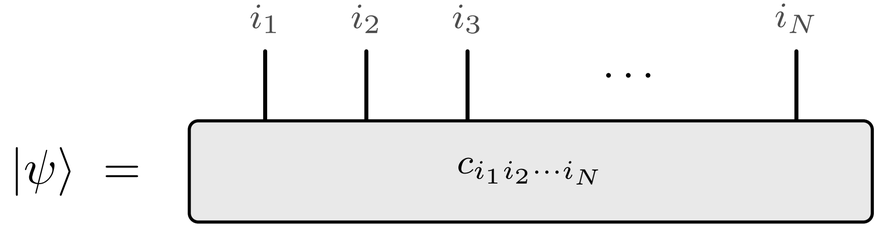
Tensor Networks for Quantum Computing

School on Quantum Simulation in the NISQ era — ICTP-SAIFR, São Paulo, 2026

Mario Collura (SISSA)

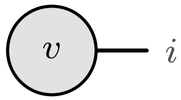
Lecture 1
Tensor networks in 1D: MPS, MPO and the Pauli basis

Tensors, contractions and gates

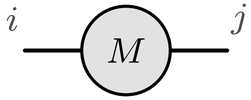


one tensor with N legs: d^N entries

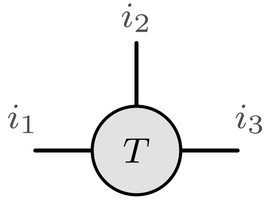
$$\#\{c_{i_1 \ldots i_N}\} = d^N$$



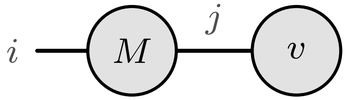
vector v_i

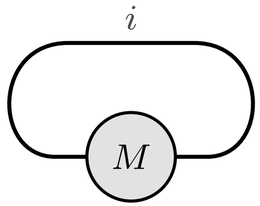


matrix M_{ij}

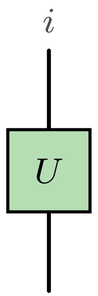


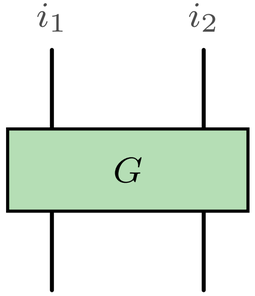
tensor $T_{i_1 i_2 i_3}$





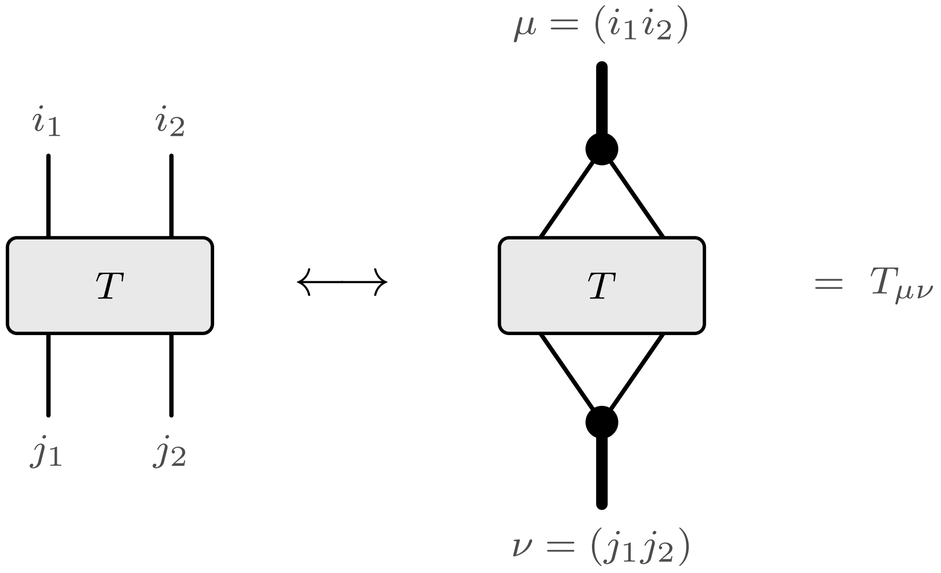
out

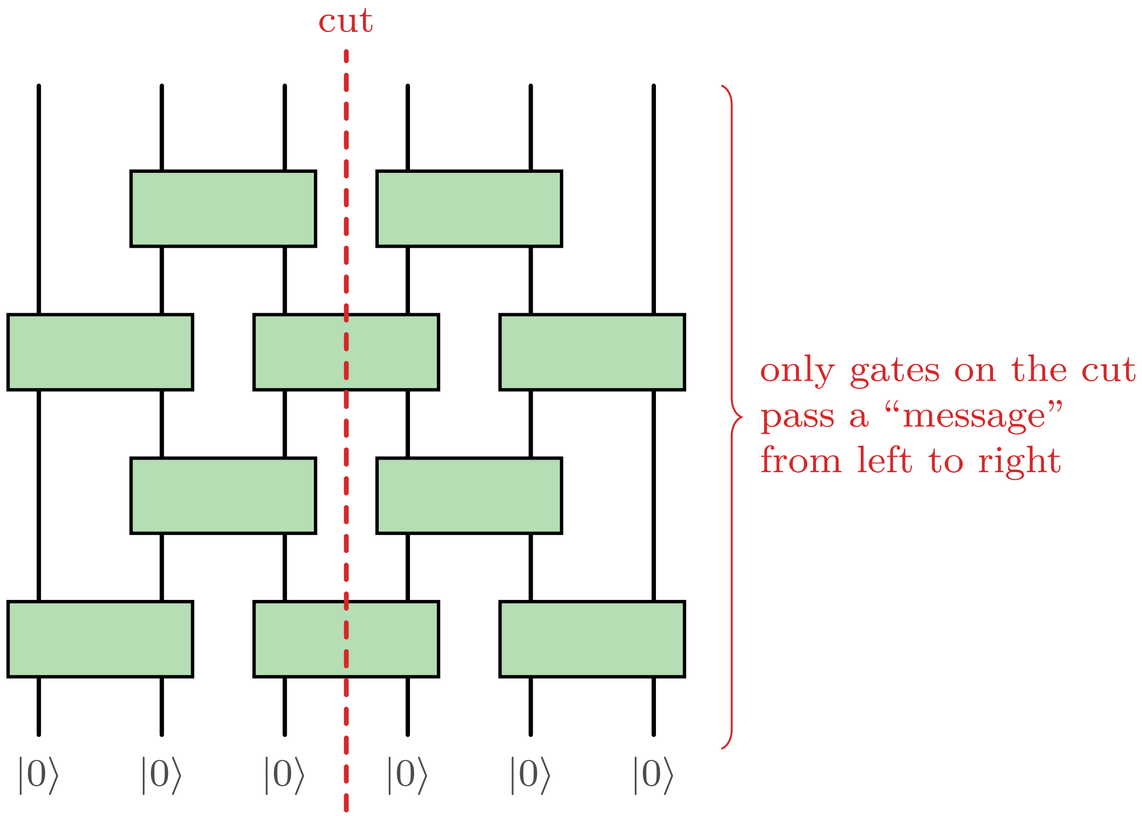




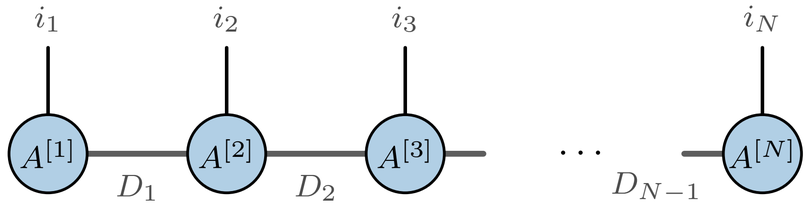
in

Index fusion, and why MPS should exist

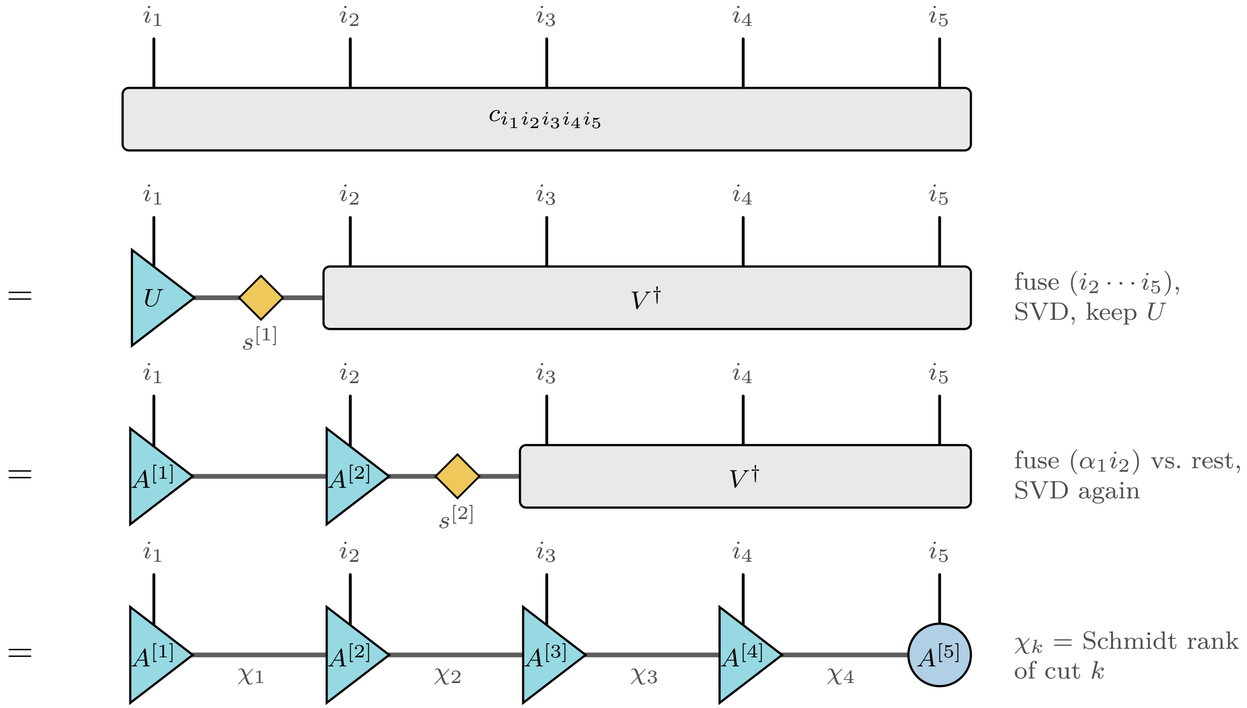




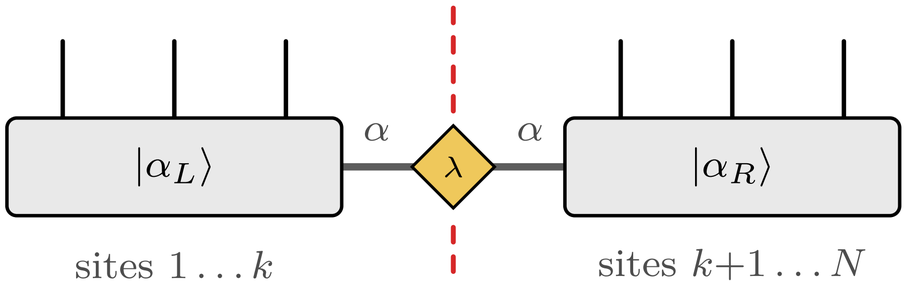
MPS: exact construction by iterated SVD



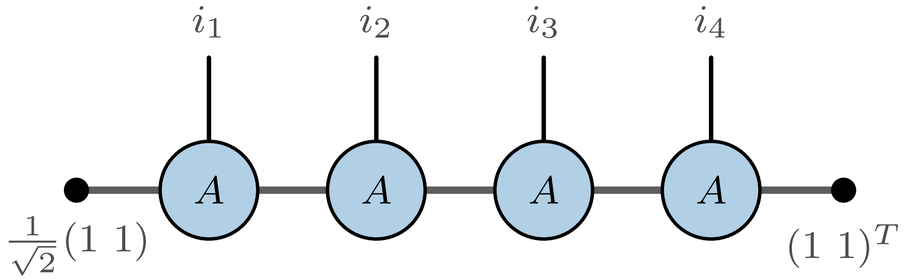
$$c_{i_1 \ldots i_N} = A^{[1]i_1} A^{[2]i_2} \cdots A^{[N]i_N}$$

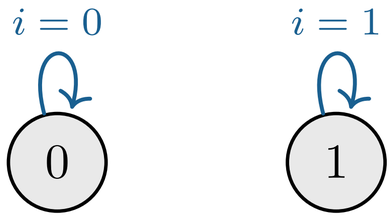


Entanglement, GHZ and W

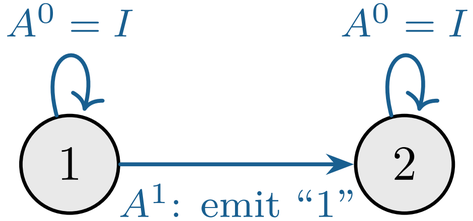


$$S(\rho_L) = -\sum_\alpha \lambda_\alpha^2 \log \lambda_\alpha^2 \leq \log \chi_k$$



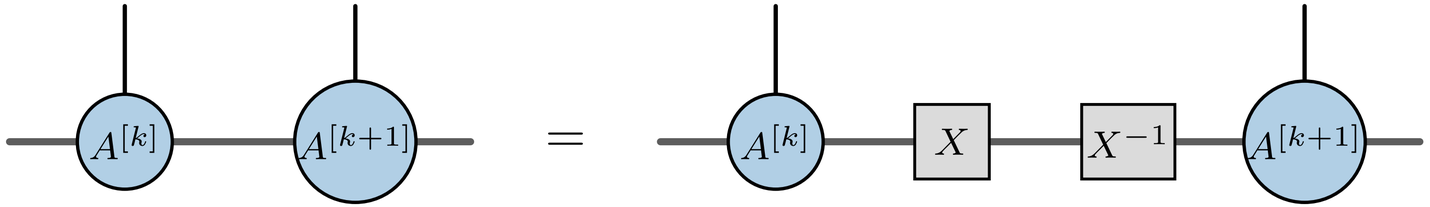


GHZ



W

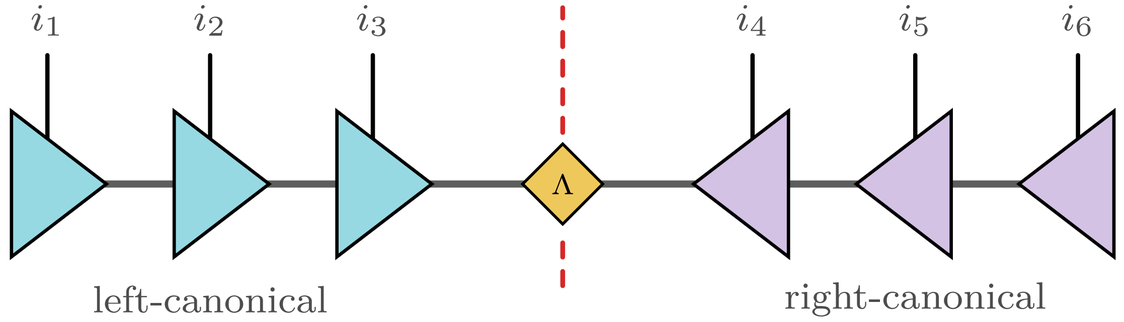
Gauge freedom and canonical forms





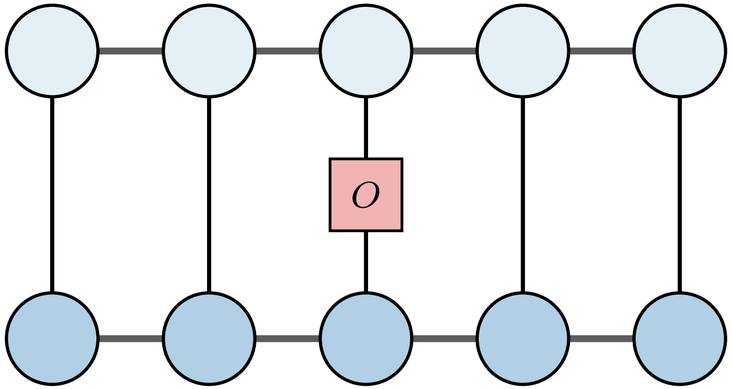
left-canonical: $\sum_i (A^i)^\dagger A^i = I$

right-canonical: $\sum_i A^i (A^i)^\dagger = I$



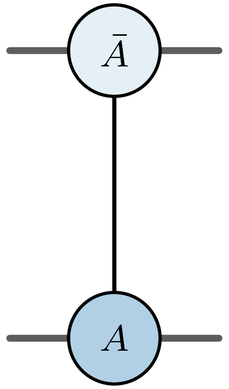
Expectation values and transfer matrices

$\langle O_3 \rangle =$

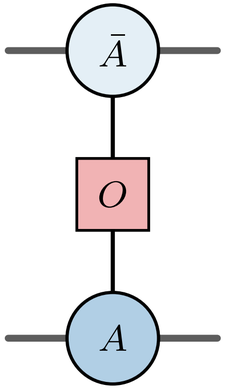


insertion at site $k = 3$

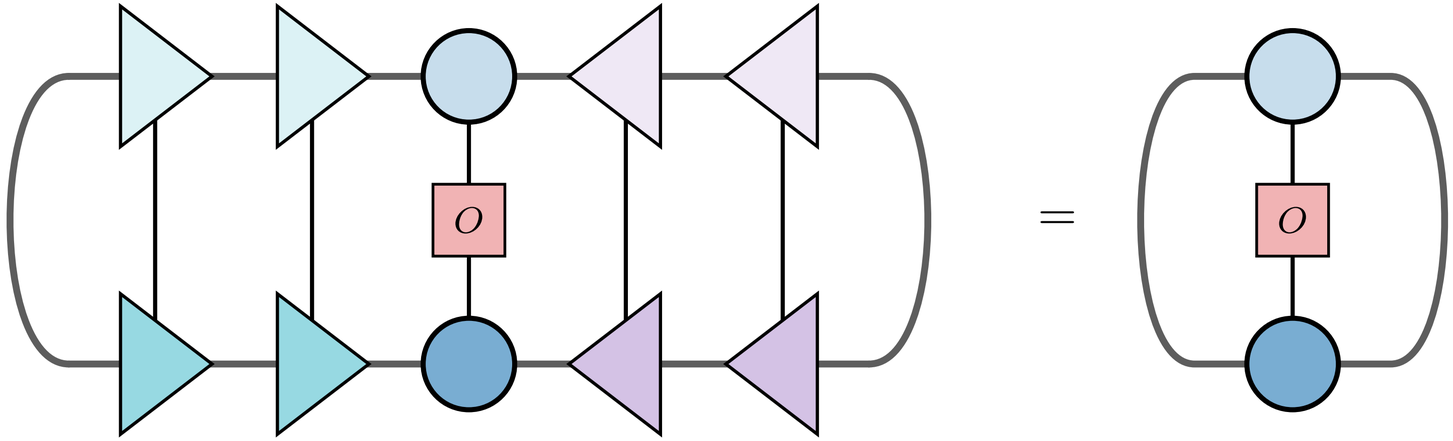
$E^{[\ell]} =$



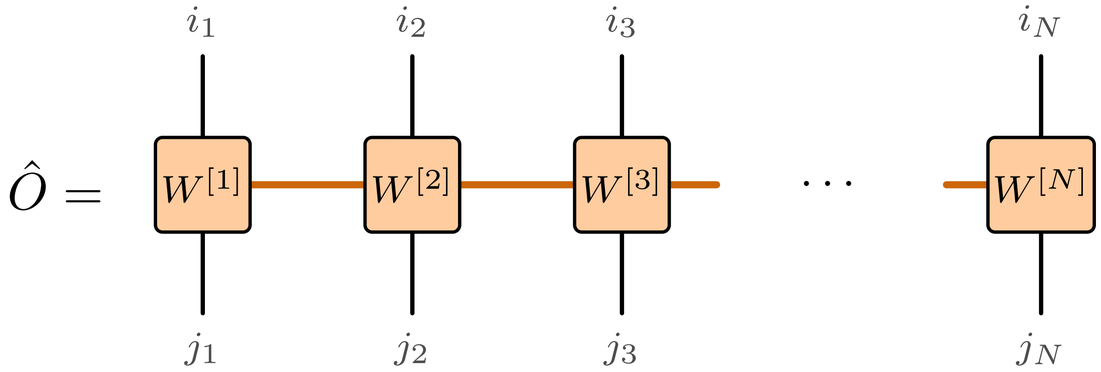
$E_O^{[k]} =$



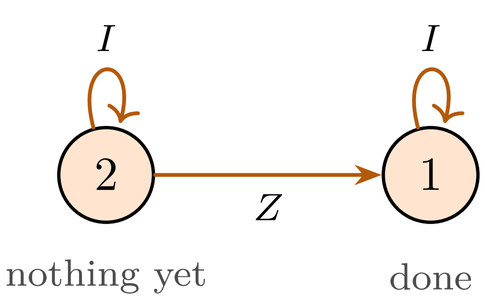
$$\langle O_k \rangle = E^{[1]} \dots E_O^{[k]} \dots E^{[N]}, \qquad E^{[\ell]} = \sum_i \overline{A^{[\ell]i}} \otimes A^{[\ell]i}$$



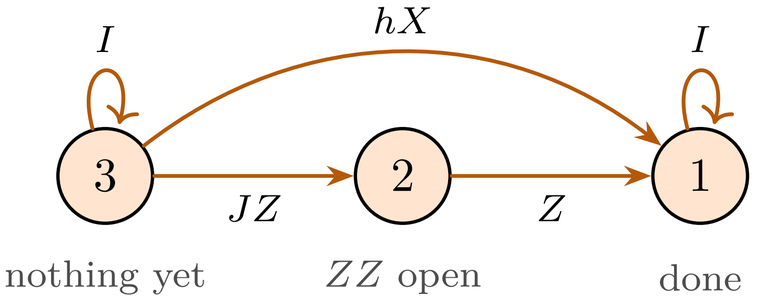
MPOs: exact examples



$$H = \sum_k J Z_k Z_{k+1} + \sum_k h X_k : \qquad W^{[k]} = \begin{pmatrix} I & 0 & 0 \\ Z & 0 & 0 \\ hX & JZ & I \end{pmatrix}$$



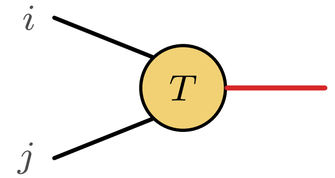
$\sum_k Z_k \quad (\chi = 2)$



transverse-field Ising $(\chi = 3)$

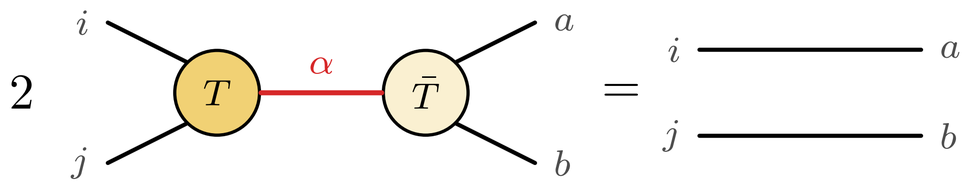
The Pauli basis: the change-of-basis tensor T

(a)



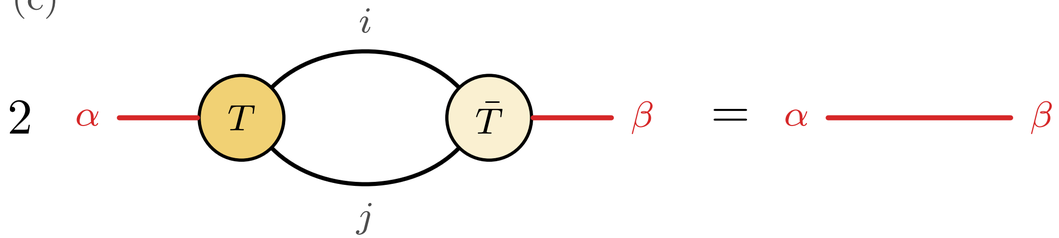
$= \; T^{\alpha}_{ij} = \frac{1}{2}\langle j|P_\alpha|i\rangle, \qquad \overline{T^{\alpha}_{ij}} = \frac{1}{2}\langle i|P_\alpha|j\rangle$

(b)



$2\sum_\alpha T^{\alpha}_{ij}\,\overline{T^{\alpha}_{ab}} = \delta_{ia}\delta_{jb}$

(c)

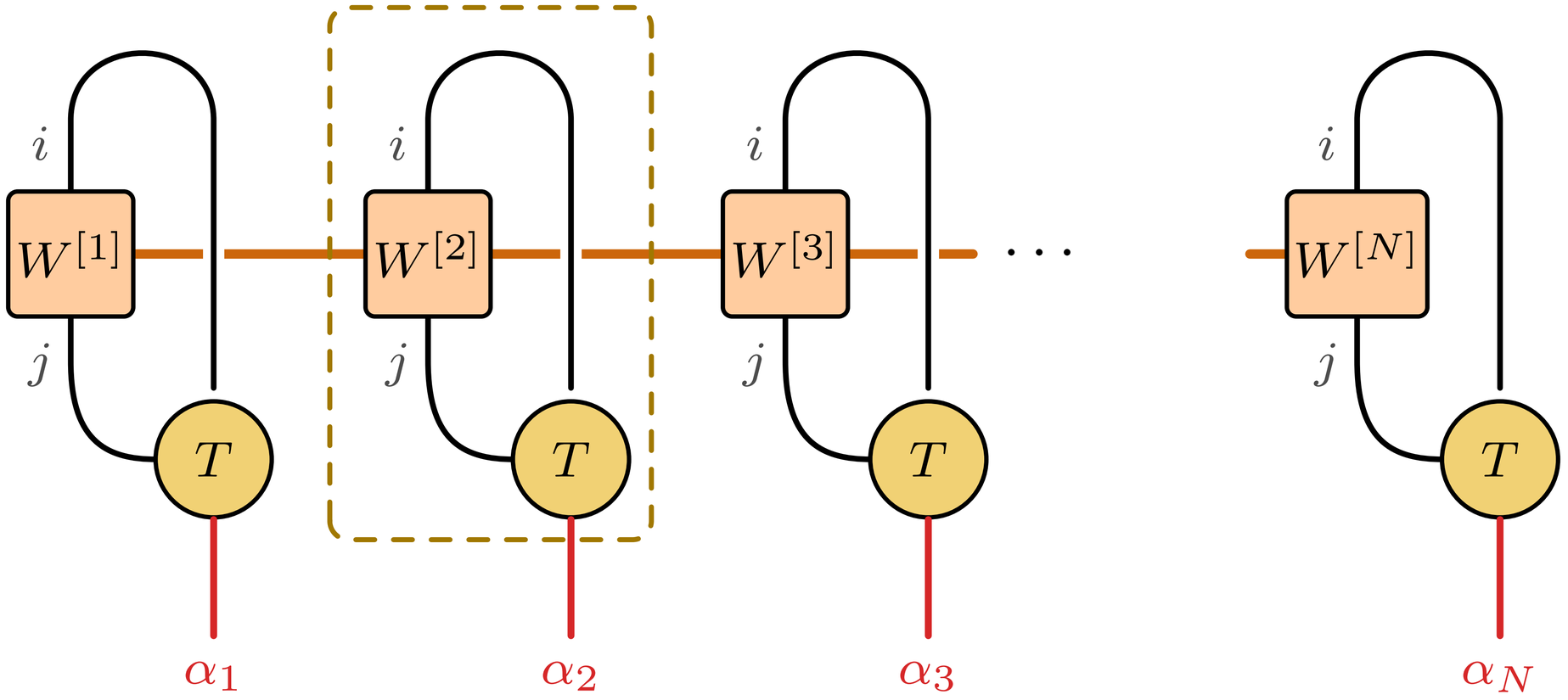


$2\sum_{ij} T^{\alpha}_{ij}\,\overline{T^{\beta}_{ij}} = \frac{1}{2}\mathrm{Tr}(P_\alpha P_\beta) = \delta_{\alpha\beta}$

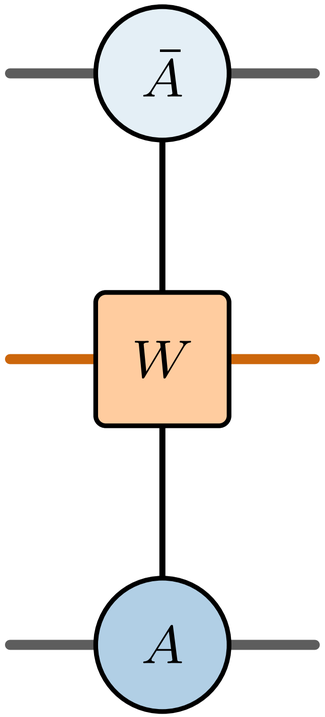
An MPO in the Pauli basis

$$\widetilde{W}^{[k]\alpha_k} = \sum_{ij} T_{ij}^{\alpha_k} W^{[k]ij}$$

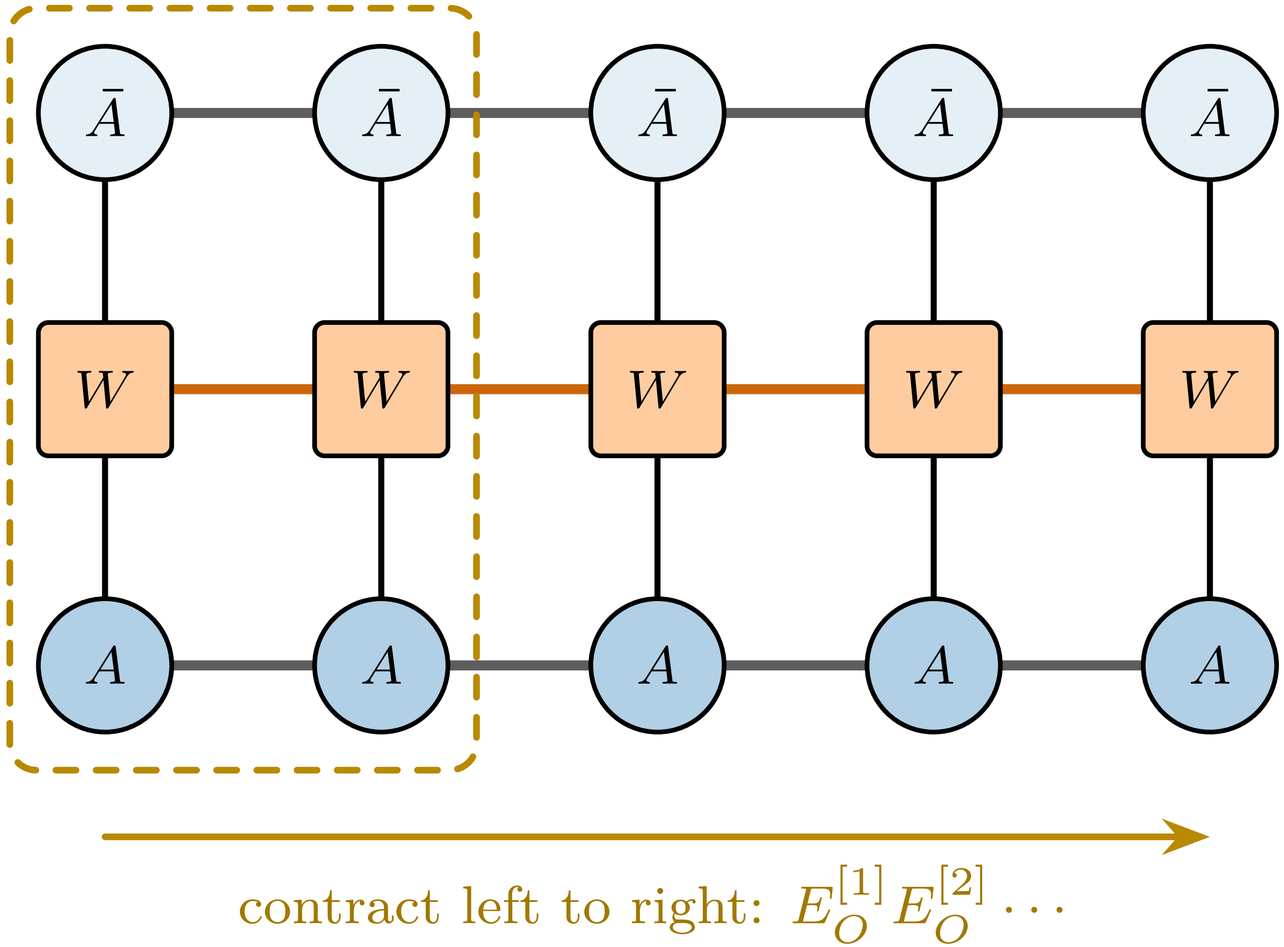
$$c_{\alpha_1 \cdots \alpha_N} =$$



Expectation values with MPOs



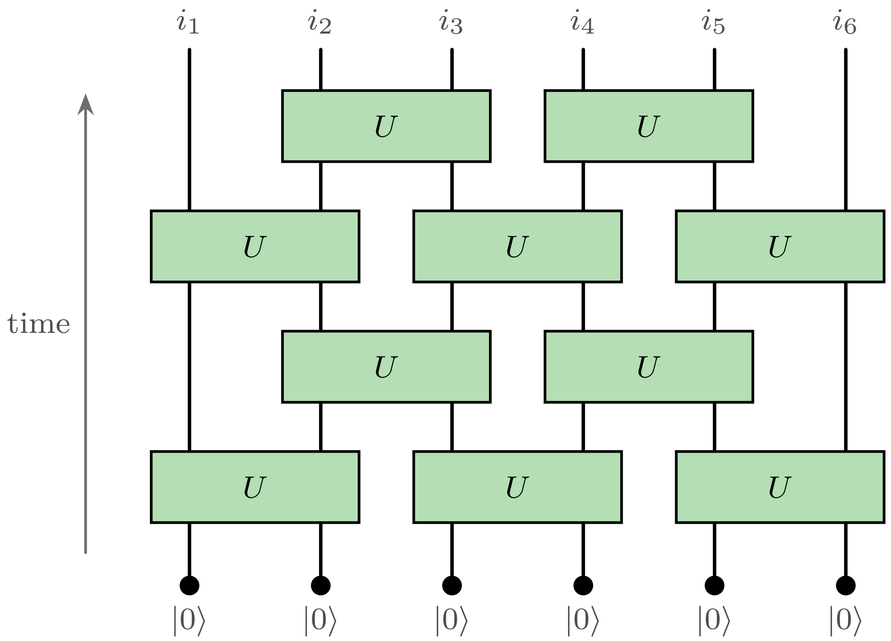
$$\langle\psi|\hat{O}|\psi\rangle =$$

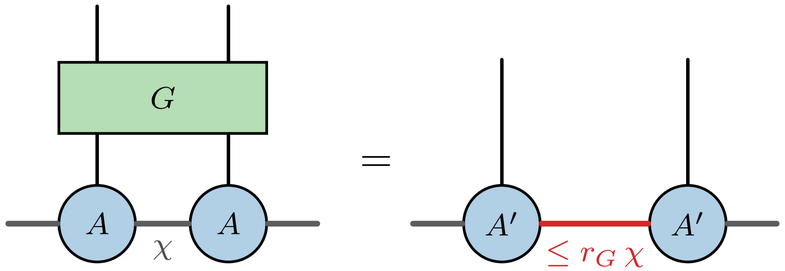


$$\langle\psi|\hat{O}|\psi\rangle = E_O^{[1]} E_O^{[2]} \cdots E_O^{[N]}$$

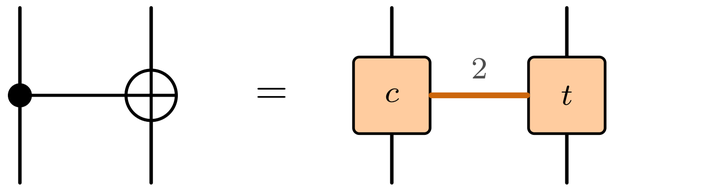
Lecture 2
Simulating quantum circuits with tensor networks: gates, measurements and noise

Circuits are tensor networks



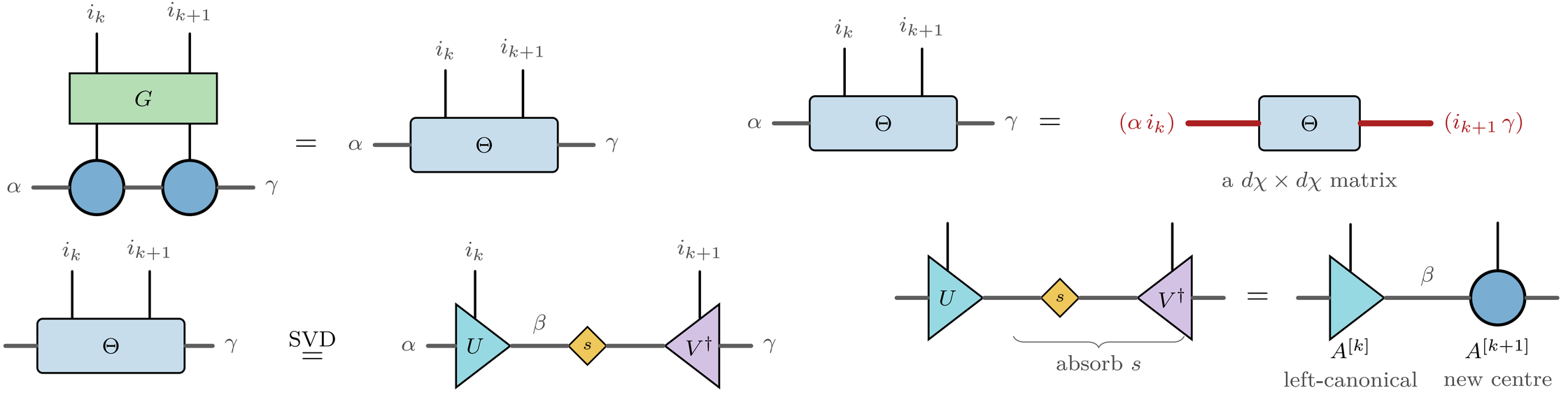


$$G = \sum_{a=1}^{r_G} L_a \otimes R_a : \quad \text{CNOT } (r=2), \quad e^{-i\theta ZZ} \ (r=2), \quad \text{SWAP } (r=4)$$



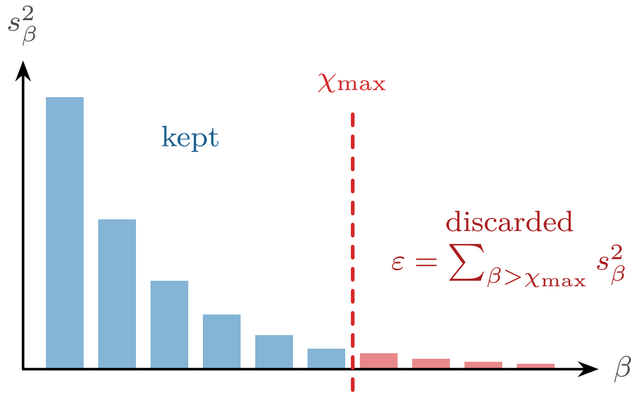
$$c = \big(|0\rangle\langle 0|,\ |1\rangle\langle 1|\big) \quad t = \big(I,\ X\big)^T$$

The two-site update

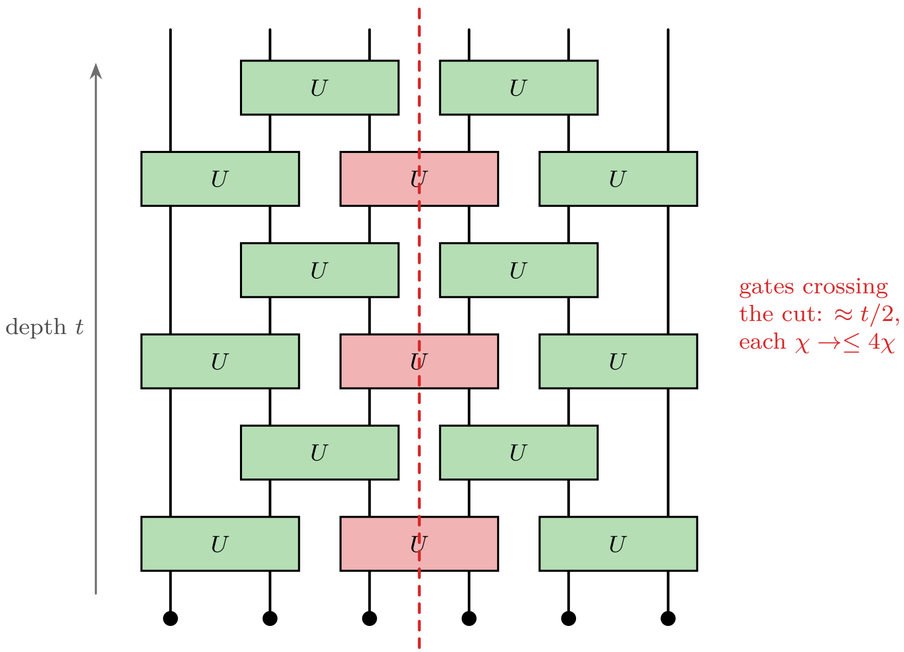


cost $O(d^3\chi^3)$ per gate, memory $O(Nd\chi^2)$

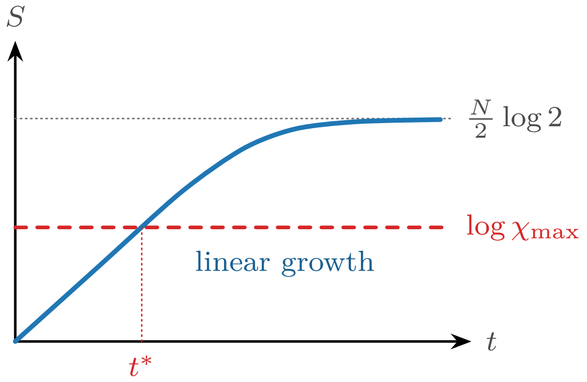
Truncation and the entanglement barrier

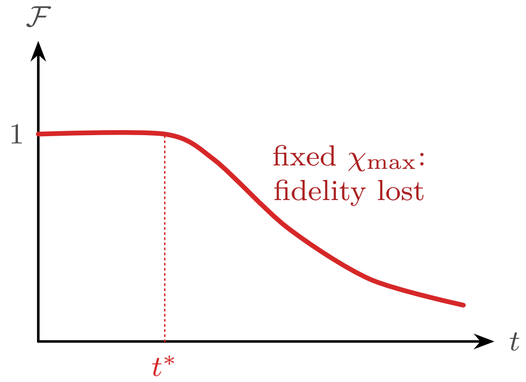


$$\varepsilon = \sum_{\beta>\chi} s_\beta^2, \quad |\langle\psi|\tilde{\psi}\rangle|^2 = 1 - \varepsilon$$

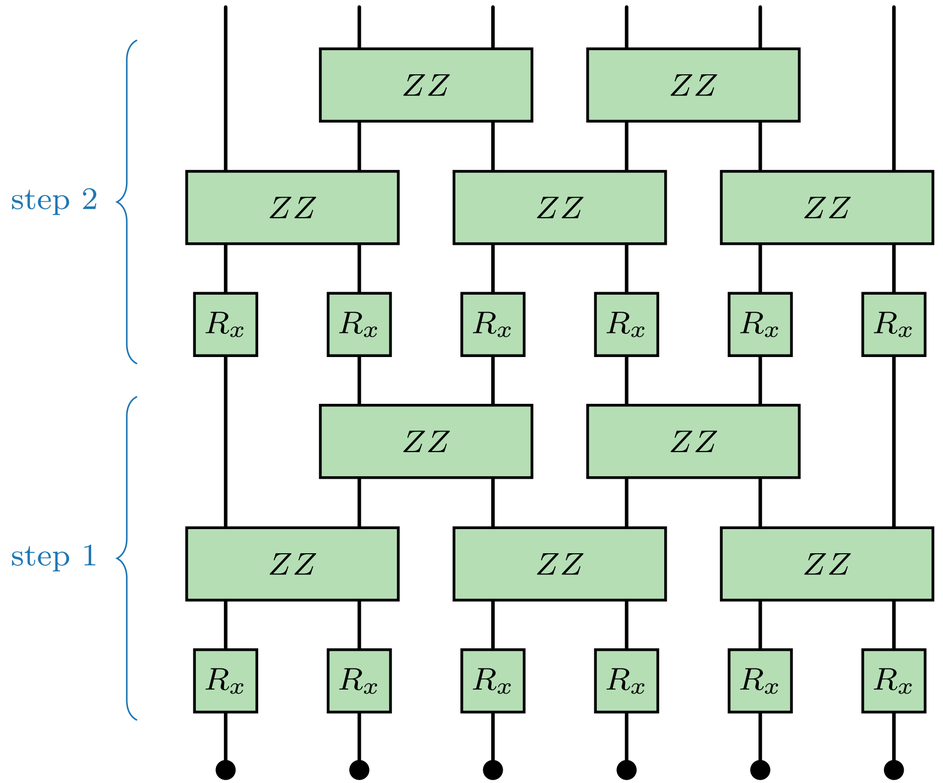


gates crossing the cut: $\approx t/2$, each $\chi \to \leq 4\chi$





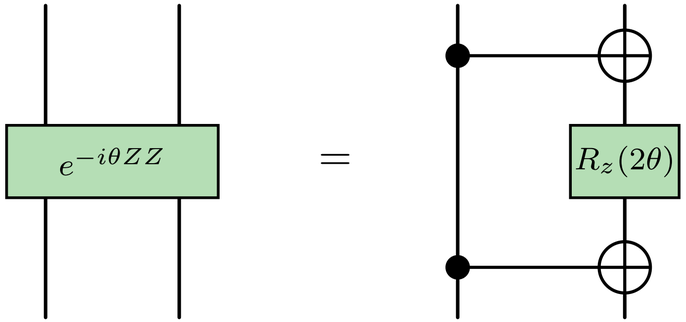
Hamiltonian dynamics as a circuit



$R_x = e^{-ih\delta X}$

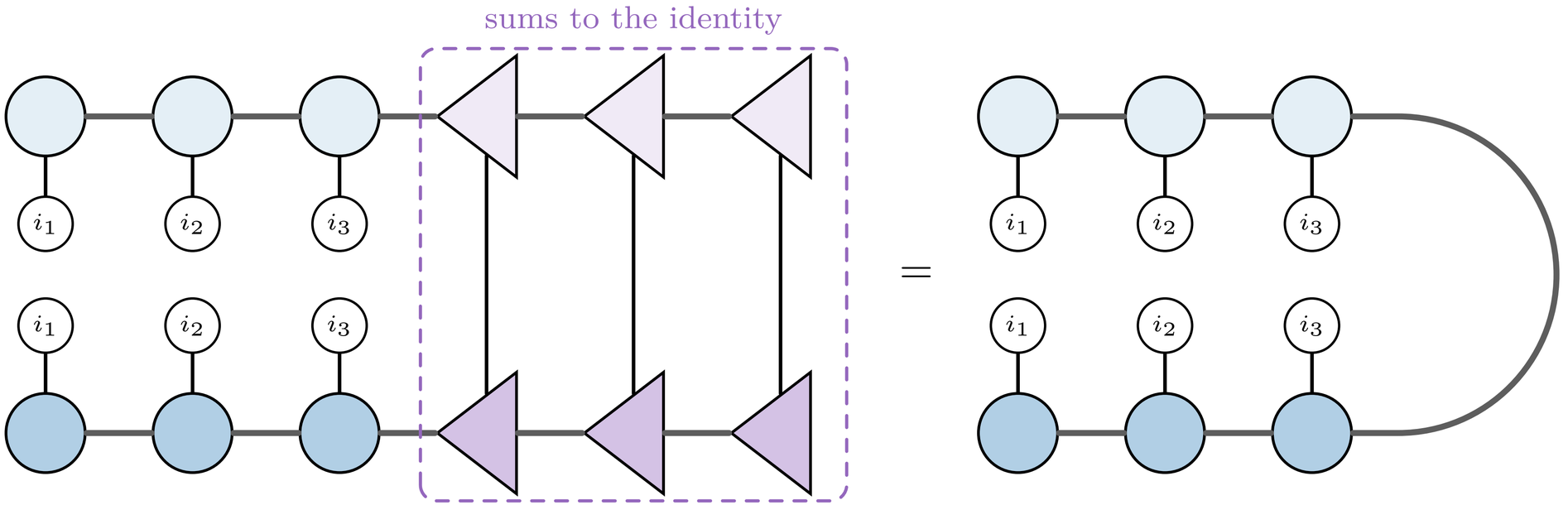
$ZZ = e^{-iJ\delta ZZ}$

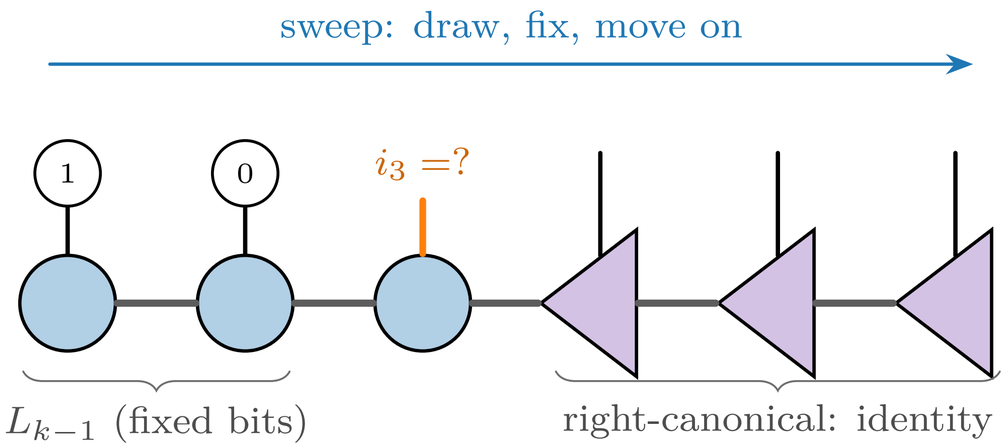
(all ZZ commute: that layer is exact)



$$e^{-i\delta H} = e^{-i\delta H_{ZZ}} e^{-i\delta H_X} + O(\delta^2)$$

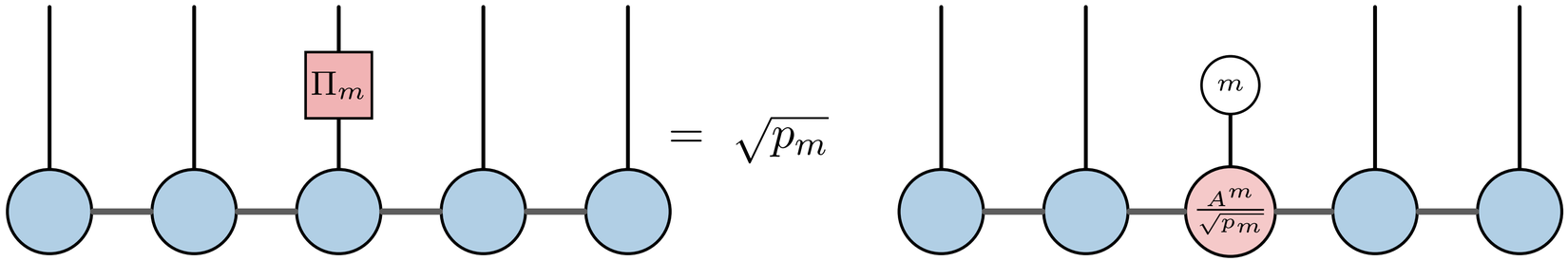
Born probabilities and perfect sampling

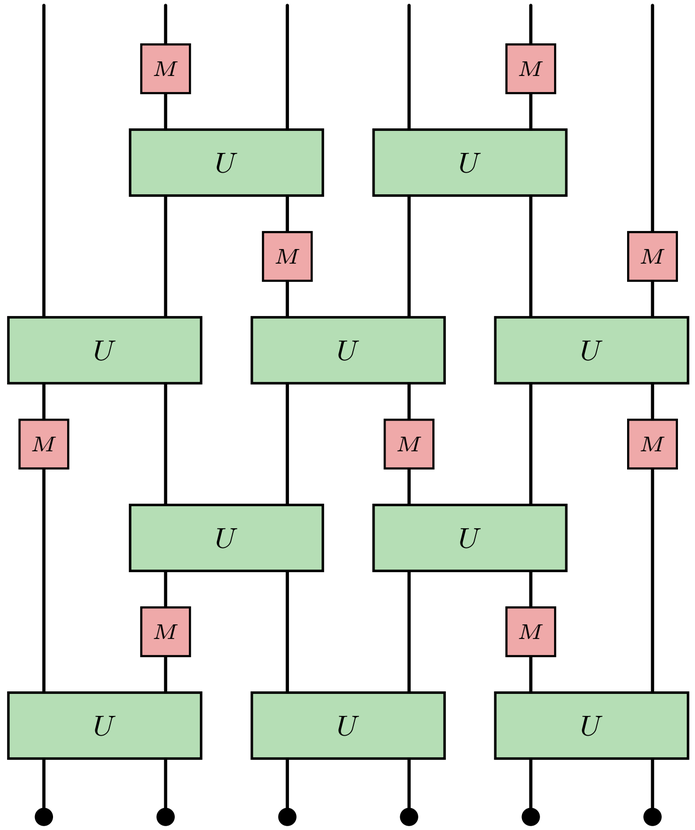




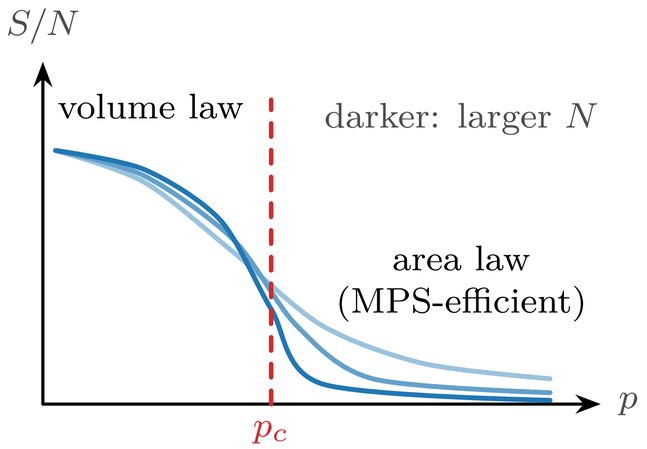
$$p(i_k|i_{<k}) = \frac{\|L_{k-1}A^{[k]i_k}\|^2}{\|L_{k-1}\|^2}$$

Measurements and monitored circuits

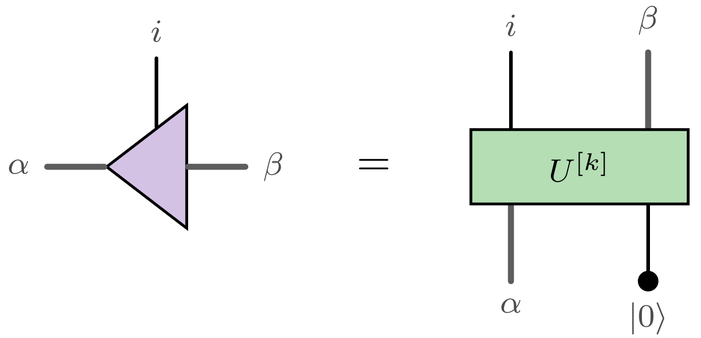


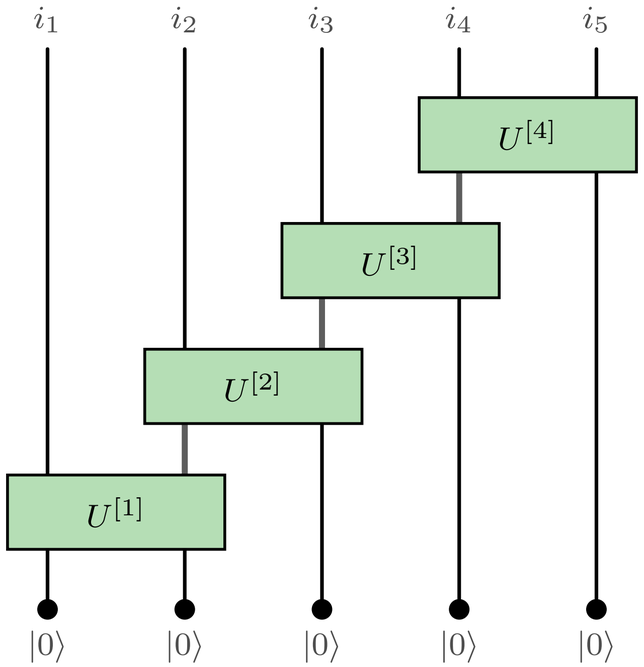


M measurement with prob. p

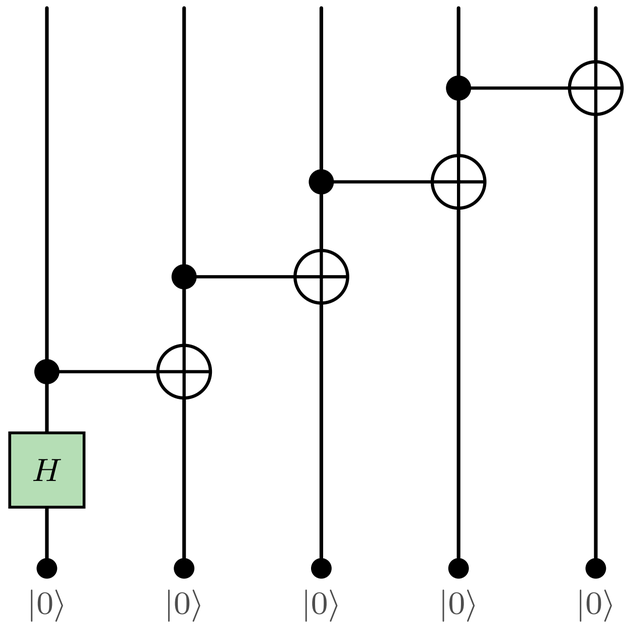


From an MPS to a circuit (χ = 2)



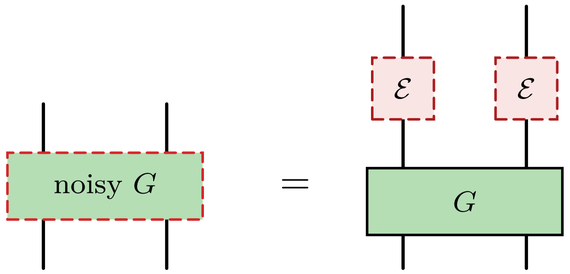


general $\chi = 2$ MPS



GHZ: Hadamard + CNOT ladder

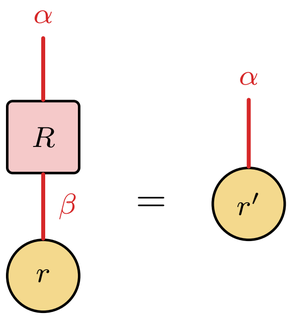
Noise and the Pauli transfer matrix



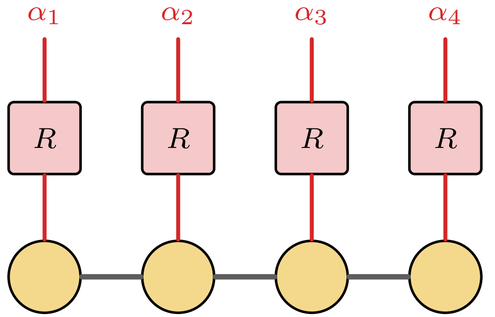
(a) one qubit

$$r_P = \mathrm{Tr}(\rho P), \qquad R_{PQ} = \tfrac{1}{2^N}\mathrm{Tr}\big[P\,\mathcal{E}(Q)\big]$$

(b) local noise on N qubits

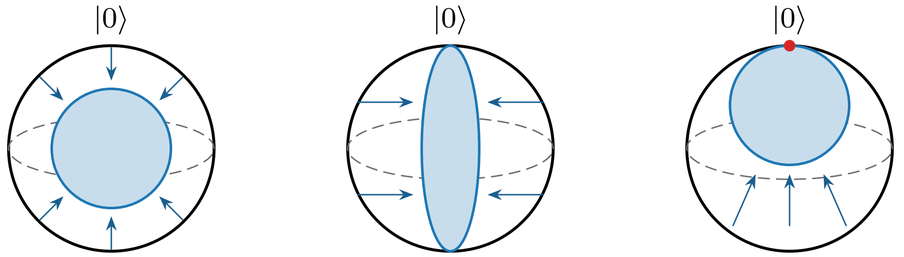


$r'_\alpha = \sum_\beta R_{\alpha\beta}\, r_\beta$



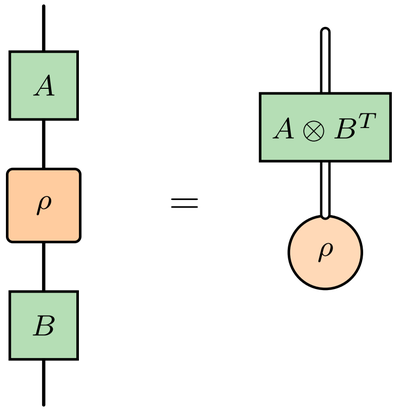
$R \otimes R \otimes \cdots \otimes R$:
no bond is touched

Pauli vector $r_{\alpha_1 \cdots \alpha_N}$
(here as an MPS)

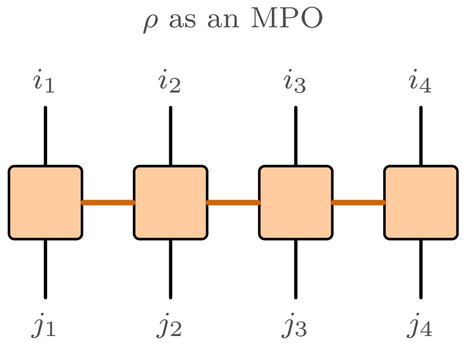


depolarizing dephasing amplitude damping

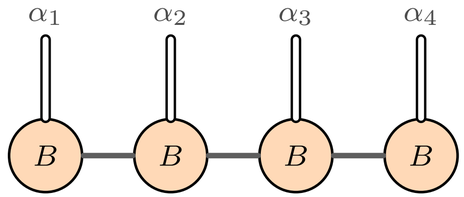
Density matrices as vectorized MPS



=

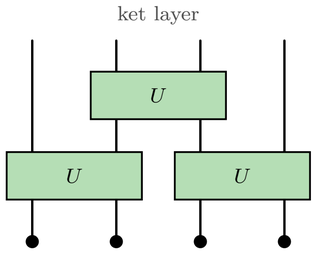


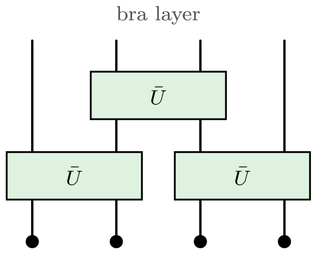
fold →



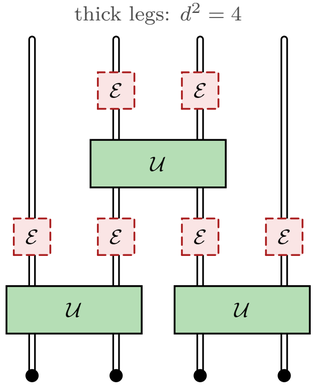
$|\rho\rangle\rangle$: MPS with $d^2 = 4$

$\alpha_k \in \{I, X, Y, Z\}$

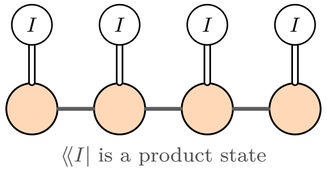




fold →

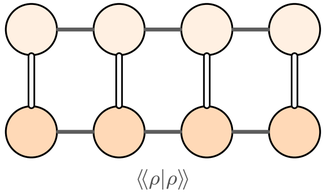


$\mathrm{Tr}\,\rho =$



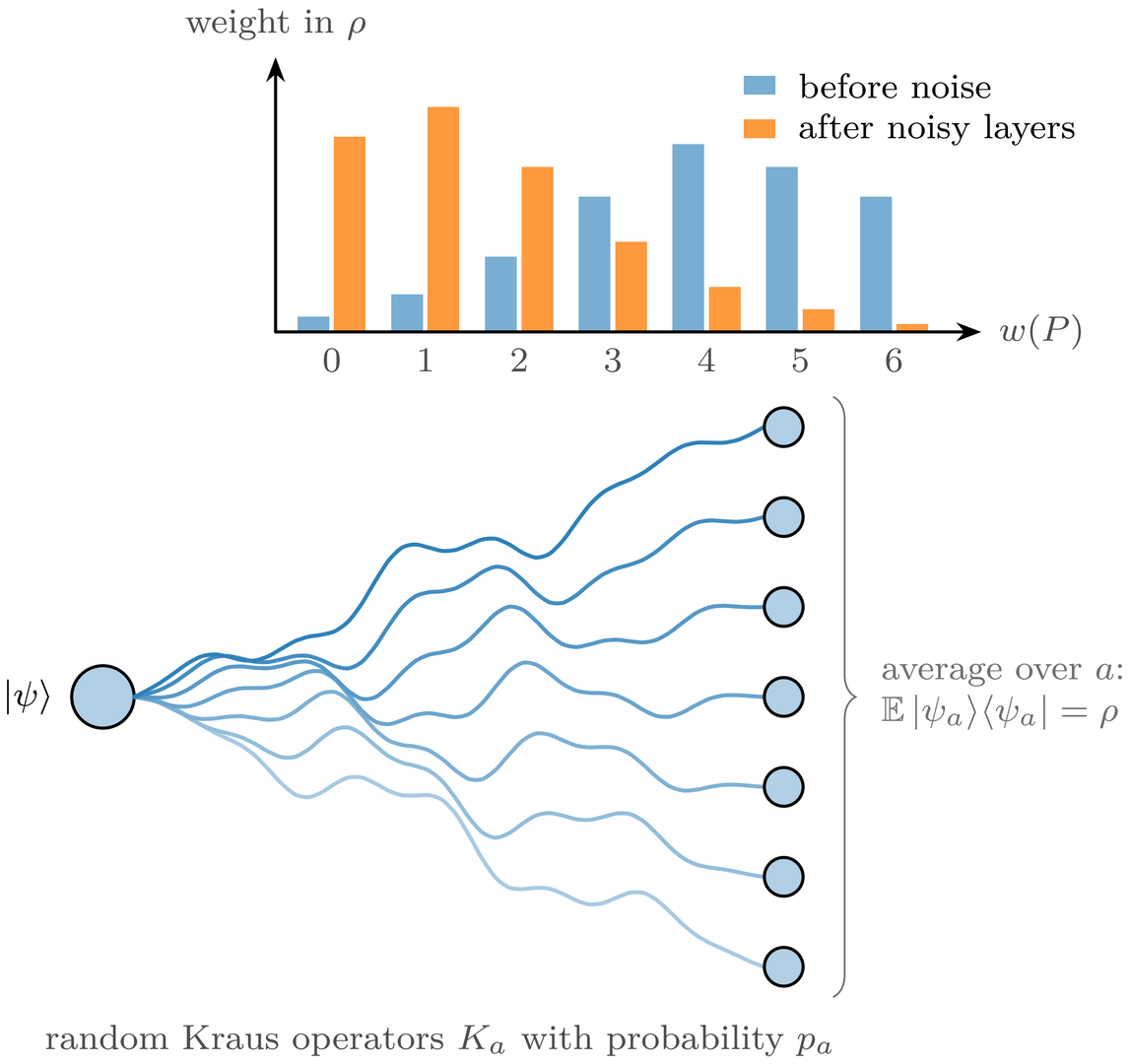
$\langle\langle I|$ is a product state

$\mathrm{Tr}\,\rho^2 =$



$\langle\langle\rho|\rho\rangle\rangle$

Why noise helps; trajectories

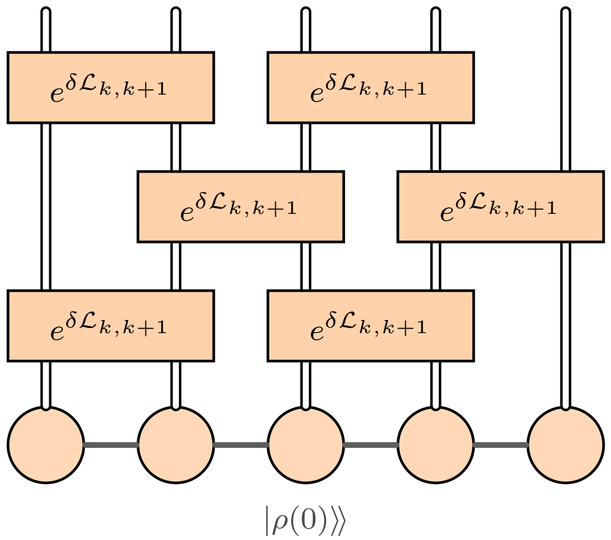


$$r_P \;\mapsto\; f^{\,w(P)}\, r_P$$

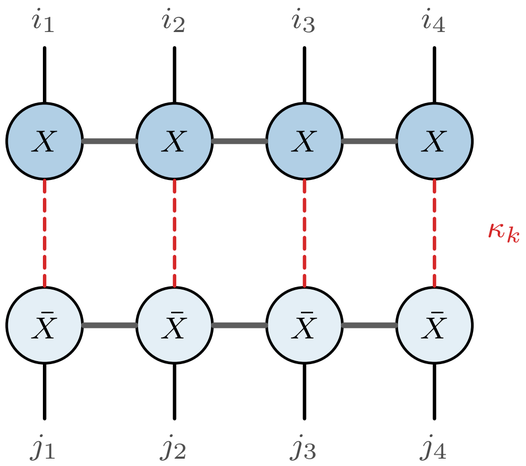
$$N_{\text{traj}} \;\geq\; \frac{2\|O\|^2}{\epsilon^2} \log \frac{2}{\delta}$$

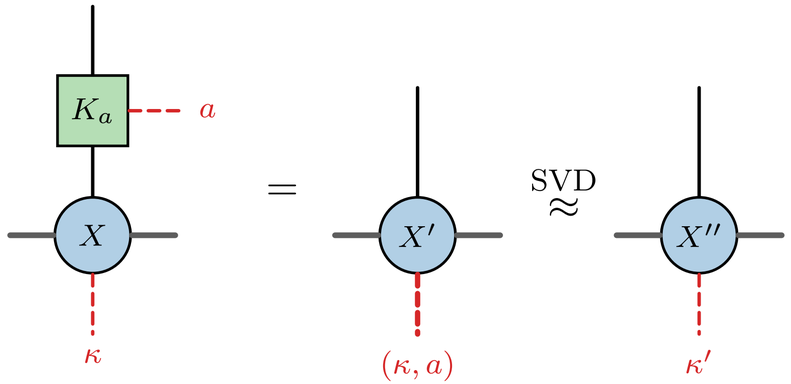
Lindblad evolution and LPTN

$$\dot{\rho} = -i[H, \rho] + \sum_{\mu} \left(L_\mu \rho L_\mu^\dagger - \tfrac{1}{2}\{L_\mu^\dagger L_\mu, \rho\} \right)$$



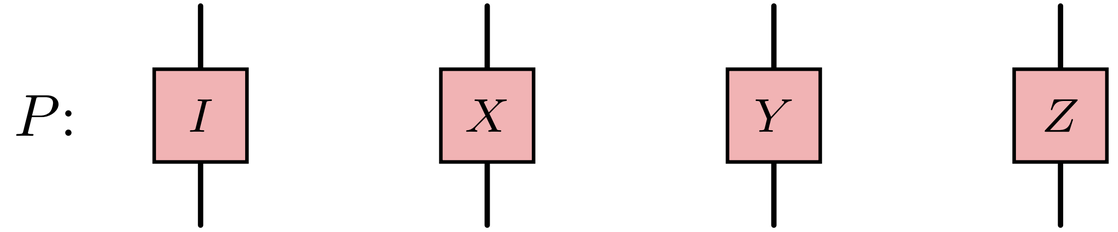
$\rho =$





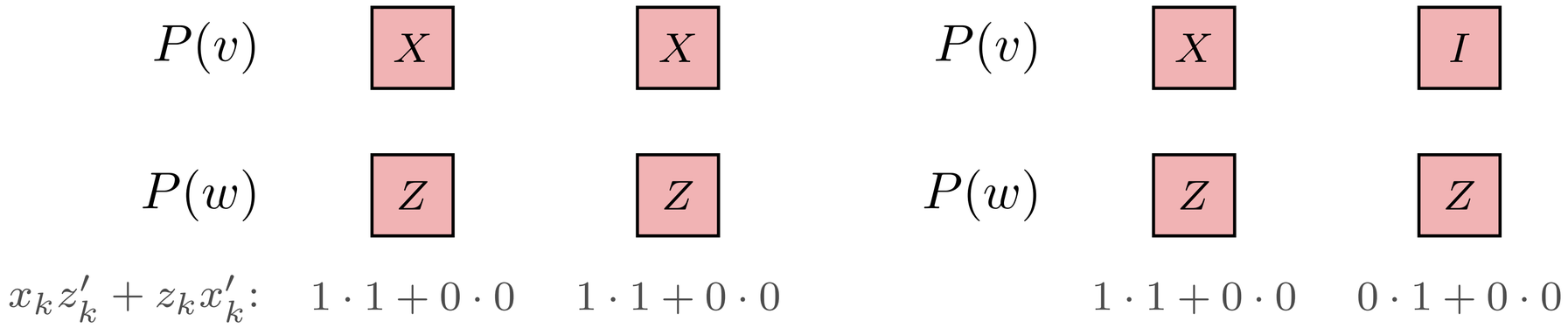
Lecture 3
Stabilizer states and Clifford circuits

The Pauli group in symplectic language



$$Y = iXZ$$

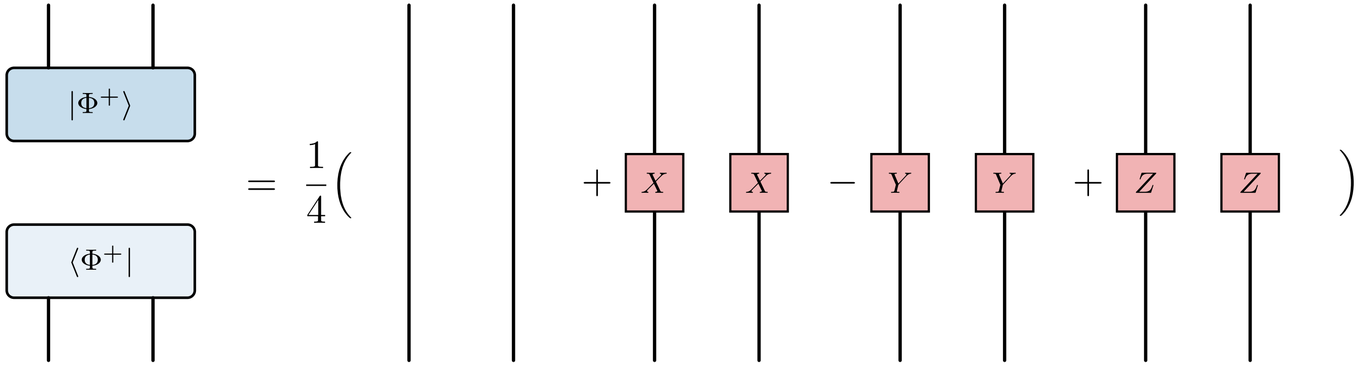
$$P(v)P(w) = \pm P(w)P(v), \qquad \omega(v,w) = x \cdot z' + z \cdot x' \mod 2$$

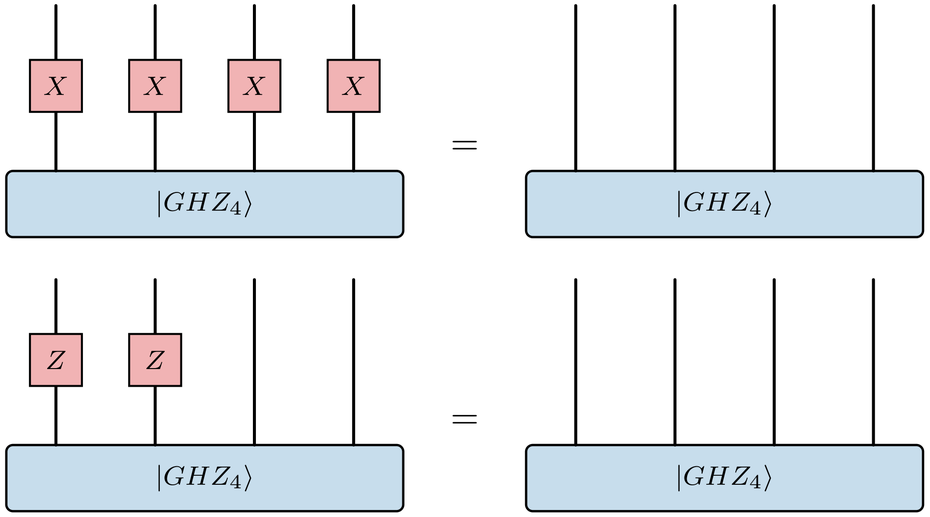


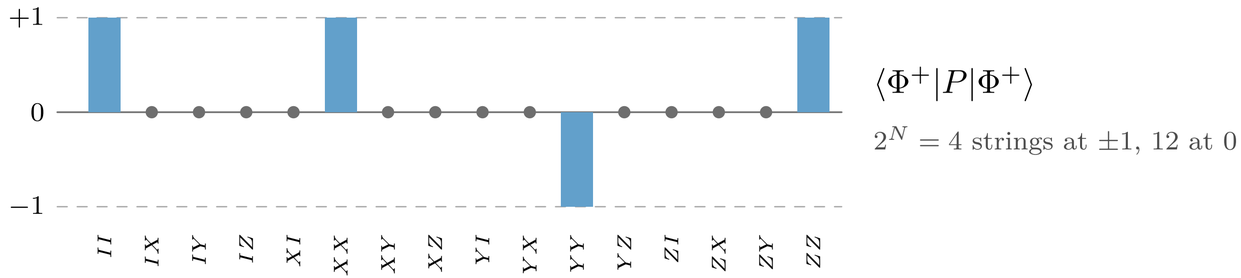
$\omega = 1 + 1 = 0 \Rightarrow$ **commute** $\qquad \omega = 1 + 0 = 1 \Rightarrow$ **anticommute**

Stabilizer states

$$|\psi\rangle\langle\psi| = \prod_{j=1}^{N} \frac{I + g_j}{2} = \frac{1}{2^N} \sum_{P \in S} P$$



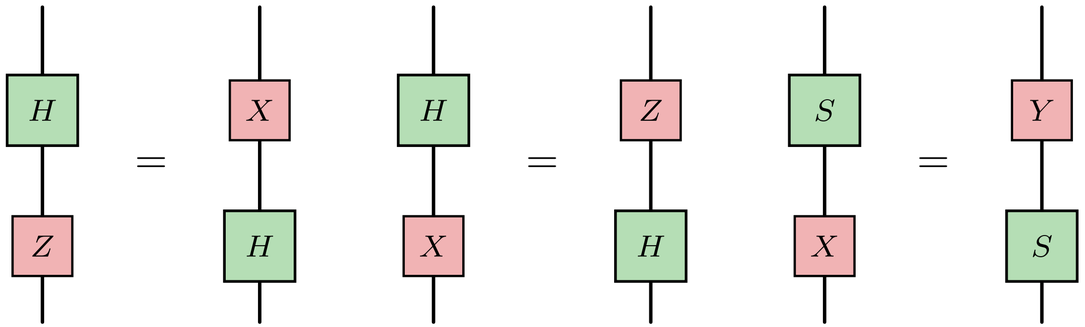




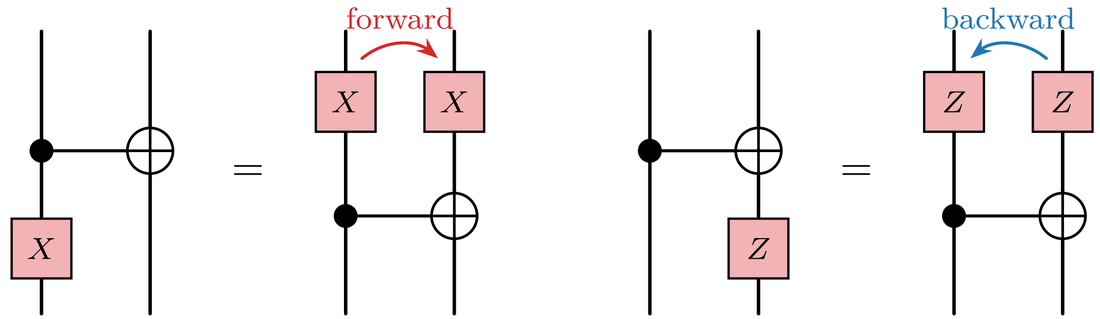
The Clifford group

	X	Y	Z
H	Z	$-Y$	X
S	Y	$-X$	Z

$\mathrm{CNOT}_{1\to 2}$	
X_1	X_1X_2
Z_1	Z_1
X_2	X_2
Z_2	Z_1Z_2



$HZH^\dagger = X$ $HXH^\dagger = Z$ $SXS^\dagger = Y$



control • left, target ⊕ right

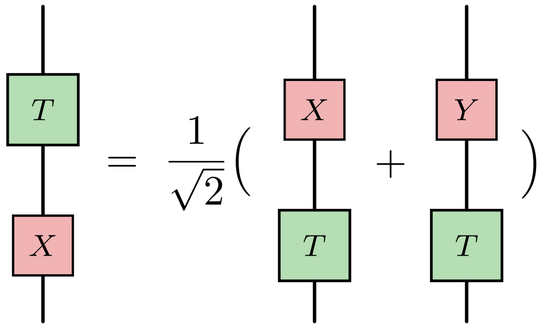
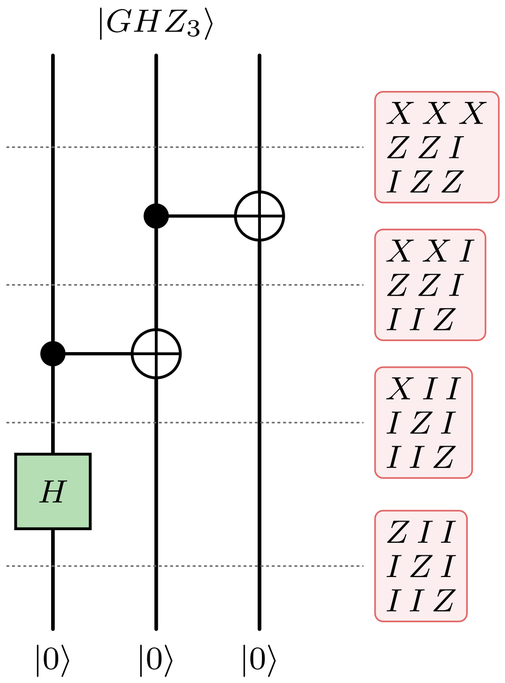


Tableau simulation (Gottesman–Knill)



$|000\rangle$

x_1	x_2	x_3	z_1	z_2	z_3	r
0	0	0	1	0	0	0
0	0	0	0	1	0	0
0	0	0	0	0	1	0

$\rightarrow$ after H_1

x_1	x_2	x_3	z_1	z_2	z_3	r
1	0	0	0	0	0	0
0	0	0	0	1	0	0
0	0	0	0	0	1	0

$\rightarrow$ after CNOT_{12}

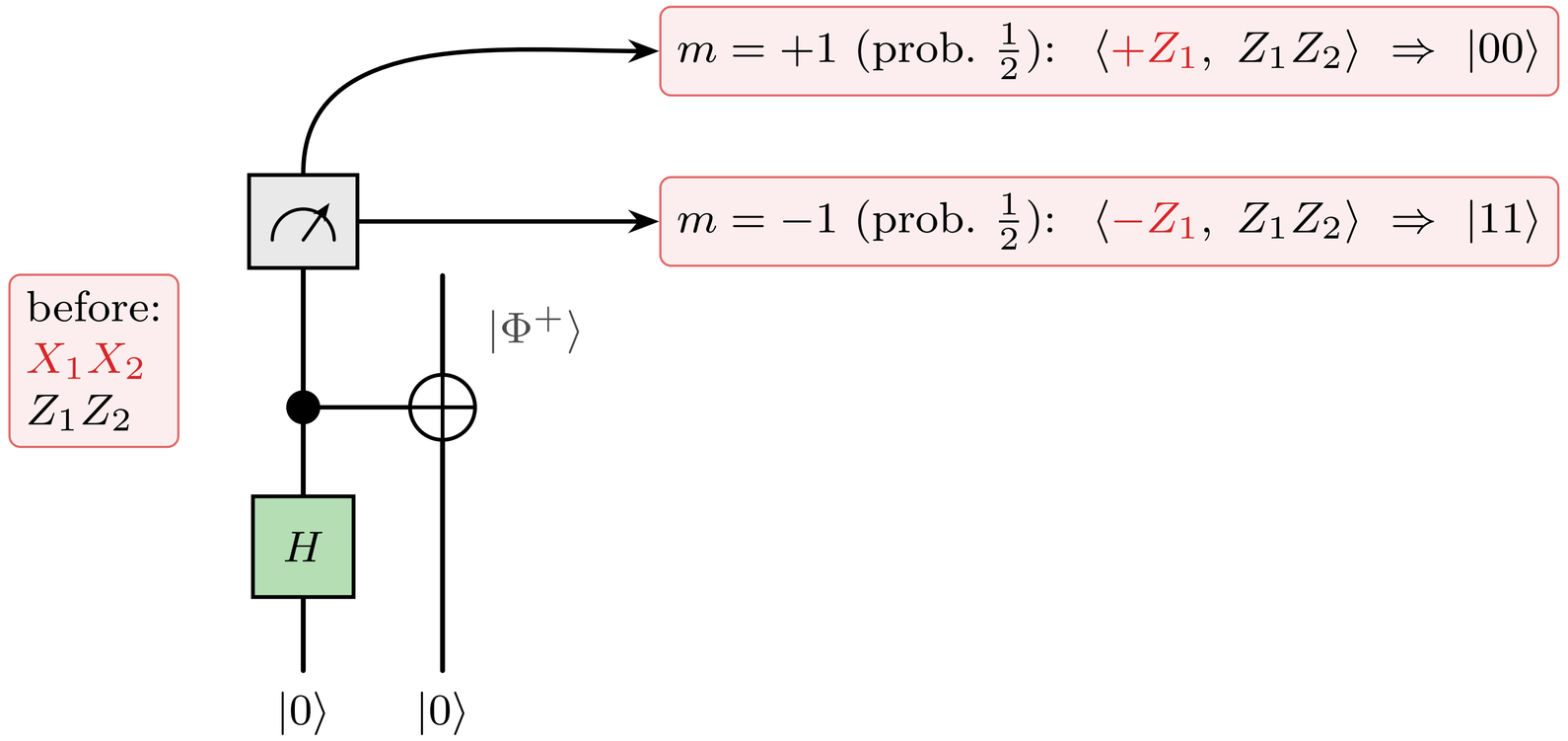
x_1	x_2	x_3	z_1	z_2	z_3	r
1	1	0	0	0	0	0
0	0	0	1	1	0	0
0	0	0	0	0	1	0

$\rightarrow$ after CNOT_{23}

x_1	x_2	x_3	z_1	z_2	z_3	r
1	1	1	0	0	0	0
0	0	0	1	1	0	0
0	0	0	0	1	1	0

$$H_a:\ r \oplus= x_a z_a,\ x_a \leftrightarrow z_a \qquad \mathrm{CNOT}_{a\to b}:\ r \oplus= x_a z_b (x_b \oplus z_a \oplus 1),\ x_b \oplus= x_a,\ z_a \oplus= z_b$$

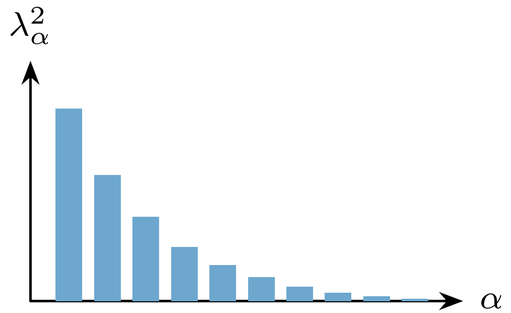
Measurements in the tableau



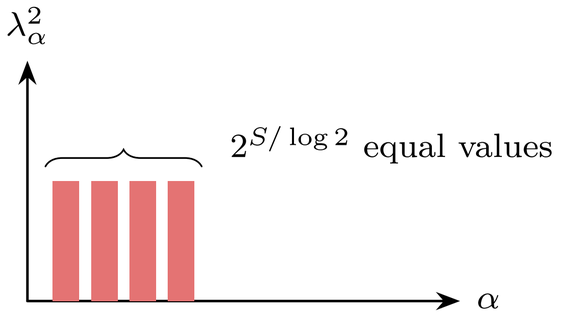
Q commutes with all g_j: deterministic. Some g_p anticommutes: outcome ± 1 with prob. $\frac{1}{2}$, $g_p \leftarrow \pm Q$.

Entanglement of stabilizer states

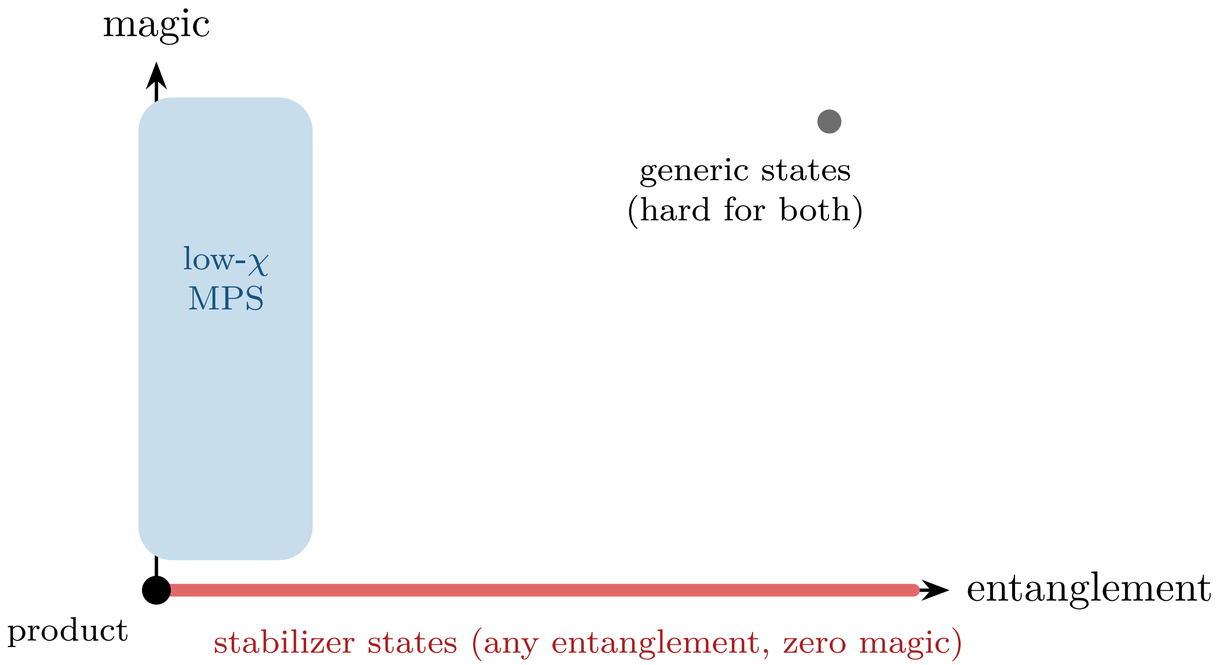
$$S(\rho_A) = \left(|A| - \log_2 |S_A|\right) \log 2$$



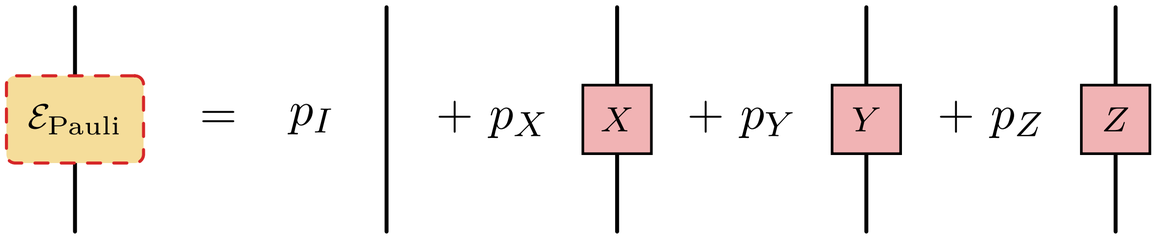
generic state



stabilizer state (flat)

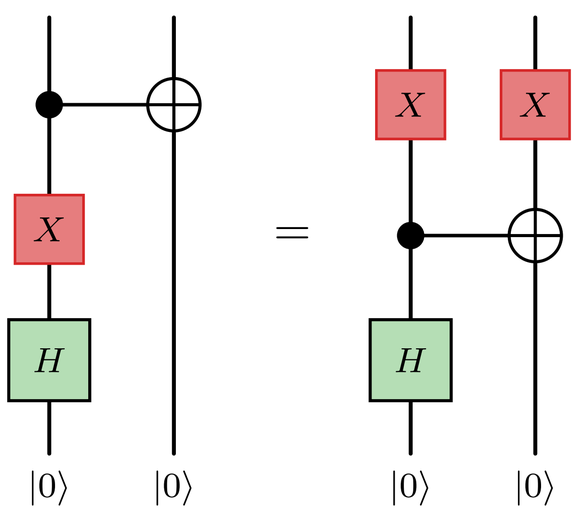


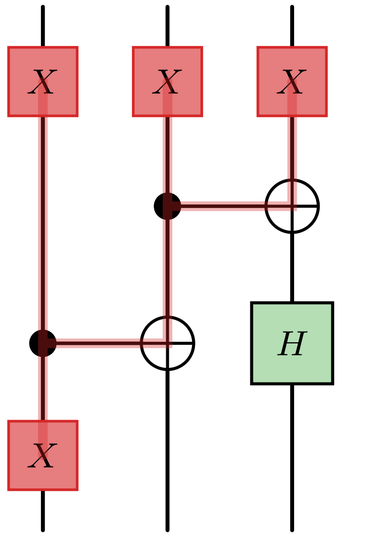
Pauli errors and the Pauli frame



$$C\,E = (CEC^\dagger)\,C, \qquad CEC^\dagger \in \mathcal{P}_N$$

with probability p_P the Pauli P is applied ($\sum_P p_P = 1$)



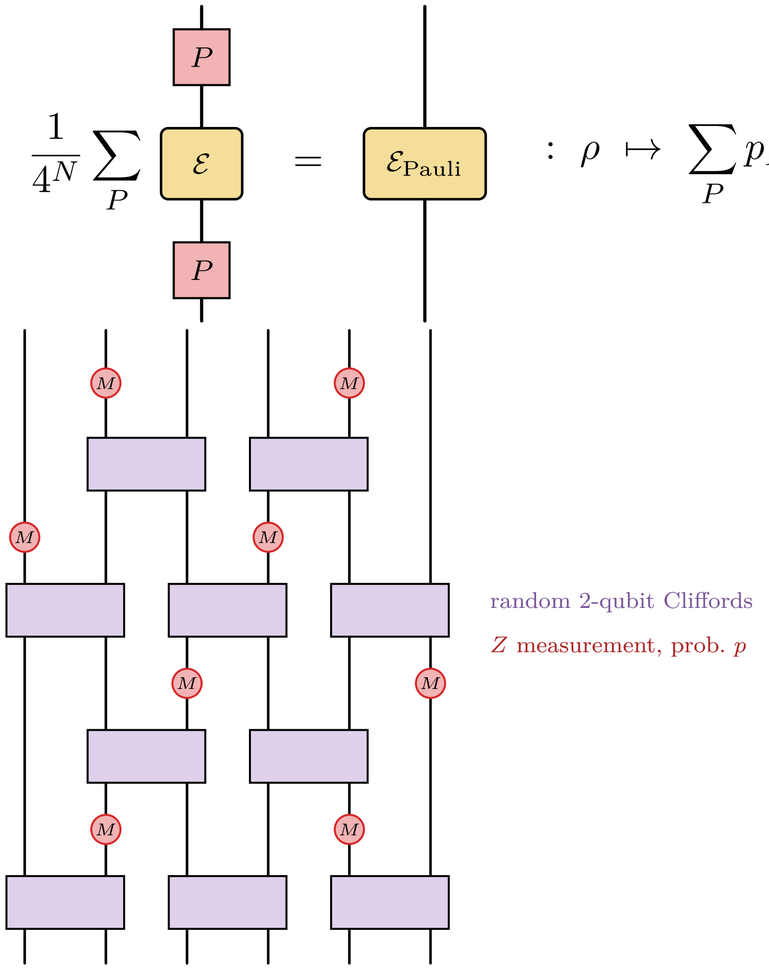


Pauli frame at the output

error in the middle

Twirling and monitored Clifford circuits

$$e^{-i\varepsilon X} \xrightarrow{\text{random } P \text{ before and after}} \rho \mapsto \cos^2\varepsilon\,\rho + \sin^2\varepsilon\, X\rho X$$

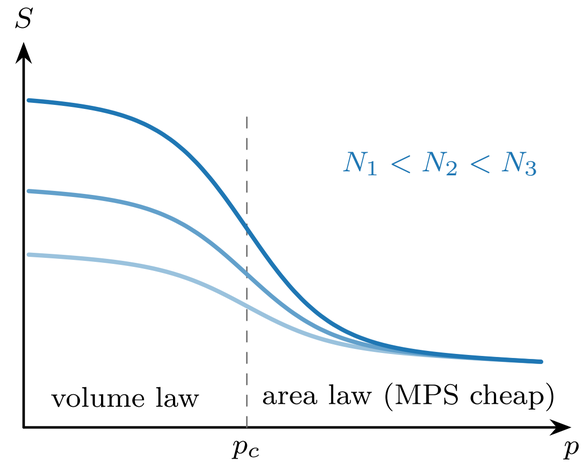


$$\frac{1}{4^N}\sum_P \quad \mathcal{E} \quad = \quad \mathcal{E}_{\text{Pauli}} \qquad : \ \rho \ \mapsto \ \sum_P p_P\, P\rho P$$

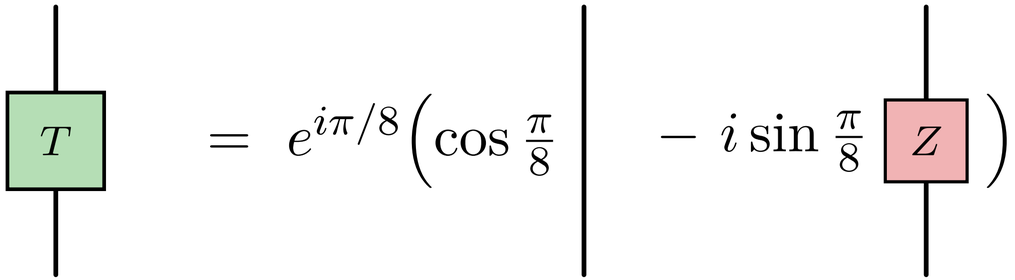
$$\left(R_{\text{tw}}\right)_{QQ'} = R_{QQ'}\,\delta_{QQ'}$$

random 2-qubit Cliffords

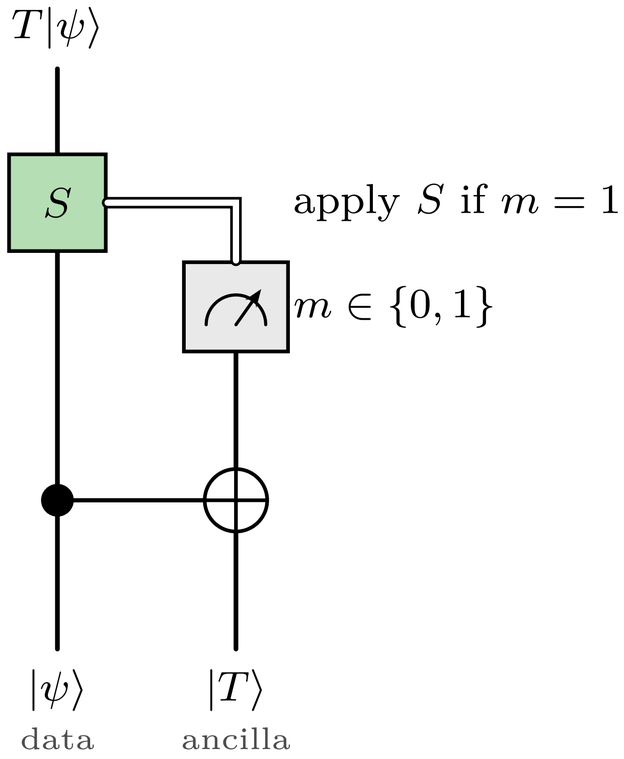
Z measurement, prob. p

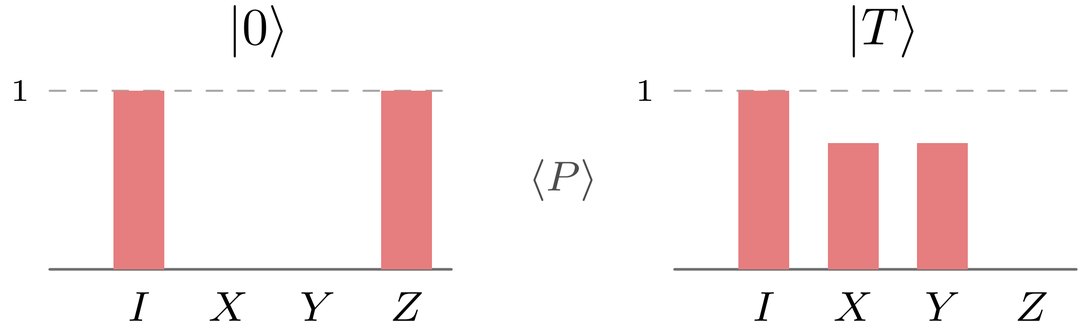


Beyond Clifford: the T gate and magic states



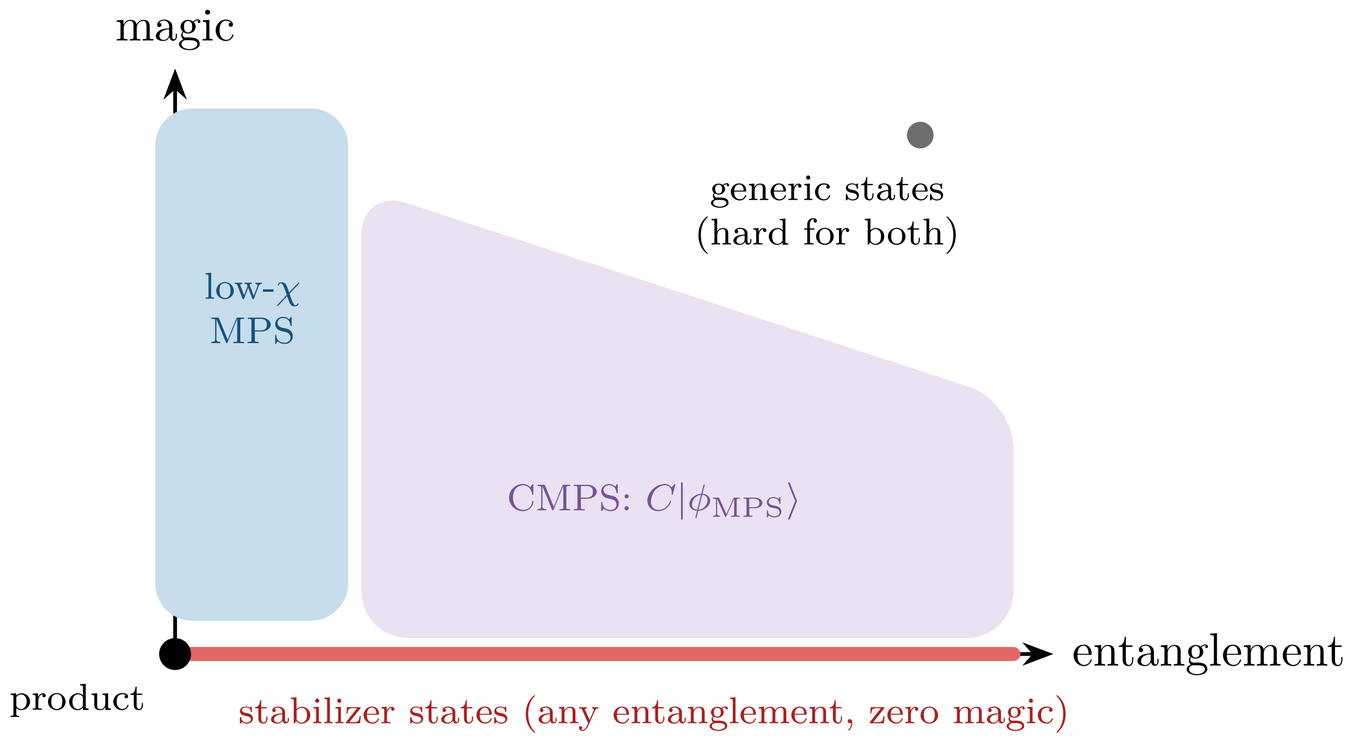
$$|T\rangle = e^{i\pi/8}\left(\cos\tfrac{\pi}{8}\,|+\rangle - i\sin\tfrac{\pi}{8}\,|-\rangle\right)$$





Two resources

	MPS	stabilizer tableau
cheap when	low entanglement	Clifford (zero magic)
cost driver	$\chi \sim e^S$	$e^{O(t)}$ $(\#T)$
fails on	random Clifford circuits	generic rotations

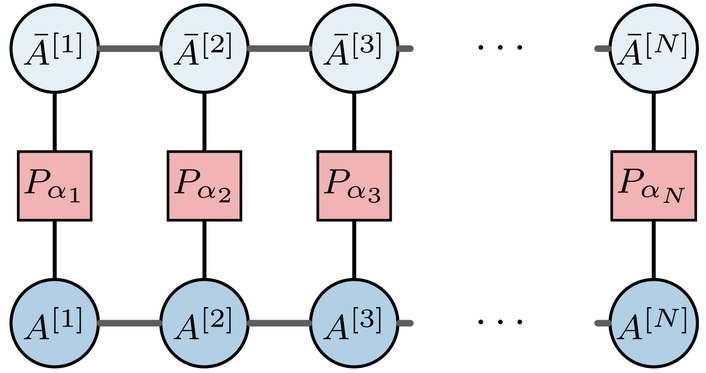


Lecture 4
Quantum magic meets tensor networks

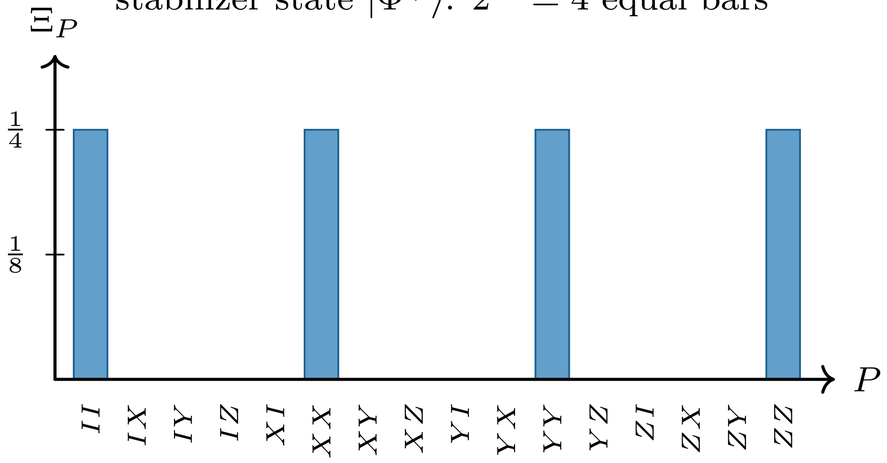
Stabilizer Rényi entropies

$$\Xi_P = \frac{\langle\psi|P|\psi\rangle^2}{2^N}, \qquad M_n = \frac{1}{1-n}\log\sum_P \Xi_P^n - N\log 2$$

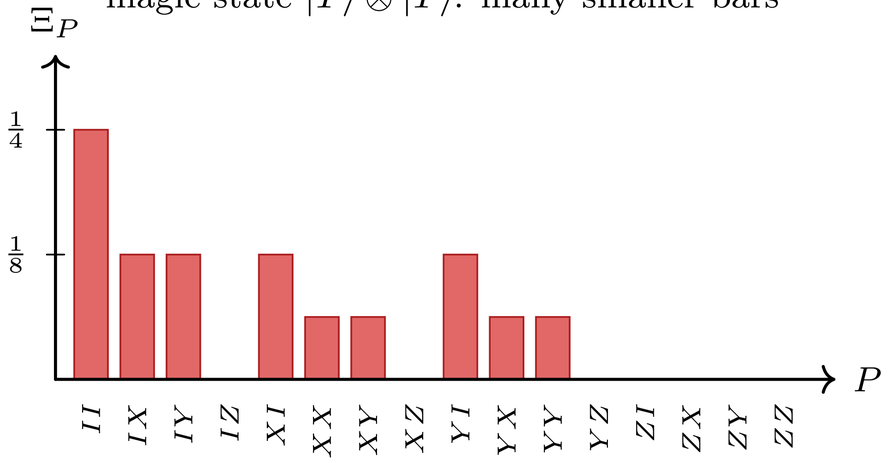
$$\langle\psi|P_{\alpha_1}\otimes\cdots\otimes P_{\alpha_N}|\psi\rangle \;=$$



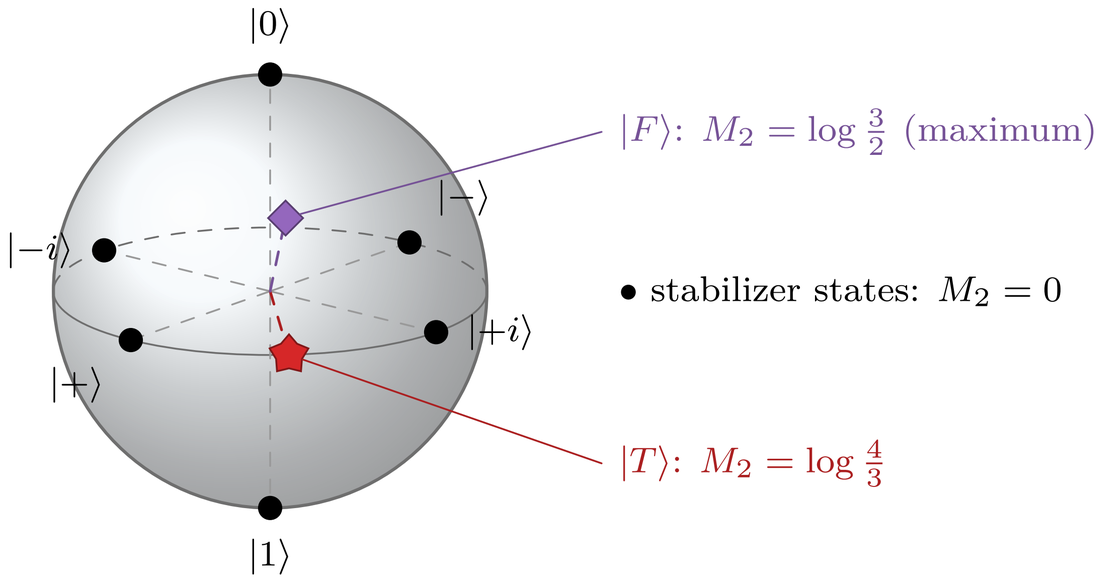
stabilizer state $|\Phi^+\rangle$: $2^N = 4$ equal bars



magic state $|T\rangle\otimes|T\rangle$: many smaller bars



Worked examples

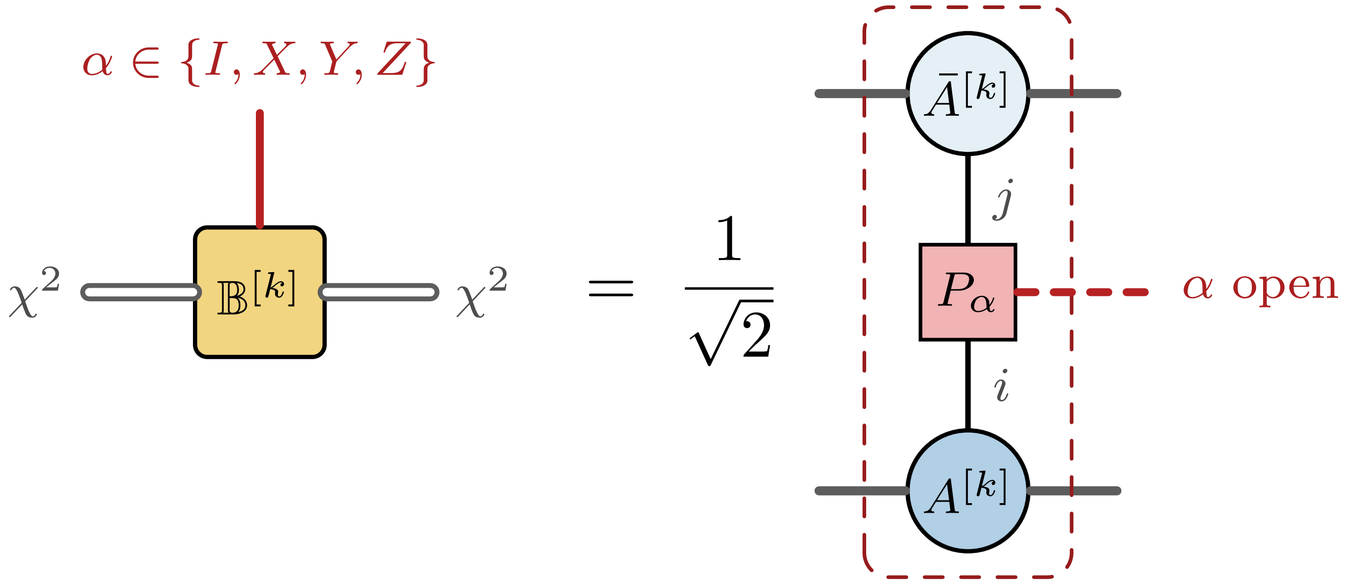


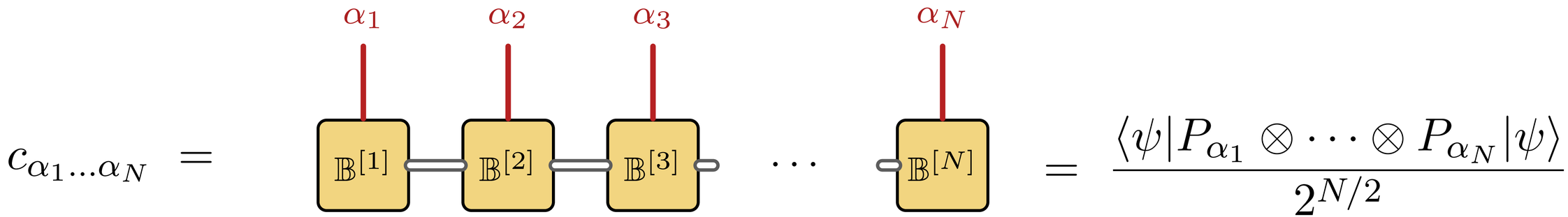
$$M_2 = -\log \frac{1 + x^4 + y^4 + z^4}{2}$$

$$M_2(|T\rangle) = \log \tfrac{4}{3}, \quad M_2(|F\rangle) = \log \tfrac{3}{2}$$

$$M_2^{\text{Haar}} \simeq \log \frac{D+3}{4}$$

The Pauli-MPS

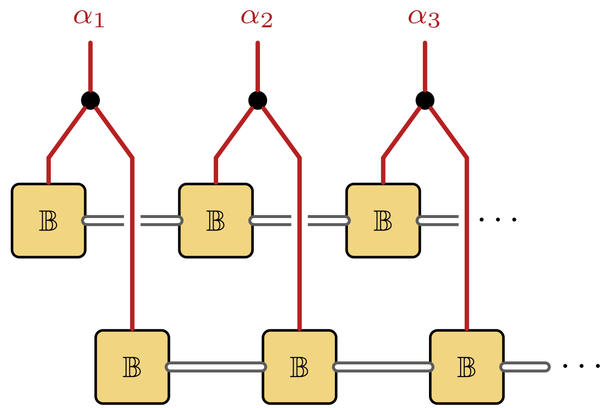




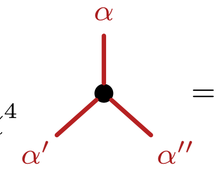
$\Xi_P = c_P^2$: the Born distribution of an MPS of bond χ^2

Exact replica contraction

$$(c \circ c)_{\alpha_1 \ldots \alpha_N} =$$

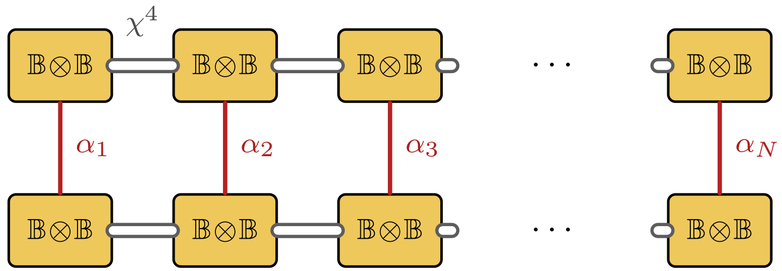


bond
$\chi^2 \cdot \chi^2 = \chi^4$



$= \delta_{\alpha\alpha'}\delta_{\alpha\alpha''}$ (copy tensor)

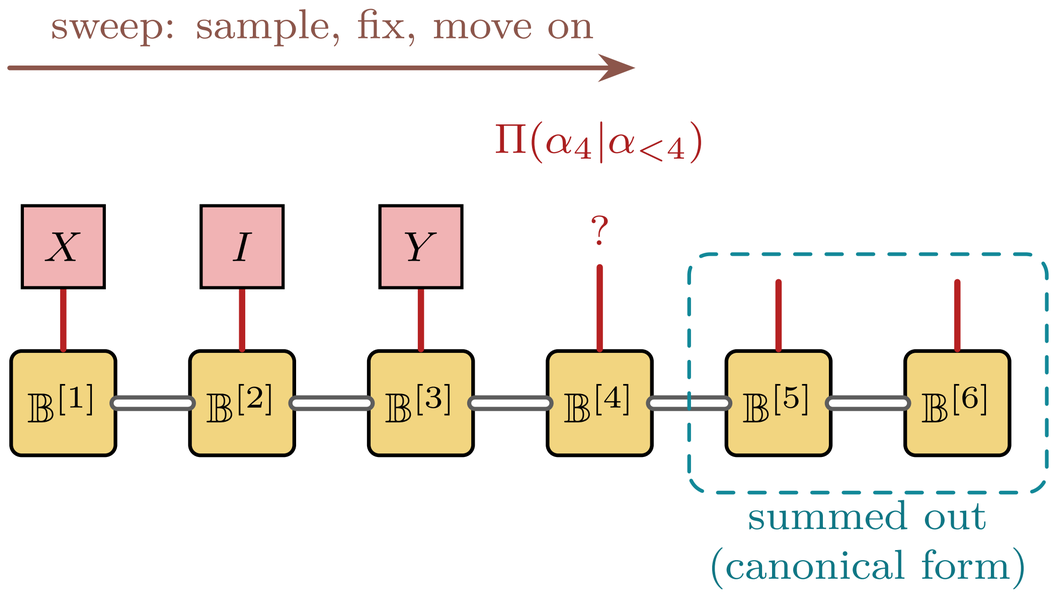
$$\sum_P \Xi_P^2 = \sum_P c_P^4 =$$



cost $O(N\chi^{12})$

$$\sum_P \Xi_P^2 = \langle\langle c \circ c | c \circ c \rangle\rangle, \quad O(N\chi^{12})$$

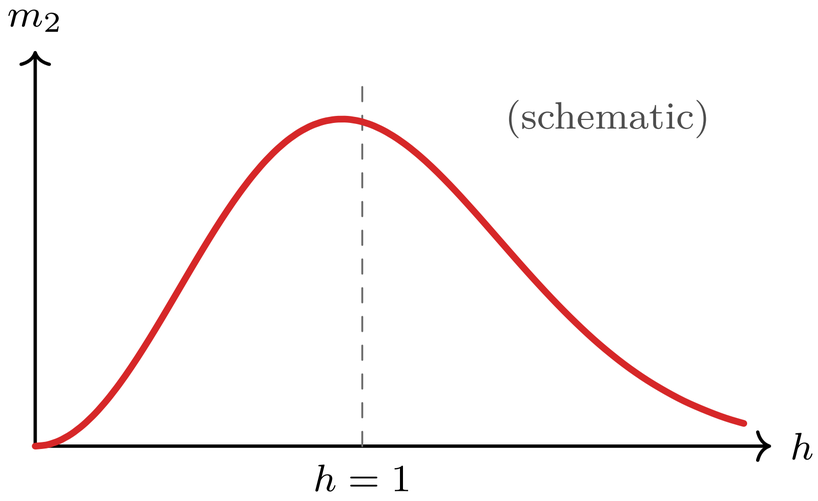
Perfect Pauli sampling

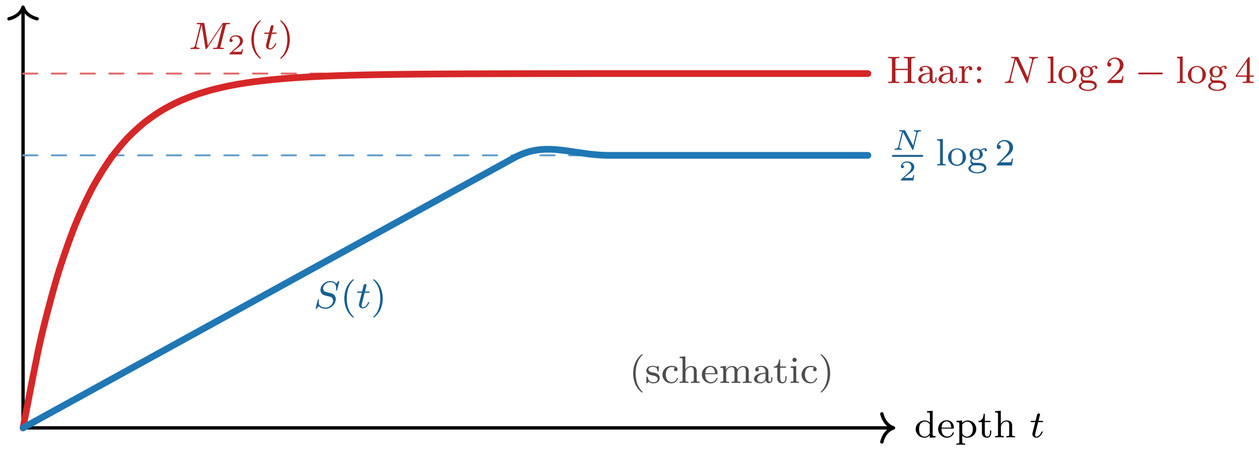


output: $P = XIY\cdots$, with $\Xi_P = \prod_k \Pi(\alpha_k|\alpha_{<k})$

$$M_n = \frac{1}{1-n}\log\left\langle \Xi_P^{n-1}\right\rangle_{P\sim\Xi} - N\log 2, \qquad M_1 = -\left\langle \log \Xi_P\right\rangle_{P\sim\Xi} - N\log 2$$

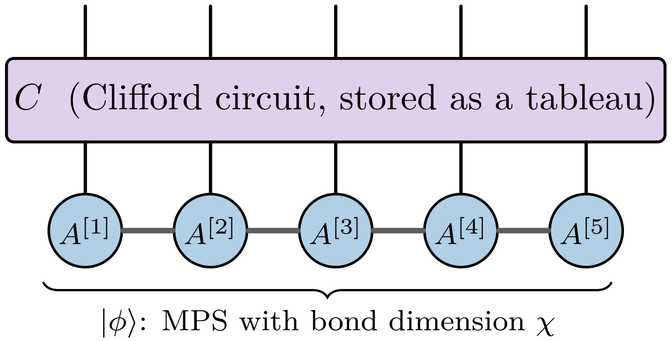
Magic in many-body states and circuits





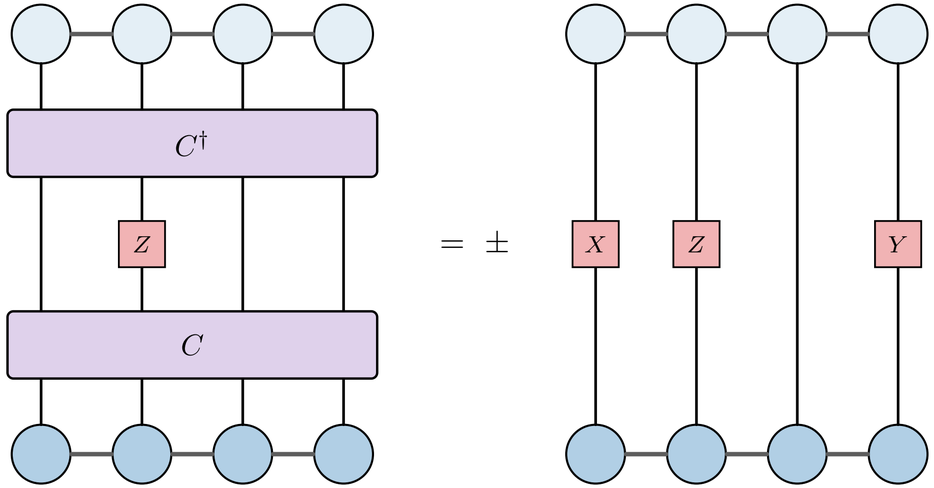
Clifford-enhanced MPS

$|\Psi\rangle =$



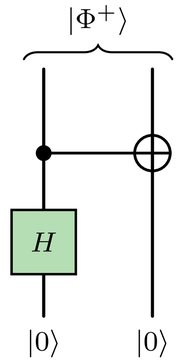
$$\langle\Psi|P|\Psi\rangle = \pm\langle\phi|P'|\phi\rangle, \quad M_n(\Psi) = M_n(\phi)$$

$\langle\Psi|P|\Psi\rangle =$

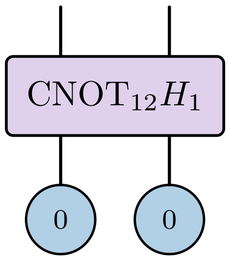


$= \pm$

$P' = C^\dagger P C$: non-local, but an MPO of bond 1

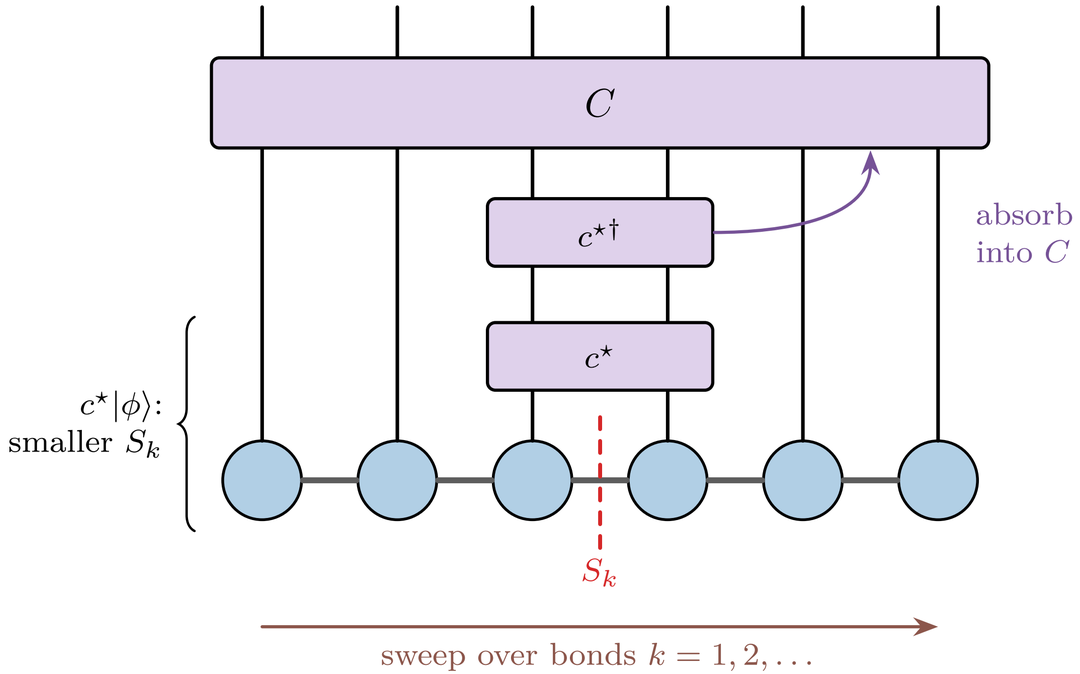


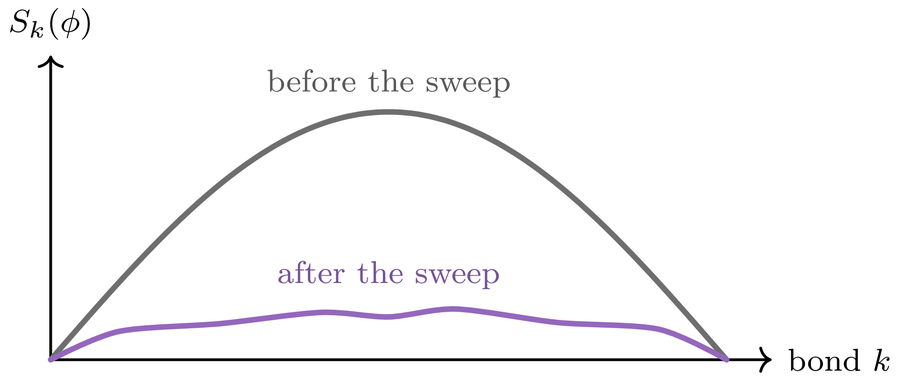
$=$

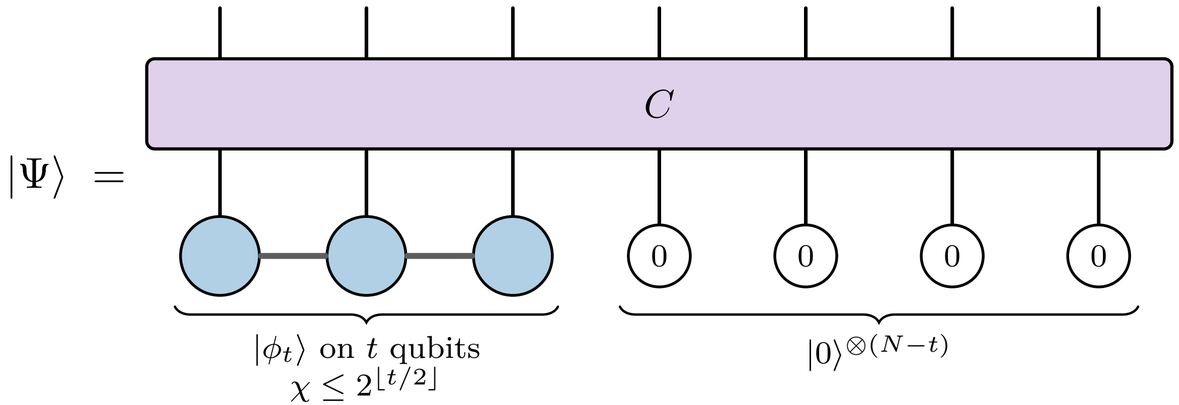


$|\phi\rangle = |00\rangle$, $\chi = 1$

Finding the Clifford: the disentangling sweep

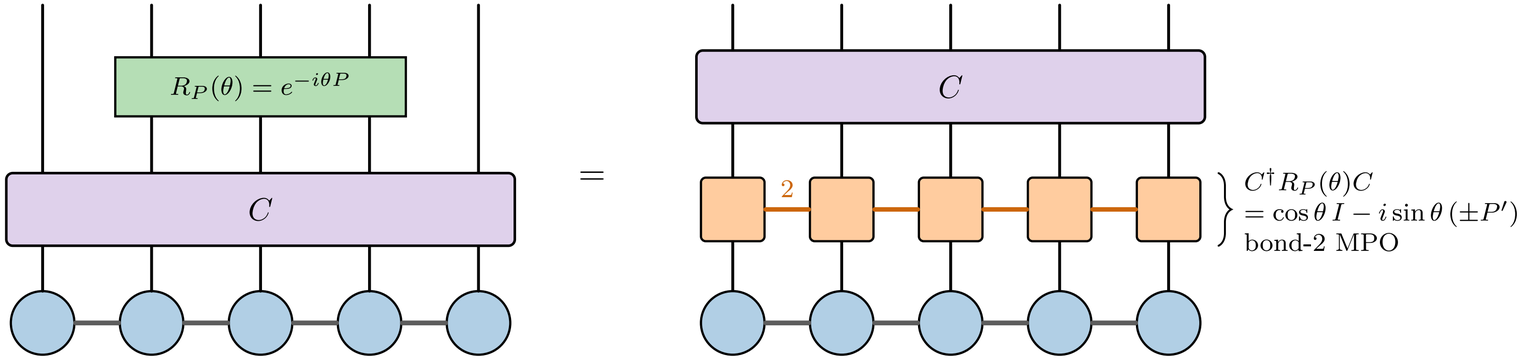




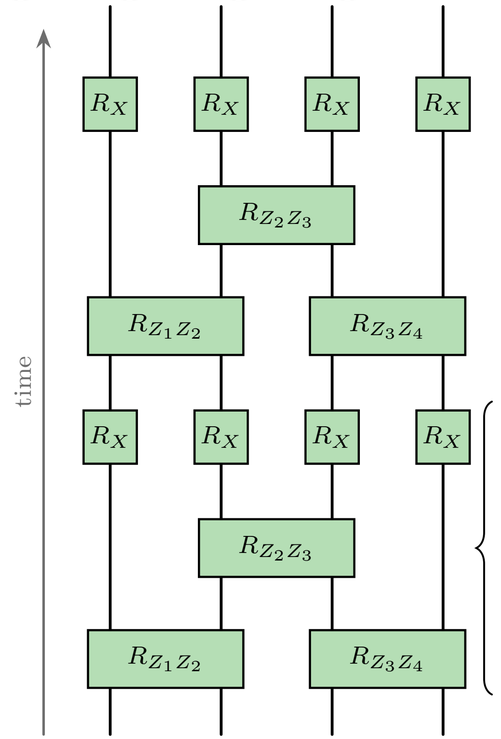


$$c^\star = \arg\min_{c \in \mathcal{C}_2} S_k(c\,\Theta), \quad 20 \text{ cosets}$$

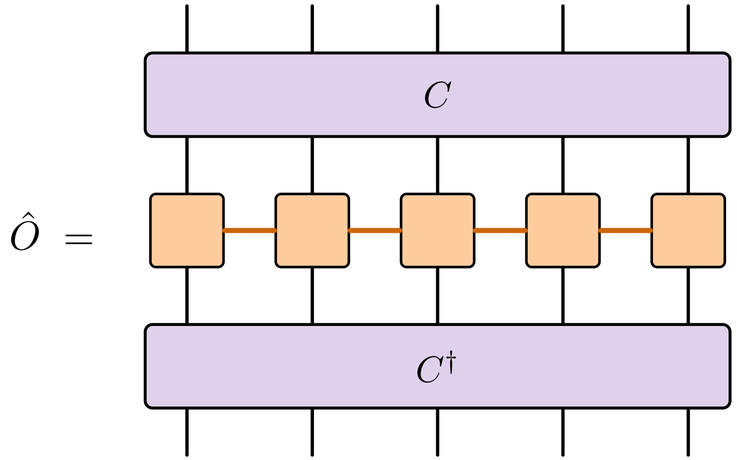
Dynamics with CMPS



$$C^\dagger e^{-i\theta P} C = \cos\theta\, I - i \sin\theta\, (\pm P')$$

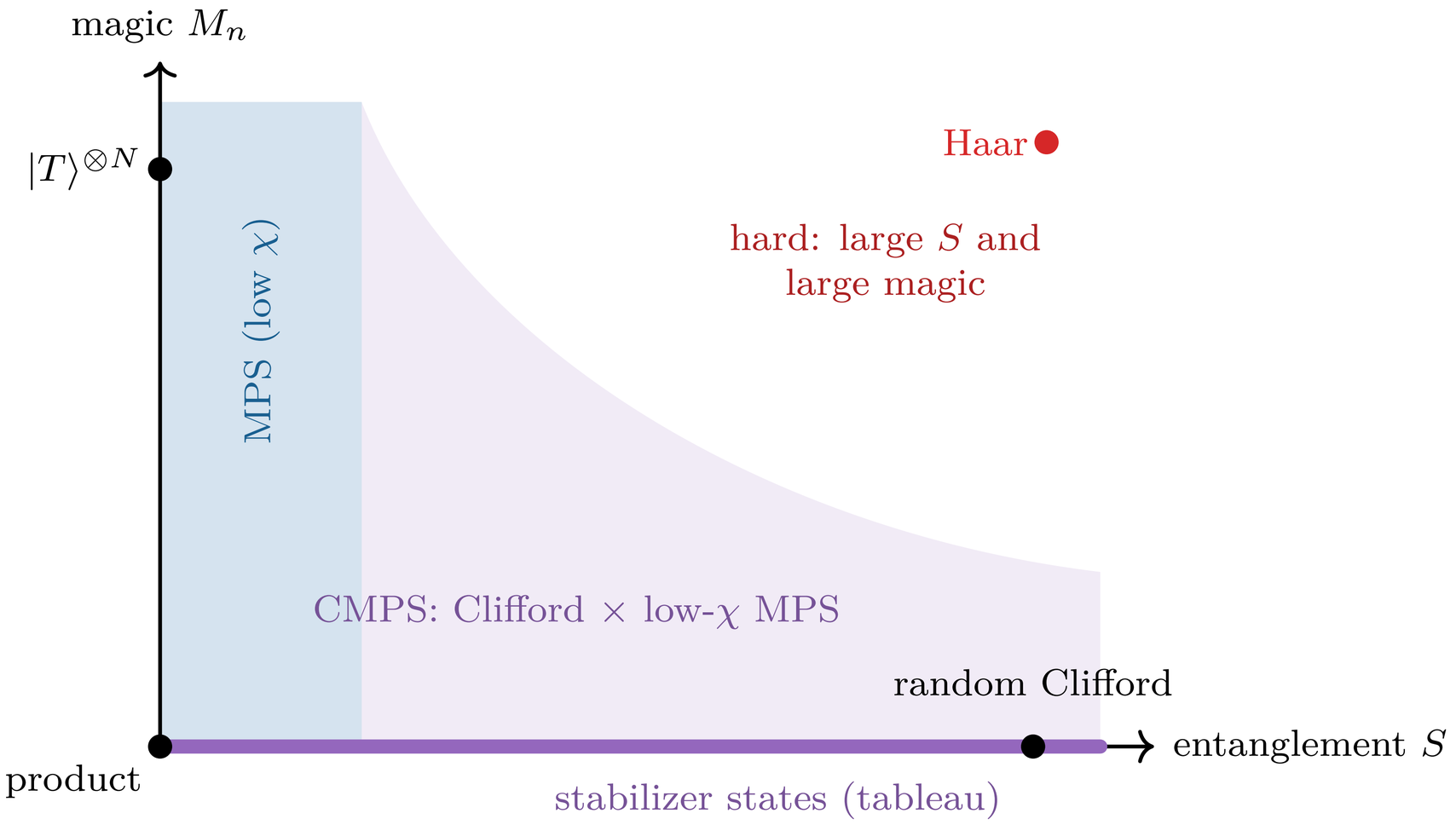


one Trotter step:
$\prod_j R_{P_j}(h_j \delta)$



W: MPO in the Pauli basis
(hybrid stabilizer MPO)

Synthesis



MPS: $\chi \sim e^{S}$ tableau: $e^{O(t)}$ CMPS: $\chi \sim e^{S(\phi)},\ S(\phi) \leq S(\Psi)$