and (4.6) vanish. However, atom-atom correlations increase as $\langle s_y \rangle_{ss}$ saturates, and for the limit $\Omega/\Omega_0 \rightarrow \infty$

$$\langle \sigma_+^j \sigma_-^k \rangle_{ss} = 2 \langle \sigma_z^j \sigma_z^k \rangle_{ss} = \begin{cases} \frac{1}{6} & j \neq k \\ \frac{1}{2} & j = k. \end{cases} \quad (4.9)$$

4.3. Second-order correlation function

In single-atom resonance fluorescence considerable attention has been paid to the second-order correlation function $g_{ss}^{(2)}(0)$ and the phenomenon of photon antibunching (Carmichael and Walls 1976, Kimble and Mandel 1976, Kimble *et al* 1977, 1978, Jakeman *et al* 1977, Carmichael *et al* 1978, Dagenais and Mandel 1978, Walls 1979). In terms of atomic averages we have

$$g_{ss}^{(2)}(0) = \langle S_+^2 S_-^2 \rangle_{ss} / \langle S_+ S_- \rangle_{ss}^2. \quad (4.10)$$

Antibunching effects ($g_{ss}^{(2)}(0) < 1$) are known to be critically dependent on the number of atoms in the scattering volume. Just two independent atoms give $g_{ss}^{(2)}(0) = 1$ (see for example Carmichael *et al* (1978), equation (4)) and for many independent scattering centres $g_{ss}^{(2)}(0) \rightarrow 2$ (Carmichael and Walls 1976). In the context of cooperative resonance fluorescence Agarwal *et al* (1977) have shown that a two-atom system preserves a reduced antibunching effect. Figure 5 extends their results to three, four, five, ten and fifty atoms. For $N = 50$ we find good agreement with the asymptotic expressions from § 3. Photon antibunching is fully eliminated in the asymptotic limit. Here, below threshold $g_{ss}^{(2)}(0) = 1$ compared with $g_{ss}^{(2)}(0) = 2$ for independent atoms. Loss of this second-order coherence above threshold complements the loss of first-order coherence evidenced in (4.3). These predictions agree qualitatively with

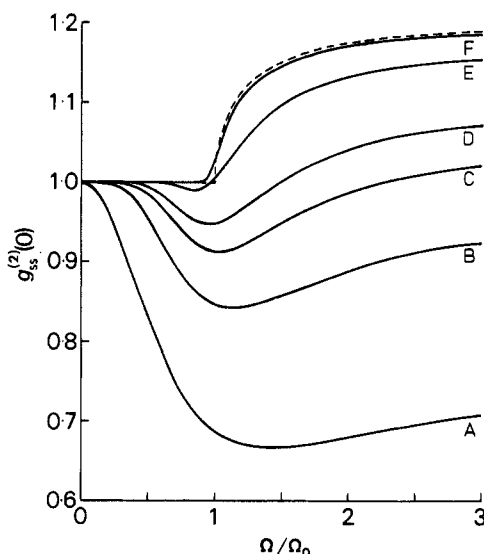


Figure 5. The second-order correlation function $g_{ss}^{(2)}(0)$ for: A, $N = 2$; B, $N = 3$; C, $N = 4$; D, $N = 5$; E, $N = 10$ and F, $N = 50$. The broken curve gives the asymptotic result.

previously published results (Walls *et al* 1978) although the limiting value $g_{ss}^{(2)}(0) \xrightarrow{\Omega/\Omega_0 \rightarrow \infty} 1.2$ is somewhat lowered.

4.4. Atom-atom correlations below threshold

In both this and the previous section little has been said about quantum fluctuations below threshold since these merely add a correction of order N^{-1} to the steady state (2.20). However, we have seen in § 2 that the omission of atom-atom correlations leads to a first-order transition at $\Omega/\Omega_0 = 1/\sqrt{2}$. In order to recover the correct behaviour it is necessary that atom-atom correlations exert a significant influence below threshold. We have sought to identify features in our numerical results which indicate their role and in figure 6 a comparison of the factorisation schemes described in § 2 is made for $\langle s_+ s_- \rangle_{ss}$ and $\langle s_+ s_z \rangle_{ss}$ (from (4.1) $\langle s_x s_z \rangle_{ss} = 0$). These are the averages factored in (2.11) and (2.12). We shall denote the correlation $\langle \sigma_+^j \sigma_-^k \rangle - \langle \sigma_+^j \rangle \langle \sigma_-^k \rangle$ by $C_{+,-}^{(1)}$ for $j = k$ and $C_{+,-}^{(2)}$ for $j \neq k$. Then, apart from the normalisation by $\langle S_+ S_- \rangle_{ss}$ the broken curves in figure 6(a) correspond to $N(N-1)C_{+,-}^{(2)}$ while the full curves give $NC_{+,-}^{(1)} + N(N-1)C_{+,-}^{(2)}$. With an equivalent definition for $C_{+,z}^{(1)}$ and $C_{+,z}^{(2)}$ a similar statement holds in relation to figure 6(b). The important feature to note below threshold is that unlike atoms are negatively correlated and their contribution substantially cancels that from single-atom fluctuations. The trend for large N suggests that

$$C_{+,-}^{(2)} \sim -N^{-1}C_{+,-}^{(1)} \quad C_{+,z}^{(2)} \sim -N^{-1}C_{+,z}^{(1)}. \quad (4.11)$$

It is this conspiracy between single-atom fluctuations and atom-atom correlations which suppresses the instability at $\Omega/\Omega_0 = 1/\sqrt{2}$. However, as shown by figure 7, this preference for full factorisation does not extend to all operator products. For $\langle s_z^2 \rangle_{ss}$

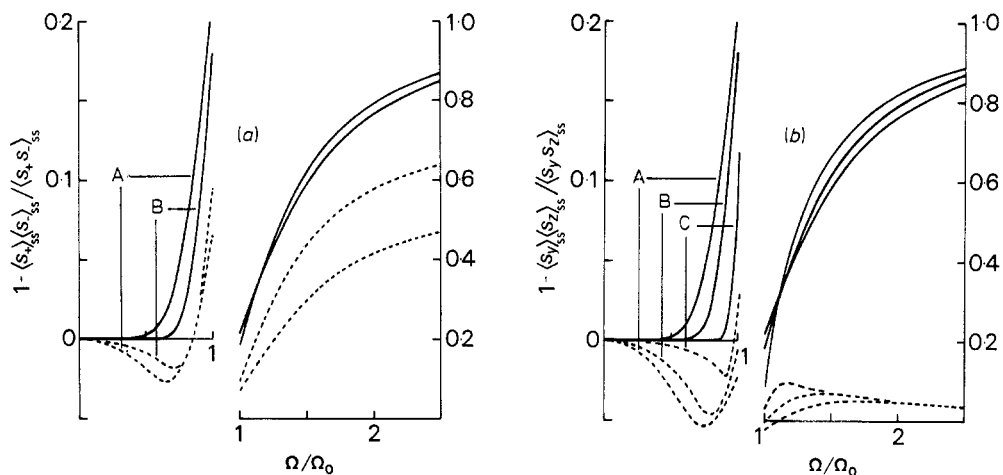


Figure 6. Comparison of the two factorisation schemes from § 2 for (a) $\langle s_+ s_- \rangle_{ss}$ and (b) $\langle s_y s_z \rangle_{ss}$. Full curves correspond to the full factorisation indicated by the labelled axes while the broken curves are obtained by factorising only unlike atoms. Results are for: A, $N = 5$; B, $N = 10$ and C, $N = 50$.

unlike atoms are not negatively correlated and the inclusion of single-atom fluctuations brings significant improvement.

In the asymptotic limit the broken and full curves in figure 6(a) and figure 7 will coincide as they differ only by something of order N^{-1} . In figure 6(b) this is true below threshold, while above threshold $\langle s_y s_z \rangle_{ss}$ is itself of order N^{-1} (see equation (3.40)) and the asymptotic limit to these curves has not been calculated.

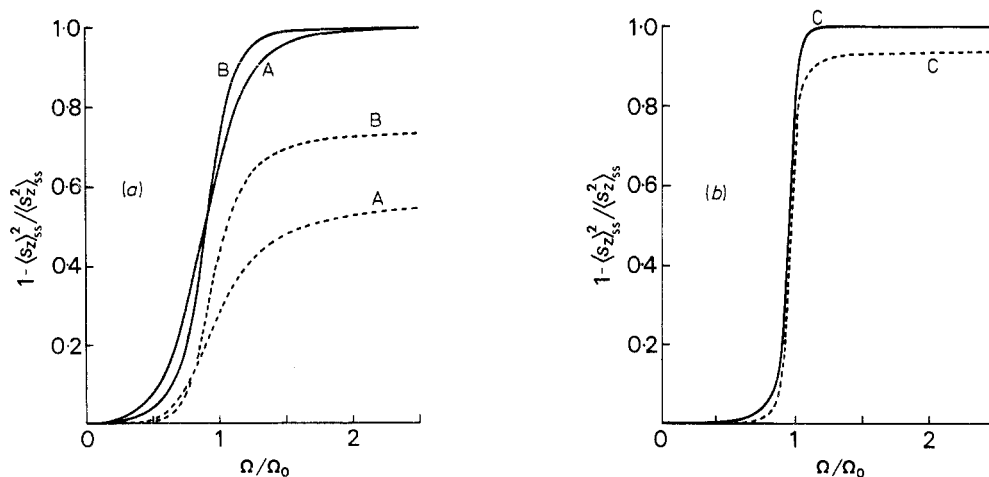


Figure 7. Comparison of the two factorisation schemes from § 2 for $\langle s_z^2 \rangle_{ss}$. Full curves correspond to the full factorisation indicated by the labelled axes while the broken curves are obtained by factoring only unlike atoms. Results are for: A, $N = 5$; B, $N = 10$ and C, $N = 50$.

5. Summary and conclusions

We have discussed the steady state in cooperative resonance fluorescence from both analytical and numerical perspectives. Particular attention has been given to quantum fluctuations and their significance in relation to an appropriate background of deterministic dynamics. Atom–atom correlations have been shown to play a central role in establishing features which characterise cooperative resonance fluorescence. In the asymptotic limit $\Omega \rightarrow \infty$, $N \rightarrow \infty$, Ω/Ω_0 constant, behaviour analogous to a second-order phase transition is predicted in agreement with previous authors (Drummond and Carmichael 1978, Narducci *et al* 1978a). Suppression of the first-order instability suggested by an inappropriate factorisation (Carmichael and Walls 1977) depends critically on atom–atom correlations. The preservation of a fluorescent intensity proportional to N^2 above threshold further evidences their role. Steady-state and saturation features in cooperative resonance fluorescence sharply contrast the periodic trajectories obtained with factorised classical equations. To explain this difference Senitzky (1972) proposed a mechanism whereby quantum fluctuations in the radiation reaction field contribute a fluctuating component to the frequency of oscillation. Our detailed Fokker–Planck description has clarified the relationship between classical and quantum-mechanical dynamics and substantially supports Senitzky's proposal. Phase diffusion around classical trajectories corresponds directly to his frequency fluctuations. However there also exists a diffusion between trajectories. This is of equal importance for a quantitative description of the steady state. Explicit expressions for operator averages have been derived within the Fokker–Planck formalism. In particular, closed expressions for atom–atom correlations above threshold have been obtained, together with expressions for the fluorescent intensity and second-order correlation function. The validity of these results for the asymptotic limit has been strongly supported by numerical calculations for fifty atoms. Further support for this Fokker–Planck approach has been gained through a direct comparison with the exact steady-state density matrix derived by Puri and Lawande (1979).

Acknowledgments

The author gratefully acknowledges stimulating discussions with Professor W C Schieve and Dr H J Kimble, and valuable correspondence with Dr D F Walls. Thanks are also extended to Dr P D Drummond who kindly made a copy of his doctoral thesis available. For his tutoring in numerical techniques, many thanks to Professor M Lax at the City University of New York where much of the numerical work was completed.

Appendix 1. General solutions to equations (2.17)–(2.19)

Equations (3.14)–(3.16) describe deterministic dynamics in the complex plane. General solutions for operator averages themselves are then available via (3.12). After implementing this transformation the results are as follows

(i) For $\Omega/\Omega_0 < 1$

$$\pm \langle s_x \rangle_\tau = \frac{\Omega_0}{\Omega} \left[1 - \left(\frac{\Omega}{\Omega_0} \right)^2 \right]^{1/2} \frac{[(\Omega/\Omega_0)^2 - K^2]^{1/2}}{\cosh \bar{\Omega}(\tau - \hat{\tau}) \mp K} \quad (\text{A.1})$$

$$\langle s_y \rangle_\tau = \frac{\Omega}{\Omega_0} \mp \frac{(\Omega_0/\Omega - \Omega/\Omega_0)K}{\cosh \bar{\Omega}(\tau - \hat{\tau}) \mp K} \quad (\text{A.2})$$

$$\langle s_z \rangle_\tau = - \left[1 - \left(\frac{\Omega}{\Omega_0} \right)^2 \right]^{1/2} \frac{\sinh \bar{\Omega}(\tau - \hat{\tau})}{\cosh \bar{\Omega}(\tau - \hat{\tau}) \mp K} \quad (\text{A.3})$$

with

$$K = \frac{\Omega}{\Omega_0} \left\{ 1 + \left[1 - \left(\frac{\Omega}{\Omega_0} \right)^2 \right] \left(\frac{\langle s_x \rangle_0}{(\Omega/\Omega_0) - \langle s_y \rangle_0} \right)^2 \right\}^{-1/2} \quad (\text{A.4})$$

$$\sinh \bar{\Omega} \hat{\tau} = \pm K \frac{\Omega_0}{\Omega} \left[1 - \left(\frac{\Omega}{\Omega_0} \right)^2 \right]^{1/2} \frac{\langle s_z \rangle_0}{(\Omega/\Omega_0) - \langle s_y \rangle_0}. \quad (\text{A.5})$$

On the left-hand side of (A.1) upper and lower signs are taken for $\langle s_x \rangle_0 \geq 0$, and elsewhere for $\langle s_y \rangle_0 \geq \Omega/\Omega_0$.

(ii) For $\Omega/\Omega_0 = 1$

$$\langle s_x \rangle_\tau = \frac{2K}{[\bar{\Omega}(\tau - \hat{\tau})]^2 + 1 + K^2} \quad (\text{A.6})$$

$$\langle s_y \rangle_\tau = 1 - \frac{2}{[\bar{\Omega}(\tau - \hat{\tau})]^2 + 1 + K^2} \quad (\text{A.7})$$

$$\langle s_z \rangle_\tau = - \frac{2\bar{\Omega}(\tau - \hat{\tau})}{[\bar{\Omega}(\tau - \hat{\tau})]^2 + 1 + K^2} \quad (\text{A.8})$$

with

$$K = \frac{\langle s_x \rangle_0}{1 - \langle s_y \rangle_0} \quad \bar{\Omega} \hat{\tau} = \frac{\langle s_z \rangle_0}{1 - \langle s_y \rangle_0}. \quad (\text{A.9})$$

(iii) For $\Omega/\Omega_0 > 1$

$$\pm \langle s_x \rangle_\tau = - \left[1 - \left(\frac{\Omega_0}{\Omega} \right)^2 \right]^{1/2} \frac{[K^2 - (\Omega/\Omega_0)^2]^{1/2}}{\cos \bar{\Omega}(\tau - \hat{\tau}) - K} \quad (\text{A.10})$$

$$\langle s_y \rangle_\tau = \frac{\Omega}{\Omega_0} + \frac{(\Omega/\Omega_0 - \Omega_0/\Omega)K}{\cos \bar{\Omega}(\tau - \hat{\tau}) - K} \quad (\text{A.11})$$

$$\langle s_z \rangle_\tau = \frac{\Omega}{\Omega_0} \left[1 - \left(\frac{\Omega_0}{\Omega} \right)^2 \right]^{1/2} \frac{\sin \bar{\Omega}(\tau - \hat{\tau})}{\cos \bar{\Omega}(\tau - \hat{\tau}) - K} \quad (\text{A.12})$$

where K is given by (A.4) and

$$\begin{aligned} \sin \bar{\Omega} \hat{\tau} &= K \left[1 - \left(\frac{\Omega_0}{\Omega} \right)^2 \right]^{1/2} \frac{\langle s_z \rangle_0}{(\Omega/\Omega_0) - \langle s_y \rangle_0} \\ \cos \bar{\Omega} \hat{\tau} &= K \frac{(\Omega_0/\Omega) - \langle s_y \rangle_0}{(\Omega/\Omega_0) - \langle s_y \rangle_0}. \end{aligned} \quad (\text{A.13})$$

In (A.10) upper and lower signs are taken for $\langle s_x \rangle_0 \geq 0$. It is useful to maintain contact with our parametrisation of cycles in the complex plane. For the oscillatory solutions we have

$$K = \frac{1+c}{2\sqrt{c}} \frac{\Omega}{\Omega_0}. \quad (\text{A.14})$$

Then $c = 1$ corresponds to a trajectory around the Bloch sphere in the plane $\langle s_x \rangle = 0$. We may show further that $\langle s_x \rangle_0 \geq 0$ corresponds to $c \geq 1$ and trajectories for $0 \leq c < 1$ and $1 < c \leq \infty$ lie entirely within eastern and western hemispheres respectively. Every trajectory has a mirror image $\langle s_x \rangle_\tau \rightarrow -\langle s_x \rangle_\tau$ which is itself an admissible solution and corresponds to the transformation $c \rightarrow 1/c$.

Solutions equivalent to those given above have been obtained by Kilin (1978).

Appendix 2. Steady-state density matrix in the asymptotic limit

Here we give details in the derivation of (3.33) and the comparison of this result with the exact steady-state density matrix (2.9).

The integrals arising in (3.32) contribute a factor

$$R_{M,M'} = (4\pi)^{-1} \int_0^\pi d\theta \sin \theta \cos(\tfrac{1}{2}\theta)^{N+M+M'} \sin(\tfrac{1}{2}\theta)^{N-M-M'} \\ \times \int_0^{2\pi} d\phi \frac{\exp[-i(M-M')\phi]}{1 - 2(\Omega_0/\Omega) \sin \theta \sin \phi + (\Omega_0/\Omega)^2 \sin^2 \theta}. \quad (\text{A.15})$$

To accomplish the integration over ϕ we invoke the result (Gradshteyn and Ryzhik 1965)

$$\int_0^\pi dx \frac{\cos(nx)}{1 - 2a \cos x + a^2} = \pi \frac{a^n}{1 - a^2} \quad a^2 < 1 \\ \int_0^\pi dx \frac{\sin(nx)}{1 - 2a \cos x + a^2} = 0 \quad (\text{A.16})$$

and following a little algebra, for $M \geq M'$

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi \frac{\exp[-i(M-M')\phi]}{1 - 2(\Omega_0/\Omega) \sin \theta \sin \phi + (\Omega_0/\Omega)^2 \sin^2 \theta} \\ = \exp[-i(M-M')\pi/2] \frac{[(\Omega_0/\Omega) \sin \theta]^{M-M'}}{1 - (\Omega_0/\Omega)^2 \sin^2 \theta}. \quad (\text{A.17})$$

If we then expand $[1 - (\Omega_0/\Omega)^2 \sin^2 \theta]^{-1}$ in a power series and take $\frac{1}{2}\theta \rightarrow \theta$, (A.15) may be written

$$\frac{1}{2} R_{M,M'} = \exp[-i(M-M')\pi/2] \sum_{k=0}^{\infty} (2\Omega_0/\Omega)^{M-M'+2k} \\ \times \int_0^{\pi/2} d\theta \cos \theta^{N+2M+2k+1} \sin \theta^{N-2M'+2k+1}. \quad (\text{A.18})$$

After integration over θ we find (Gradshteyn and Ryzhik 1965)

$$R_{M,M'} = \exp[-i(M-M')\pi/2] (2\Omega_0/\Omega)^{M-M'} \\ \times \sum_{k=0}^{\infty} (2\Omega_0/\Omega)^{2k} B(\tfrac{1}{2}N + M + k + 1, \tfrac{1}{2}N - M' + k + 1) \quad (\text{A.19})$$

where $B(\alpha, \beta)$ is the beta function.

To compare the exact steady state with (3.33) we first consider the normalisation constant $\mathcal{N}$ from (2.10). Changing the order of summation in (2.10) we have

$$\mathcal{N}^{-1} = \sum_{m=0}^N \left(\frac{N\Omega}{2\Omega_0} \right)^{2(N-m)} \sum_{n=0}^{N-m} \frac{(N-n)!(n+m)!}{n!(N-n-m)!} \quad (\text{A.20})$$

which may be written

$$\mathcal{N}^{-1} = (N\Omega/2\Omega_0)^{2N} N \sum_{k=0}^N C_k (\Omega_0/\Omega)^{2k} \quad (\text{A.21})$$

with

$$C_k = \frac{2^{2k}}{N^{2k+1}} \sum_{n=0}^{N-k} \frac{(N-n)!(n+k)!}{n!(N-n-k)!}. \quad (\text{A.22})$$

Our concern is with the asymptotic limit $\Omega \rightarrow \infty$, $N \rightarrow \infty$, Ω/Ω_0 constant. Apart from corrections of order N^{-1} the first few terms in (A.21) have $C_0 = 1$, $C_1 = \frac{2}{3}$, $C_2 = \frac{8}{15}$, and in Stirling's approximation $C_N = 2\pi(2/e)^{2N}$. Then above threshold ($\Omega_0/\Omega < 1$) we replace the finite sum in (A.21) by a convergent power series:

$$\sum_{k=0}^N C_k (\Omega_0/\Omega)^{2k} \sim \sum_{k=0}^{\infty} \bar{C}_k (\Omega_0/\Omega)^{2k}. \quad (\text{A.23})$$

Here the $\bar{C}_k$ are to be evaluated from (A.22) where for fixed k we let $N \rightarrow \infty$ and neglect terms of order kN^{-1} . We have

$$\begin{aligned} C_k &= (2^{2k}/N^{2k+1}) \sum_{n=0}^{N-k} (n+k) \dots (n+1)(N-n) \dots (N-n-k+1) \\ &= \frac{2^{2k}}{N^{2k+1}} \left[\sum_{p=k}^{2k} (-1)^{p-k} \binom{k}{p-k} N^{2k-p} \sum_{n=0}^{N-k} n^p + \sum_{p=0}^{2k} P_{k,p}(N) \sum_{n=0}^{N-k} n^p \right]. \end{aligned} \quad (\text{A.24})$$

The $P_{k,p}(N)$ are polynomials of degree less than $2k-p$. Now for the summation over n (Hansen 1975)

$$\sum_{n=0}^{N-k} n^p = (p+1)^{-1} (B_{p+1}(N-k+1) - B_{p+1}) \quad (\text{A.25})$$

where $B_{p+1}(x)$ is a Bernoulli polynomial and B_{p+1} is a Bernoulli number. Then in (A.24) the first term within the square bracket contributes a factor N^{2k+1} where N^{2k-p} multiplies the highest power in $B_{p+1}(N-k+1)$. The second term contributes only lower orders. Summing the net contribution proportional to N^{2k+1} we find (Hansen 1975)

$$\bar{C}_k = 2^{2k} \sum_{p=0}^k (-1)^p \binom{k}{p} \frac{1}{p+k+1} = 2^{2k} \frac{(k!)^2}{(2k+1)!} \quad (\text{A.26})$$

and asymptotically

$$\mathcal{N}^{-1} \sim \left(\frac{N\Omega}{2\Omega_0} \right)^{2N} N \sum_{k=0}^{\infty} 2^{2k} \frac{(k!)^2}{(2k+1)!} \left(\frac{\Omega_0}{\Omega} \right)^{2k}. \quad (\text{A.27})$$

With the help of a standard expansion for $\tan^{-1}$ (Abramowitz and Stegun 1965) we have

$$\mathcal{N}^{-1} \sim \left(\frac{N\Omega}{2\Omega_0} \right)^{2N} N \frac{\sin^{-1} \Omega_0/\Omega}{(\Omega_0/\Omega)[1 - (\Omega_0/\Omega)^2]^{1/2}}. \quad (\text{A.28})$$

Introducing this result to (2.9) and using (3.1) the exact steady state yields ($M \geq M'$)

$$\rho_{M,M'}^{ss} \sim \frac{(\Omega_0/\Omega)[1 - (\Omega_0/\Omega)^2]^{1/2}}{\sin^{-1} \Omega_0/\Omega} \left(\frac{N}{\frac{1}{2}N + M}\right)^{1/2} \left(\frac{N}{\frac{1}{2}N + M'}\right)^{1/2} \exp[-i(M - M')\pi/2] \\ \times \left(2 \frac{\Omega_0}{\Omega}\right)^{M-M'} \sum_{k=0}^{\frac{1}{2}N-M} \left(2 \frac{\Omega_0}{\Omega}\right)^{2k} \frac{(\frac{1}{2}N - M)! (\frac{1}{2}N - M')! (\frac{1}{2}N + M + k)!}{N! N^{2k+M-M'+1} (\frac{1}{2}N - M - k)!} \quad (\text{A.29})$$

A comparison between (A.29) and (3.33) reveals differences in the series component where both the coefficients and the range of summation are at variance. However, we are concerned only with the limit $N \rightarrow \infty$. A complication arises here as the ranges of $M = -\frac{1}{2}N \dots \frac{1}{2}N$ and $M' = -\frac{1}{2}N \dots M$ are themselves N dependent. We consider four separate cases. In (A.29) let us define

$$E_{M,M'}^k = \frac{(\frac{1}{2}N - M)! (\frac{1}{2}N - M')! (\frac{1}{2}N + M + k)!}{N! N^{2k+M-M'+1} (\frac{1}{2}N - M - k)!} \left(\frac{N}{\frac{1}{2}N + M}\right)^{1/2} \left(\frac{N}{\frac{1}{2}N + M'}\right)^{1/2} \quad (\text{A.30})$$

and for comparison, in (3.33) we define

$$A_{M,M'}^k = \left(\frac{N}{\frac{1}{2}N + M}\right)^{1/2} \left(\frac{N}{\frac{1}{2}N + M'}\right)^{1/2} B(\frac{1}{2}N + M + k + 1, \frac{1}{2}N - M' + k + 1). \quad (\text{A.31})$$

Then with M and M' fixed, in Stirling's approximation we find

$$E_{M,M'}^k = A_{M,M'}^k \sim N^{-1} (\frac{1}{2}M - M' + 2k). \quad (\text{A.32})$$

The finite sum in (A.29) tends to an infinite power series giving full agreement with (3.33). Specifically, both results have

$$\rho_{M,M'}^{ss} \sim \frac{1}{N} \exp[-i(M - M')\pi/2] \left(\frac{\Omega_0}{\Omega}\right)^{M-M'} \frac{(\Omega_0/\Omega)[1 - (\Omega_0/\Omega)^2]^{1/2}}{\sin^{-1} \Omega_0/\Omega} [1 - (\Omega_0/\Omega)^2]^{-1}. \quad (\text{A.33})$$

We might also consider expressions for $\rho_{-\frac{1}{2}N+p, -\frac{1}{2}N+q}^{ss}$ and $\rho_{\frac{1}{2}N-q, \frac{1}{2}N-p}^{ss}$ where $p \geq q \geq 0$. Using Stirling's approximation

$$E_{-\frac{1}{2}N+p, -\frac{1}{2}N+q}^k = A_{-\frac{1}{2}N+p, -\frac{1}{2}N+q}^k \sim (p+k)! (p!q!)^{-1/2} N^{-[\frac{1}{2}(p-q)+k+1]} \quad (\text{A.34})$$

and

$$E_{\frac{1}{2}N-q, \frac{1}{2}N-p}^k \sim (p!q!)^{1/2} [(q-k)!]^{-1} N^{-[\frac{1}{2}(p-q)+k+1]} \\ A_{\frac{1}{2}N-q, \frac{1}{2}N-p}^k \sim (p+k)! (p!q!)^{-1/2} N^{-[\frac{1}{2}(p-q)+k+1]} \quad (\text{A.35})$$

Due to the presence of a factor N^{-k} , for both of these cases the series in (A.29) and (3.33) may be truncated at the first term. Then we again have agreement and here

$$\rho_{-\frac{1}{2}N+p, -\frac{1}{2}N+q}^{ss} = \rho_{\frac{1}{2}N-q, \frac{1}{2}N-p}^{ss} \\ \sim \frac{1}{N^{\frac{1}{2}(p-q)+1}} \left(\frac{p}{q}\right)^{1/2} \exp[-i(p-q)\pi/2] \left(2 \frac{\Omega_0}{\Omega}\right)^{p-q} \frac{(\Omega_0/\Omega)[1 - (\Omega_0/\Omega)^2]^{1/2}}{\sin^{-1} \Omega_0/\Omega}. \quad (\text{A.36})$$

A final comparison may be made for the matrix elements $\rho_{\frac{1}{2}N-p, -\frac{1}{2}N+q}^{ss}$. Here expressions from (A.29) and (3.33) differ. However, both vanish asymptotically, the first as e^{-N} and the second as 2^{-2N} .

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